AN ARTINIAN POINT-CONFIGURATION QUOTIENT AND THE STRONG LEFSCHETZ PROPERTY

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Abstract. In this paper, we study an Artinian point-configuration quotient having the SLP. We show that an Artinian quotient of points in $\mathbb{P}^n$ has the SLP when the union of two sets of points has a specific Hilbert function. As an application, we prove that an Artinian linear star configuration quotient $R/(I_X + I_Y)$ has the SLP if $X$ and $Y$ are linear star-configurations in $\mathbb{P}^2$ of type $s$ and $t$ for $s \geq \binom{t}{2} - 1$ and $t \geq 3$. We also show that an Artinian $k$-configuration quotient $R/(I_X + I_Y)$ has the SLP if $X$ is a $k$-configuration of type $(1,2)$ or $(1,2,3)$ in $\mathbb{P}^2$, and $X \cup Y$ is a basic configuration in $\mathbb{P}^2$.

1. Introduction

Ideals of sets of finite points in $\mathbb{P}^n$ have been studied for a long time ([8,9,11]), and in particular we consider an ideal of a special configuration in $\mathbb{P}^n$, so called a star-configuration and a $k$-configuration in $\mathbb{P}^n$ ([1–3, 6, 7, 9–11, 15]). In 2006, Geramita, Migliore, and Sabourin introduced the notion of a star-configuration set of points in $\mathbb{P}^2$ (see [10]), the name having been inspired by the fact that 10-points in $\mathbb{P}^2$, defined by 5 general linear forms in $\mathbb{k}[x_0, x_1, x_2]$ resembles a star. In this paper, we refer to this as a “linear star-configuration”, as more general definition of star-configurations has evolved through the subsequent literature (see [1,6,7,19]). Indeed, a star-configuration in $\mathbb{P}^n$ has been studied to find the dimension of secant varieties to the variety of reducible forms in $R = \mathbb{k}[x_0, x_1, \ldots, x_n]$, where $\mathbb{k}$ is a field of characteristic 0 (see [4,5,20]).

If $R/I$ is a standard graded Artinian algebra and $\ell$ is a general linear form, we recall that $R/I$ is said to have the weak Lefschetz property (WLP) if the
multiplication map by $\ell$

$$[R/I]_d \xrightarrow{\ell} [R/I]_{d+1}$$

has maximal rank for every $d \geq 0$. Over the years, there have been several papers which have devoted to a classification of possible Artinian quotients having the WLP (see [1,8,9,13,14,16–18,21,22]). The strong Lefschetz property (SLP) says that for every $i \geq 1$ the multiplication map by $\ell^i$

$$[R/I]_d \xrightarrow{\ell^i} [R/I]_{d+i}$$

has maximal rank for every $d \geq 0$ ([13, 14, 17]). In [14] the authors proved that a complete intersection ideal in $k[x_0, x_1]$ has the SLP. Moreover, in [13], the authors give a nice description for a graded Artinian ring having the SLP by using the so-called Jordan type (see Lemma 2.2). The Jordan type is the partition of $n$ specifying the lengths of blocks in the Jordan block matrix determined by the multiplication map by $\ell$ in a suitable $k$-basis for $R/I$. Here, we apply this result often to show that some Artinian quotients of the ideals of points in $\mathbb{P}^n$ have the SLP.

We use Hilbert functions for many our arguments. Given a homogeneous ideal $I \subset R$, the Hilbert function of $R/I$, denoted $H_{R/I}$, is the numerical function $H_{R/I} : \mathbb{Z}^+ \cup \{0\} \to \mathbb{Z}^+ \cup \{0\}$ defined by

$$H_{R/I}(i) := \dim_k[R/I]_i = \dim_k[R]_i - \dim_k[I]_i,$$

where $[R]_i$ and $[I]_i$ denote the $i$-th graded component of $R$ and $I$, respectively. If $I := I_X$ is the defining ideal of a subscheme $X$ in $\mathbb{P}^n$, then we denote

$$H_{R/I_X}(i) := H_X(i) \quad \text{for} \quad i \geq 0,$$

and call it the Hilbert function of $X$.

Let $R = k[x_0, x_1, \ldots, x_n]$ be a polynomial ring over a field $k$ of characteristic 0. For positive integers $r$ and $s$ with $1 \leq r \leq \min\{n, s\}$, suppose $F_1, \ldots, F_s$ are general forms in $R$ of degrees $d_1, \ldots, d_s$, respectively. Here $s$ general forms $F_1, \ldots, F_s$ in $R$ means that all subsets of size $1 \leq r \leq \min\{n+1, s\}$ are regular sequences in $R$, and if $\mathcal{H} = \{F_1, \ldots, F_s\}$ is a collection of distinct hypersurfaces in $\mathbb{P}^n$ corresponding to general $F_1, \ldots, F_s$ respectively, then the hypersurfaces meet properly, by which we mean that the intersection of any $r$ of these hypersurfaces with $1 \leq r \leq \min\{n, s\}$ has codimension $r$. We call the variety $X$ defined by the ideal

$$\bigcap_{1 \leq i_1 < \cdots < i_r \leq s} (F_{i_1}, \ldots, F_{i_r})$$

a star-configuration in $\mathbb{P}^n$ of type $(r, s)$. In particular, if $X$ is a star-configuration in $\mathbb{P}^n$ of type $(n, s)$, then we simply call a point star-configuration in $\mathbb{P}^n$ of type $s$ for short.
Notice that each \( n \)-forms \( F_{i_1}, \ldots, F_{i_n} \) of \( s \)-general forms \( F_1, \ldots, F_s \) in \( R \) define \( d_{i_1} \cdots d_{i_n} \) points in \( \mathbb{P}^n \) for each \( 1 \leq i_1 < \cdots < i_n \leq s \). Thus the ideal
\[
\bigcap_{1 \leq i_1 < \cdots < i_n \leq s} (F_{i_1}, \ldots, F_{i_n})
\]
defines a finite set \( X \) of points in \( \mathbb{P}^n \) with
\[
\deg(X) = \sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq s} d_{i_1} d_{i_2} \cdots d_{i_n}.
\]
Furthermore, if \( F_1, \ldots, F_s \) are general linear (quadratic, cubic, quartic, quintic, etc) forms in \( R \), then we call \( X \) a linear (quadratic, cubic, quartic, quintic, etc) star-configuration in \( \mathbb{P}^n \) of type \( s \), respectively.

To provide some additional focus to this paper, we consider the following questions.

**Question 1.1.** Let \( X \) and \( Y \) be finite sets of points in \( \mathbb{P}^n \) and \( R = k[x_0, x_1, \ldots, x_n] \).

(a) Does an Artinian ring \( R/(I_X + I_Y) \) have the WLP?
(b) Does an Artinian ring \( R/(I_X + I_Y) \) have the SLP?

**Question 1.2.** More precisely, let \( X \) and \( Y \) be finite point star configurations in \( \mathbb{P}^n \), or \( X \) be a \( k \)-configuration in \( \mathbb{P}^n \) such that \( X \cup Y \) is a basic configuration in \( \mathbb{P}^n \).

(a) Does an Artinian ring \( R/(I_X + I_Y) \) have the WLP?
(b) Does an Artinian ring \( R/(I_X + I_Y) \) have the SLP?

In [1], the authors proved that an Artinian linear star-configuration quotient in \( \mathbb{P}^2 \) has the WLP, which is a partial answer to Question 1.2(a). Indeed, it is still true that any finite number of an Artinian linear point star-configuration quotient in \( \mathbb{P}^n \) has the WLP. In [8,9], the authors show that Question 1.2(a) is true in general if \( X \) is a \( k \)-configuration in \( \mathbb{P}^n \) and \( X \cup Y \) is a basic configuration in \( \mathbb{P}^n \) with the condition \( 2\sigma(X) \leq \sigma(X \cup Y) \), where
\[
\sigma(X) = \min\{i | H_X(i-1) = H_X(i)\}.
\]

In this paper, we focus on Questions 1.1(b) and 1.2(b). More precisely, we first find a condition in which an Artinian quotient of two sets of points in \( \mathbb{P}^n \) has the SLP (see Lemma 2.4 and Proposition 2.5). Next we find some Artinian linear star configuration quotient in \( \mathbb{P}^2 \) that has the SLP (see Corollary 2.9). Then, we find an Artinian \( k \)-configuration quotient having the SLP (see Proposition 3.4 and Theorem 3.6). Unfortunately, we do not have any counter example of an Artinian quotient \( R/(I_X + I_Y) \) of two point sets in \( \mathbb{P}^n \), which does not have the SLP, and thus we expect Question 1.1(a) and (b) are true in general, especially when \( X \) and \( Y \) are sets of general points in \( \mathbb{P}^n \).

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2. Artinian linear star-configuration quotients in \( \mathbb{P}^2 \)

In this section, we shall show that an Artinian ring \( R/(I_X + I_Y) \) has the SLP if \( X \) and \( Y \) are linear star-configurations in \( \mathbb{P}^2 \) of type \( s \) and \( t \) with \( s \geq \left( \frac{t^2}{4} \right) - 1 \) and \( t \geq 3 \), respectively.

We first introduce the following two results of a star-configuration in \( \mathbb{P}^n \) in [13, 22].

Remark 2.1. Let \( k \) be a field of characteristic zero and let \( F \in k[x_0, x_1, \ldots, x_n] = R = \bigoplus_{i \geq 0} R_i \) be a homogeneous polynomial (form) of degree \( d \), i.e., \( F \in R_d \). It is well known that in this case each \( R_i \) has a basis consisting of \( i \)-th powers of linear forms. Thus we may write

\[
F = \sum_{i=1}^{r} \alpha_i L_i^d, \quad \alpha_i \in k, \ L_i \in R_i.
\]

If \( k \) is algebraically closed (which we now assume for the rest of the paper), then each \( \alpha_i = \beta_i^d \) for some \( \beta_i \in k \) and so we can write

\[
(2.1) \quad F = \sum_{i=1}^{r} (\beta_i L_i)^d = \sum_{i=1}^{r} M_i^d, \quad M_i \in R_i.
\]

We call a description of \( F \) as in equation (2.1), a Waring Decomposition of \( F \). The least integer \( r \) such that \( F \) has a Waring Decomposition with exactly \( r \) summands is called the Waring Rank (or simply the rank) of \( F \).

Lemma 2.2 ([13]). Assume \( A \) is graded and \( H_A \) is unimodal. Then

(a) \( A \) has the WLP if and only if the number of parts of the Jordan type \( J_\ell = \max \{ H_A(i) \} \) (The Sperner number of \( A \));

(b) \( \ell \) is a strong Lefschetz element of \( A \) if and only if \( J_\ell = H_\ell^X \).

Proposition 2.3 ([22, Proposition 2.5]). Let \( X \) and \( Y \) be linear star-configurations in \( \mathbb{P}^2 \) of type \( s \) and \( t \), respectively, with \( 3 \leq t \) and \( s \geq \left\lfloor \frac{1}{2} \left( \frac{t^2}{4} \right) \right\rfloor \). Then \( X \cup Y \) has generic Hilbert function.

Recall that

\[
H_A : h_0 \ h_1 \ \cdots \ h_s
\]

is said to be unimodal if there exists \( j \) such that

\[
\begin{cases} 
    h_i \leq h_{i+1} & (i < j), \\
    h_i \geq h_{i+1} & (j \leq i).
\end{cases}
\]

Lemma 2.4. Let \( X \) be a finite set of points in \( \mathbb{P}^n \) and let \( A \) be an Artinian quotient of the coordinate ring of \( X \). Assume that \( H_A(i) = H_X(i) \) for every \( 0 \leq i \leq s-1 \) and \( A_s = 0 \). Then an Artinian ring \( A \) has the SLP.

Proof. First, we assume that the Hilbert function of \( A \) is of the form

\[
H_A : h_0 \ h_1 \ \cdots \ h_{\sigma-1} \ h_\sigma \ \cdots \ h_{s-1} \ 0.
\]
where \( h_{\sigma-2} < h_{\sigma-1} = h_\sigma = \cdots = h_{s-1} \).

Let \( \ell \) be a general linear form in \( A_1 \). Since \( \ell \) is not a zero divisor of \( A \), we see that the multiplication map by \( \ell^{s-1} \)
\[
[R/I_x]_0 = [A]_0 \times^{\ell^{s-1}} [A]_{s-1} = [R/I_x]_{s-1}
\]
is injective. Hence we have a string of length \( s \)
\[1, \ell, \ldots, \ell^{s-1},\]
and so the Jordan type \( J_{\ell} \) for \( H_A \) is of the form
\[J_{\ell} = (s, \ldots, \cdot \cdot \cdot).\]

(i) Let \( i = 1 \). Then the multiplication map by \( \ell^{s-2} \)
\[
[R/I_x]_1 = [A]_1 \times^{\ell^{s-2}} [A]_{s-1} = [R/I_x]_{s-1}
\]
is injective. Hence there are \( g_1 \) := \((h_1 - h_0) = (h_1 - 1)\) linear forms
\( F_{1,1}, F_{1,2}, \ldots, F_{1,g_1} \in [A]_1 \) such that the \( h_1 \) linear forms
\( \ell, F_{1,1}, F_{1,2}, \ldots, F_{1,g_1} \)
are linearly independent. Hence there are \( g_1 \)-strings of length \((s - 1)\)
\[F_{1,1}, F_{1,1}\ell, \ldots, F_{1,1}\ell^{s-2}, \quad \text{and} \quad F_{1,2}, F_{1,2}\ell, \ldots, F_{1,2}\ell^{s-2}; \]
\[\vdots \]
\[F_{1,g_1}, F_{1,g_1}\ell, \ldots, F_{1,g_1}\ell^{s-2}. \]

(ii) For \( 1 \leq i < \sigma - 1 \) and \( 1 \leq j \leq i \), define
\[g_j := h_j - h_{j-1}\]
for such \( j \). Assume that there are \( g_j \)-forms \( F_{j,1}, \ldots, F_{j,g_j} \in [A]_j \) and there are \( g_j \)-strings of length \((s - j)\)
\[F_{j,1}, F_{j,1}\ell, \ldots, F_{j,1}\ell^{s-j-1}, \]
\[F_{j,2}, F_{j,2}\ell, \ldots, F_{j,2}\ell^{s-j-1}, \]
\[\vdots \]
\[F_{j,g_j}, F_{j,g_j}\ell, \ldots, F_{j,g_j}\ell^{s-j} \]
such that the \((1 + \sum_{k=1}^{i} g_k)\)-forms
\[\ell^i, F_{1,1}\ell^{i-1}, \ldots, F_{i,g_i}\ell^{i-1}, \ldots, F_{j-1,1}\ell, \ldots, F_{j-1,g_{j-1}}\ell, F_{j,1}, \ldots, F_{j,g_j} \]
are linearly independent for such \( j \).

Since the multiplication map by \( \ell^{(s-1)-(i+1)} \)
\[
[R/I_x]_{i+1} = [A]_{i+1} \times^{\ell^{(s-1)-(i+1)}} [A]_{s-1} = [R/I_x]_{s-1}
\]
is injective, there are linearly independent \( g_{i+1} := (h_{i+1} - h_i) \)-forms \( F_{i+1,1}, \ldots, F_{i+1,g_{i+1}} \in [A]_{i+1} \). Then the following \((1 + \sum_{k=1}^{i+1} g_k)\)-forms

\[
\ell^{i+1}, F_{1,1}, \ell^i, \ldots, F_{1,g_{i+1}} \ell^i, \ldots, F_{i-1,1}, \ell^2, F_{i-1,g_i} \ell^2, F_{i,1}, \ell, F_{i,g_i} \ell, F_{i+1,1}, \ldots, F_{i+1,g_{i+1}} \]
\]

are linearly independent as well. Hence we have \( g_{i+1}\)-strings of length \((s - i - 1)\)

\[
\begin{align*}
F_{i+1,1}, F_{i+1,1} \ell, & \ldots, F_{i+1,1} \ell^{s-i-2}, \\
F_{i+1,2}, F_{i+1,2} \ell, & \ldots, F_{i+1,2} \ell^{s-i-2}, \\
& \vdots \\
F_{i+1,g_{i+1}}, F_{i+1,g_{i+1}} \ell, & \ldots, F_{i+1,g_{i+1}} \ell^{s-i-2}.
\end{align*}
\]

It is from (i) \( \sim \) (ii) that the Jordan type

\[
J_{\ell} = (s, s-1, \ldots, s-1, s-i, \ldots, s-i, \ldots, s-\sigma+1, \ldots, s-\sigma+1) = H_{A_Y}^\ell,
\]

as we wished. Therefore, by Lemma 2.2, an Artinian ring has the SLP, which completes the proof. \( \square \)

The following proposition is immediate from Lemma 2.4.

**Proposition 2.5.** Let \( X \) and \( Y \) be linear star-configurations in \( \mathbb{P}^2 \) of type \( t \) and \( s \) with \( t \geq 2 \) and \( s \geq (t-1)_2 \). Then an Artinian ring \( R/(I_X + I_Y) \) has the SLP. 

**Proof.** First, note that the Hilbert functions of \( R/I_X, R/I_Y, \) and \( R/(I_X \cap I_Y) \) (see Proposition 2.3) are

\[
\begin{align*}
H_{R/I_X} & : 1 \ 3 \ \cdots \ \binom{(t-2)-nd}{2} \ \binom{(t-1)-nd}{2} \to, \\
H_{R/I_Y} & : 1 \ 3 \ \cdots \ \binom{(t-2)-nd}{2} \ \binom{(t+1)-nd}{2} \ \binom{(s-2)-nd}{2} \ \binom{s-1-st}{2} \to, \\
H_{R/(I_X \cap I_Y)} & : 1 \ 3 \ \cdots \ \binom{(t-2)-nd}{2} \ \binom{(t+1)-nd}{2} \ \binom{(s-2)-nd}{2} \ \binom{s-1-st}{2} \ = \binom{t}{2} \ \binom{s}{2} \ \to,
\end{align*}
\]

respectively. Using the exact sequence

\[
0 \to R/I_X \cap I_Y \to R/I_X \oplus R/I_Y \to R/(I_X + I_Y) \to 0,
\]

the Hilbert function of \( R/(I_X + I_Y) \) is

\[
H_{R/(I_X + I_Y)} : 1 \ 3 \ \cdots \ \binom{(t-2)-nd}{2} \ \cdots \ \binom{(s-2)-nd}{2} \ 0 \to,
\]

and so by Lemma 2.4, an Artinian linear star configuration quotient \( R/(I_X + I_Y) \) has the SLP, which completes the proof. \( \square \)

**Example 2.6.** Let \( X \) and \( Y \) be linear star-configurations in \( \mathbb{P}^2 \) of type \( 5 \) and \( 9 \), respectively. Note that \( 9 = (\binom{5}{2}) - 1 \). By Proposition 2.3 the Hilbert function of an Artinian ring \( A := R/(I_X + I_Y) \) is

\[
(1, 3, 6, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 8-nd).
\]
(a) By Waring decomposition, there is a general linear form $\ell \in [A]_1$ such that
$$\ell^8 \in [A]_8,$$
i.e., we have a string of length 9
$$1, \ell, \ldots, \ell^8.$$Hence the Jordan type $J_\ell$ is of the form
$$J_\ell = (9, \ldots).$$

(b) Note that the multiplication map by $\ell^6$
$$[A]_1 \times^{\ell^6} \to [A]_7$$is injective, and the multiplication map by $\ell^7$
$$[A]_1 \times^{\ell^7} \to [A]_8$$is surjective. Then we can choose a basis $\{\ell, F_{1,1}, F_{1,2}\}$ for $[A]_1$ such that
$$F_{1,1}\ell^6, F_{1,2}\ell^6 \neq 0, \quad \text{and} \quad F_{1,1}\ell^7, F_{1,2}\ell^7 = 0.$$Moreover, since $\{F_{1,1}\ell^6, F_{1,2}\ell^6\}$ is linearly independent, we have 2-strings of length 7
$$F_{1,1}, F_{1,1}\ell, \ldots, F_{1,1}\ell^6, \quad \text{and} \quad F_{1,2}, F_{1,2}\ell, \ldots, F_{1,2}\ell^6.$$Note that the multiplication map by $\ell^5$
$$[A]_2 \times^{\ell^5} \to [A]_7$$is injective, and the multiplication map by $\ell^6$
$$[A]_2 \times^{\ell^6} \to [A]_8$$is surjective. Then we can choose a basis $\{\ell^2, F_{1,1}\ell, F_{1,2}\ell, F_{2,1}, F_{2,2}, F_{2,3}\}$ for $[A]_2$ such that
$$F_{2,1}\ell^5, F_{2,2}\ell^5, F_{2,3}\ell^5 \neq 0, \quad \text{and} \quad F_{2,1}\ell^6, F_{2,2}\ell^6, F_{2,3}\ell^6 = 0.$$Moreover, since $\{F_{2,1}\ell^5, F_{2,2}\ell^5, F_{2,3}\ell^5\}$ is linearly independent, we have 3-strings of length 6
$$F_{2,1}, F_{2,1}\ell, \ldots, F_{2,1}\ell^5, \quad F_{2,2}, F_{2,2}\ell, \ldots, F_{2,2}\ell^5, \quad \text{and} \quad F_{2,3}, F_{2,3}\ell, \ldots, F_{2,3}\ell^5.$$Note that the multiplication map by $\ell^4$
$$[A]_3 \times^{\ell^4} \to [A]_7$$is injective, and the multiplication map by $\ell^6$
$$[A]_3 \times^{\ell^6} \to [A]_8$$
is surjective. Then we can choose a basis \{\ell^3, F_{1,1}\ell^2, F_{1,2}\ell^2, F_{2,1}\ell, F_{2,2}\ell, F_{2,3}\ell, F_{3,1}, \ldots, F_{3,4}\} for \([A]_3\) such that
\[ F_{3,1}\ell^4, \ldots, F_{3,4}\ell^4 \neq 0, \quad \text{and} \quad F_{3,1}\ell^5, \ldots, F_{3,4}\ell^5 = 0. \]

Moreover, since \{\ell^3, F_{3,1}\ell^4, \ldots, F_{3,4}\ell^4\} is linearly independent, we have 4-strings of length 5
\[ F_{3,1}, F_{3,2}, F_{3,3}, F_{3,4}, \ldots, F_{3,1}\ell^4, F_{3,2}\ell^4, F_{3,3}\ell^4, F_{3,4}\ell^4, \ldots, F_{3,1}\ell^5, F_{3,2}\ell^5, F_{3,3}\ell^5, F_{3,4}\ell^5. \]

This shows that the Jordan type of \(H_{R/(I_X+I_Y)}\) is
\[ J_\ell = (9, 7, 7, 6, 6, 6, 6, 6, 5, 5, 5) = H_{R/(I_X+I_Y)}. \]

Thus, by Lemma 2.2, an Artinian quotient of two linear star-configurations in \(\mathbb{P}^2\) of type 5 and 9 has the SLP, as we wished.

Example 2.6 motivates the following proposition.

**Proposition 2.7.** Let \(\mathcal{X}\) be a finite set of points in \(\mathbb{P}^n\) and let \(A\) be an Artinian quotient of the coordinate ring of \(\mathcal{X}\). Assume that \(H_A(i) = H_{\mathcal{X}}(i)\) for every \(0 \leq i \leq s - 2\) with \(A_s = 0\), and the Hilbert function of \(A\) is of the form
\[ H_A : h_0 \ h_1 \ldots \ h_{\sigma-1} \ h_{\sigma} \ldots \ h_{s-1}^{(s-2)-nd} \ h_{s} \ h_{s-1} \ 0 \]
where \(h_{\sigma-2} < h_{\sigma-1} = h_{\sigma}\) and \(h_{s-1} = 1\). Then an Artinian ring \(A\) has the SLP.

**Proof.** We first define
\[ g_i := h_i - h_{i-1} \quad \text{for} \quad i = 1, \ldots, \sigma - 1. \]

(a) By Waring decomposition, there is a linear form \(\ell \in [A]_1\) such that
\[ \ell^{s-1} \in [A]_{s-1}. \]

In other words, there is a string of length \(s\) as
\[ 1, \ell, \ldots, \ell^{s-1}. \]

Hence Jordan type of \(H_{R/(I_X+I_Y)}\) is of the form
\[ J_\ell = (s, \ldots). \]

(b) Note that the multiplication map by \(\ell^{s-3}\)
\[ [R/I_X]_1 = [A]_1 \times_{[A]_{s-3}} [A]_{s-2} = [R/I_X]_{s-2} \]
is injective, and the multiplication map by \(\ell^{s-2}\)
\[ [A]_1 \times_{[A]_{s-2}} [A]_{s-1} \]
is surjective. Then we can choose a basis \( \{ \ell, F_{1,1}, F_{1,2}, \ldots, F_{1,g_1} \} \) for \([A]_i\) such that
\[
F_{1,1}^{s-3}, F_{1,2}^{s-3}, \ldots, F_{1,g_1}^{s-3} \neq 0, \quad \text{and} \quad F_{1,1}^{s-2}, F_{1,2}^{s-2}, \ldots, F_{1,g_1}^{s-2} = 0.
\]
Moreover, since \( \{ F_{1,1}^{s-3}, F_{1,2}^{s-3}, \ldots, F_{1,g_1}^{s-3} \} \) is linearly independent, we have \( g_1 \)-strings of length \((s-2)\)
\[
F_{1,1}, F_{1,1}^{s-3}, \ldots, F_{1,1}^{s-3},
F_{1,2}, F_{1,2}^{s-3}, \ldots, F_{1,2}^{s-3},
\vdots
F_{1,g_1-1}, F_{1,g_1-1}^{s-3}, \ldots, F_{1,g_1-1}^{s-3}, \quad \text{and} \quad
F_{1,g_1}, F_{1,g_1}^{s-3}, \ldots, F_{1,g_1}^{s-3}.
\]
This means that Jordan type of \( H_{R/(I_x+I_y)} \) is of the form
\[
J_\ell = (s, s-2, \ldots, s-2, \ldots).
\]

(c) Let \( 1 \leq i \leq \sigma - 1 \). Note that the multiplication map by \( \ell^{s-i-2} \)
\[
[R/I_x]_i = [A]_i \times^{(s-i-2)} [A]_{s-2} = [R/I_x]_{s-2}
\]
is injective, and the multiplication map by \( \ell^{s-i-1} \)
\[
[R/I_x]_i = [A]_i \times^{(s-i-1)} [A]_{s-1}
\]
is surjective. Then we can choose a basis \( B_i \)
\[
B_i = \left\{ \ell, F_{1,1}^{s-i-1}, \ldots, F_{1,g_1}^{s-i-1}, F_{2,1}^{s-i-2}, \ldots, F_{2,g_2}^{s-i-2}, \ldots, F_{i-1,1}^{s-i-1}, \ldots, F_{i-1,g_{i-1}}^{s-i-1}, F_{i,1}, \ldots, F_{i,g_i} \right\}
\]
for \([A]_i\) such that
\[
F_{1,1}^{s-i-2}, \ldots, F_{i,g_i}^{s-i-2} \neq 0, \quad \text{and} \quad F_{1,1}^{s-i-1}, \ldots, F_{i,g_i}^{s-i-1} = 0.
\]
Moreover, since \( \{ F_{1,1}^{s-i-2}, \ldots, F_{i,g_i}^{s-i-2} \} \) is linearly independent, we have \( g_i \)-strings of length \((s-i-1)\)
\[
F_{1,1}, F_{1,1}^{s-i-2}, \ldots, F_{1,1}^{s-i-2},
F_{1,2}, F_{1,2}^{s-i-2}, \ldots, F_{1,2}^{s-i-2},
\vdots
F_{i-1,1}, F_{i-1,1}^{s-i-2}, \ldots, F_{i-1,1}^{s-i-2}, \quad \text{and} \quad
F_{i,1}, F_{i,1}^{s-i-2}, \ldots, F_{i,1}^{s-i-2}.
\]
Hence Jordan type of \( H_{R/(I_x+I_y)} \) is of the form
\[
J_\ell = (s, s-2, s-2, \ldots, s-2, \ldots, s-i-1, s-1, \ldots, s-1, \ldots)
\]
for such $i$.

It is from (a) $\sim$ (c) that the Jordan type $J_{\ell}$ of $H_{R/(I_X+I_Y)}$ is

$$
J_{\ell} = H_{R/(I_X+I_Y)} \bigvee
(s, s-2, s-2, \ldots, s-2, \ldots, s-i, s-i, \ldots, s-i, \ldots, g_{s-1}, s, \ldots, s, s)\text{,}
$$

Therefore, by Lemma 2.2, an Artinian ring $R/(I_X + I_Y)$ has the SLP, as we wished. $\square$

The following two corollaries are immediate from Proposition 2.7.

**Corollary 2.8.** Let $X$ and $Y$ be finite sets of general points in $\mathbb{P}^n$ with $n \geq 2$ and $s \geq t \geq n$. Assume that

$$
\binom{s}{n} \leq \deg(X) < \binom{s+1}{n}, \quad \binom{t}{n} \leq \deg(Y) < \binom{t+1}{n},
$$

and

$$
\deg(X) + \deg(Y) = \binom{s+1}{n} + 1.
$$

Then an Artinian ring $R/(I_X + I_Y)$ has the SLP.

**Proof.** Since $X$ and $Y$ are finite sets of general points in $\mathbb{P}^n$, we get that the Hilbert functions of $R/I_X$, $R/I_Y$, and $R/(I_X \cap I_Y)$ are

$$
H_{R/I_X} : 1 \quad \binom{s+1}{n} \quad \binom{s+1}{n} \quad \binom{s-1}{n} \quad \binom{s-1}{n} \quad \deg(X) \quad \rightarrow,
$$

$$
H_{R/I_Y} : 1 \quad \binom{t+1}{n} \quad \binom{t+1}{n} \quad \binom{t-1}{n} \quad \binom{t-1}{n} \quad \deg(Y) \quad \deg(Y) \quad \rightarrow,
$$

$$
H_{R/(I_X \cap I_Y)} : 1 \quad \binom{s+1}{n} \quad \binom{s+1}{n} \quad \binom{s}{n} \quad \binom{s}{n} \quad \binom{s-1}{n} \quad \deg(Y) \quad \deg(Y) \quad \deg(Y) \quad [\deg(X) + \deg(Y)] - 1 \quad \rightarrow.
$$

respectively. Using the exact sequence

$$
0 \rightarrow R/(I_X \cap I_Y) \rightarrow R/I_X \oplus R/I_Y \rightarrow R/(I_X + I_Y) \rightarrow 0,
$$

the Hilbert function of $R/(I_X + I_Y)$ is

$$
H_{R/(I_X+I_Y)} : 1 \quad 3 \quad \binom{t-n}{n} \quad \binom{t-n}{n} \quad \binom{s-n}{n} \quad \deg(Y) \quad \deg(Y) \quad [\deg(X) + \deg(Y)] - 1 \quad \rightarrow,
$$

and so by Proposition 2.7, an Artinian ring $R/(I_X + I_Y)$ has the SLP, which completes the proof. $\square$

**Corollary 2.9.** Let $X$ and $Y$ be linear star-configurations in $\mathbb{P}^2$ of type $s$ and $t$ with $s \geq \binom{t}{2} - 1$ and $t \geq 3$. Then an Artinian linear star-configuration quotient $R/(I_X + I_Y)$ has the SLP.
Proof. By Proposition 2.5, it holds for \( s \geq \binom{t}{2} \). So we assume that \( s = \binom{t}{2} - 1 \).

First note that
\[
\left[ \deg(X) + \deg(Y) \right] - \left( \frac{s + 1}{2} \right) = \left( \frac{s}{2} + \frac{t}{2} \right) - \left( \frac{s + 1}{2} \right) = \left( \frac{s}{2} + s + 1 \right) - \left( \frac{s + 1}{2} \right) = 1.
\]

Hence the Hilbert functions of \( R/I_X, R/I_Y, \) and \( R/(I_X \cap I_Y) \) (see Proposition 2.3) are
\[
\begin{align*}
H_{R/I_X} &: 1 \ 3 \ \cdots \ \binom{t}{2} \ \cdots \ \binom{s+1}{2} \ \cdots \ \binom{s-2}{2} \\
H_{R/I_Y} &: 1 \ 3 \ \cdots \ \binom{t}{2} \ \cdots \ \binom{t+1}{2} \ \cdots \ \binom{s}{2} \\
H_{R/(I_X \cap I_Y)} &: 1 \ 3 \ \cdots \ \binom{t}{2} \ \cdots \ \binom{s-2}{2} \ \cdots \ \binom{s-1}{2} = \binom{s}{2} + \binom{t}{2} - 1 \rightarrow,
\end{align*}
\]
respectively. Using the exact sequence
\[
0 \rightarrow R/(I_X \cap I_Y) \rightarrow R/I_X \oplus R/I_Y \rightarrow R/(I_X + I_Y) \rightarrow 0,
\]
the Hilbert function of \( R/(I_X + I_Y) \) is
\[
H_{R/(I_X + I_Y)} : 1 \ 3 \ \cdots \ \binom{t}{2} \ \cdots \ \binom{t}{2} \ 1 \rightarrow,
\]
and so by Proposition 2.7, an Artinian linear star-configuration quotient \( R/(I_X + I_Y) \) has the SLP, as we wished. \( \square \)

3. Artinian \( k \)-configuration quotients in \( \mathbb{P}^2 \)

In this section, we shall introduce another Artinian quotient having the SLP. We first recall a definition of a \( k \)-configuration in \( \mathbb{P}^2 \) and some preliminary result.

Definition 3.1. A \( k \)-configuration of points in \( \mathbb{P}^2 \) is a finite set \( \mathbb{X} \) of points in \( \mathbb{P}^2 \) which satisfy the following conditions: there exist integers \( 1 \leq d_1 < \cdots < d_m \), and subsets \( \mathbb{X}_1, \ldots, \mathbb{X}_m \) of \( \mathbb{X} \), and distinct lines \( L_1, \ldots, L_m \subseteq \mathbb{P}^2 \) such that
\begin{itemize}
  \item[(a)] \( \mathbb{X} = \bigcup_{i=1}^{m} \mathbb{X}_i \),
  \item[(b)] \( |\mathbb{X}_i| = d_i \) and \( \mathbb{X}_i \subseteq L_i \) for each \( i = 1, \ldots, m \), and
  \item[(c)] \( L_i \) (\( 1 < i \leq m \)) does not contain any points of \( \mathbb{X}_j \) for all \( j < i \).
\end{itemize}

In this case, the \( k \)-configuration in \( \mathbb{P}^2 \) is said to be of type \((d_1, \ldots, d_m)\).

Recall that a finite complete intersection set of points \( \mathbb{Z} \) in \( \mathbb{P}^n \) is said to be a basic configuration in \( \mathbb{P}^n \) (see [11, 12]) if there exist integers \( r_1, \ldots, r_n \) and distinct hyperplanes \( L_{ij} (1 \leq i \leq n, 1 \leq j \leq r_i) \) such that
\[
\mathbb{Z} = \mathbb{H}_1 \cap \cdots \cap \mathbb{H}_n \text{ as schemes, where } \mathbb{H}_i = L_{i1} \cup \cdots \cup L_{ir_i}.
\]

In this case \( \mathbb{Z} \) is said to be of type \((r_1, \ldots, r_n)\).

Before we prove our main theorem, we first introduce two lemmas.
Lemma 3.2. Let \( X \) be a \( k \)-configuration in \( \mathbb{P}^2 \) of type \((1, 2, \ldots, d)\) (see Figure 1), and let \( L_i \) and \( M_j \) be lines in \( \mathbb{P}^2 \) defined by linear forms \( x_0 - (i - 1)x_2 \) and \( x_1 - (j - 1)x_2 \) for \( 1 \leq i, j \leq d - 1 \), respectively. Then the multiplication map by \( L_1 := x_0 \)

\[
[R/I_X]_i \times L_1^i \to [R/I_X]_{i+1}
\]

is injective for \( i \geq 0 \). In particular, for \( j \geq 1 \), the multiplication map by \( L_j \)

\[
[R/I_X]_i \times L_j^i \to [R/I_X]_{i+j}
\]

is injective for every \( i \geq 0 \).

\[\begin{array}{cccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \cdots & \bullet & \bullet \\
\bullet & \bullet & \bullet & \cdots & \cdots & \bullet \\
M_1 & M_2 & M_3 & \cdots & M_{d-1} & M_d \\
\end{array}\]

Figure 1

Proof. If \( d = 1 \), then \( X \) is a set of a single point in \( \mathbb{P}^2 \), so it is immediate. Hence we assume that \( d > 1 \).

Note that

\[
I_X = (L_1 \cdots L_d, M_1L_2 \cdots L_d, M_1M_2L_3 \cdots L_d, \ldots, M_1 \cdots M_{d-1}L_d, M_1M_2 \cdots M_d)
\]

(see [9, 11]) and the Hilbert function of \( R/I_X \) is

\[
H_X : 1 \begin{pmatrix} 1 + 2 \\ 2 \end{pmatrix} \cdots \begin{pmatrix} (d - 1) + 2 \\ 2 \end{pmatrix} \begin{pmatrix} d + 1 \\ 2 \end{pmatrix} \to
\]

(see Theorems 2.7 and 3.6 in [9]).

First, it is obvious that the multiplication map by \( L_1 := x_0 \)

\[
[R/I_X]_i \times L_1^i \to [R/I_X]_{i+1}
\]

is injective for \( 0 \leq i \leq d - 2 \).

Let \( i = d - 1 = j_1 + j_2 + j_3 \) with \( 0 \leq j_1, j_2, j_3 \leq d \).

(i) Assume \( j_2 = 0 \) and

\[
x_0^{j_1} x_2^{j_3} L_1 \in [I_X]_d = \langle L_1 \cdots L_d, M_1L_2 \cdots L_d, M_1M_2L_3 \cdots L_d, \ldots, M_1 \cdots M_{d-1}L_d, M_1M_2 \cdots M_d \rangle,
\]

that is,

\[
x_0^{j_1} x_2^{j_3} L_1 = \alpha_1 L_1 \cdots L_d + \alpha_2 M_1L_2 \cdots L_d + \alpha_3 M_1M_2L_3 \cdots L_d + \cdots
\]
\[ + \alpha d M_1 \cdots M_{d-1} L_d + \alpha_{d+1} M_1 M_2 \cdots M_d \]

for some \( \alpha_i \in k \). Let \( \wp_{i,j} \) be a point defined by two linear forms \( L_i \) and \( M_j \). Since two linear forms \( L_1 \) and \( M_2 \) vanish on a point \( \wp_{1,2} \), we get that \( \alpha_2 = 0 \).

Moreover, since two forms \( L_1 \) and \( M_3 \) vanish on a point \( \wp_{1,3} \), we have \( \alpha_3 = 0 \).

By continuing this procedure, one can show that

\[ \alpha_2 = \cdots = \alpha_d = 0. \]

Hence

\[ x_0^{j_1} x_2^{j_2} L_1 = \alpha_1 L_1 \cdots L_d + \alpha_{d+1} M_1 M_2 \cdots M_d, \]

that is,

\[ L_1 \mid \alpha_{d+1} M_1 M_2 \cdots M_d \quad \text{and so,} \quad \alpha_{d+1} = 0. \]

It follows that

\[ x_0^{j_1} x_2^{j_2} L_1 = \alpha_1 L_1 \cdots L_d, \quad \text{and thus,} \quad \alpha_1 = 0. \]

(ii) Assume \( j_2 > 0 \) and

\[ x_0^{j_1} x_1^{j_2} x_2^{j_3} L_1 = \alpha_1 L_1 \cdots L_d + \alpha_2 M_1 L_2 \cdots L_d + \alpha_3 M_1 M_2 L_3 \cdots L_d + \cdots + \alpha_d M_1 \cdots M_{d-1} L_d + \alpha_{d+1} M_1 M_2 \cdots M_d \]

for some \( \alpha_i \in k \). Recall that \( M_1 := x_1 \). Thus

\[ M_1 \mid \alpha_1 L_1 \cdots L_d, \quad \text{and hence,} \quad \alpha_1 = 0. \]

By the analogous argument as in (i), one can show that

\[ \alpha_2 = \cdots = \alpha_d = \alpha_{d+1} = 0. \]

It is from (i) and (ii) that

\[ x_0^{j_1} x_1^{j_2} x_2^{j_3} L_1 \notin [I_X]_d, \]

which means that the multiplication map by \( L_1 \)

\[ [R/I_X]_{d-1} \xrightarrow{\times L_1} [R/I_X]_d \]

is injective, and surjective as well. Thus the multiplication map by \( L_1 \)

\[ [R/I_X]_1 \xrightarrow{\times L_1} [R/I_X]_{i+1} \]

is injective and surjective for every \( i \geq d - 1 \), as we wished.

So it follows that the multiplication map by \( L_1 \)

\[ [R/I_X]_1 \xrightarrow{\times L_1} [R/I_X]_{i+j} \]

is injective for every \( i \geq 0 \). This completes the proof. \( \square \)
The following lemma is immediate from Proposition 2.7. But we introduce another elementary proof here.

**Lemma 3.3.** Let $X$ be a $k$-configuration in $P^2$ of type $(1, 2)$ in a basic configuration $Z$ in $P^2$ of type $(a, 2)$ with $a \geq 2$, and let $Y := Z - X$, ($X$ is a set of solid 3-points in $Z$ in Figure 2). Then an Artinian $k$-configuration quotient $R/(I_X + I_Y)$ has the SLP.

\[
\begin{array}{ccccccc}
\bullet & \circ & \circ & \cdots & \circ & \circ & L_2 \\
\bullet & \bullet & \circ & \cdots & \circ & \circ & L_1 \\
M_1 & M_2 & M_3 & \cdots & M_{a-1} & M_a
\end{array}
\]

**Figure 2**

**Proof.** First, if $a = 2$, then the Hilbert function of $R/(I_X + I_Y)$ is

\[
H_{R/(I_X + I_Y)} : 1 \ 1 \ 0,
\]

(see [12, Theorem 2.1]) and so it follows that $R/(I_X + I_Y)$ has the SLP.

Now suppose $a \geq 3$ and assume that $L_i$ and $M_j$ are lines defined by linear forms $L_i = x_0 - (i - 1)x_2$ and $M_j = x_1 - (j - 1)x_2$ for $i$ and $j$, respectively. Let $\varphi_{i,j}$ be a point defined by two linear forms $L_i$ and $M_j$. Then

\[
\begin{align*}
I_X &= (L_1L_2, L_1M_1, M_1M_2), \\
I_Y &= (L_1L_2, L_2M_3M_4 \cdots M_a, M_2M_3M_4 \cdots M_a)
\end{align*}
\]

(see [9,11]) and an ideal $I_X + I_Y$ has 5-minimal generators, i.e.,

\[
I_X + I_Y = (L_1L_2, L_1M_1, M_1M_2, L_2M_3M_4 \cdots M_a, M_2M_3M_4 \cdots M_a).
\]

By [12, Theorem 2.1], the Hilbert function of $R/(I_X + I_Y)$ is

\[
H_{R/(I_X + I_Y)} : 1 \ 3 \ 3 \ \cdots \ (a-2) \ \text{ad} \ 3 \ 1 \ 0 \ \to.
\]

Note that

\[
H_{R/(I_X + I_Y)}(i) = H_{R/I_X}(i)
\]

for $0 \leq i \leq a - 2$.

(i) Assume $x_0L_1^{a-2} = L_1^{a-1} \in [I_X + I_Y]_{a-1}$. Then

\[
x_0L_1^{a-2} = L_1^{a-1} = F_1L_1L_2 + F_2L_1M_1 + F_3M_1M_2 + \beta_1L_2M_3M_4 \cdots M_a + \beta_2M_2M_3M_4 \cdots M_a
\]

for some $F_i \in R_{a-3}$ and $\beta_j \in k$. Since two linear forms $L_1$ and $M_2$ vanish on a point $\varphi_{1,2}$, we get that $\beta_1 = 0$. Similarly, we have $\beta_2 = 0$ as well. This means that

\[
x_0L_1^{a-2} = L_1^{a-1} = F_1L_1L_2 + F_2L_1M_1 + F_3M_1M_2 \in [I_X]_{a-1},
\]

which is a contradiction (see Lemma 3.2). Hence the Jordan type of $H_{R/(I_X + I_Y)}$ is of the form

\[
J_{L_1} = (a, \ldots).
\]
(ii) Similarly, it is from Lemma 3.2 that
\[ x_1 L_1^{a-3}, x_2 L_2^{a-3} \notin [I_X]_{a-2} = [I_X + I_Y]_{a-2}. \]
Furthermore, it is obvious that two forms \( x_1 L_1^{a-3}, x_2 L_2^{a-3} \) are linearly independent in \( [R/(I_X + I_Y)]_{a-2} = [R/I_X]_{a-2} \). So it is from (i) and (ii) that the Jordan type \( J_{L_1} \) of \( H_{R/(I_X + I_Y)} \) is
\[ J_{L_1} = H_{R/(I_X + I_Y)} = (a, a - 2, a - 2). \]
Therefore, by Lemma 2.2, an Artinian \( k \)-configuration quotient \( R/(I_X + I_Y) \) has the SLP.

The following proposition can be obtained using Proposition 2.7. However, we also introduce a different proof here.

**Proposition 3.4.** Let \( \mathcal{X} \) be a \( k \)-configuration of type \((1, 2)\) contained in a basic configuration \( \mathcal{Z} \) in \( \mathbb{P}^2 \) of type \((a, b)\) with \( 2 \leq b \leq a \). Define \( Y := Z - X \). (\( \mathcal{X} \) is a set of solid 3-points in Figure 3.) Then an Artinian \( k \)-configuration quotient \( R/(I_X + I_Y) \) has the SLP.

![Figure 3](image-url)

**Proof.** First, if \( a = b = 2 \), then it is immediate. If \( a \geq 3 \) and \( b = 2 \), by Lemma 3.3 it holds.

Now suppose \( a \geq b \geq 3 \) and assume that \( L_i \) is a line defined by a linear form \( L_i = x_0 - (i-1)x_1 \) and \( M_j \) is a line defined by a linear form \( M_j = x_1 - (j-1)x_2 \) for \( i \) and \( j \). Let \( \varphi_{ij} \) be a point defined by two linear forms \( L_i \) and \( M_j \). Then it is from [9, 11] that
\[
I_X = (L_1 L_2, L_1 M_1, M_1 M_2), \quad \text{and} \quad I_Y = (L_1 L_2 \cdots L_b, L_2 L_3 \cdots L_b M_3 \cdots M_a, L_3 \cdots L_b M_2 M_3 \cdots M_a, M_1 M_2 \cdots M_a).
\]
Then an ideal \( I_X + I_Y \) has 5-minimal generators, i.e.,
\[
I_X + I_Y = (L_1 L_2, L_1 M_1, M_1 M_2, L_2 L_3 \cdots L_b M_3 \cdots M_a, L_3 \cdots L_b M_2 M_3 \cdots M_a),
\]
and by [12, Theorem 2.1] the Hilbert function of \( R/(I_X + I_Y) \) is
\[
H_{R/(I_X + I_Y)} : \begin{array}{cccccc}
1 & 3 & 3 & \cdots & 3 & (a+b-4)st \\
3 & 1 & 0 & \rightarrow.
\end{array}
\]
(i) Assume \( x_0L_1^{a+b-4} = L_1^{a+b-3} \in [I_X + I_Y]_{a+b-3} \). Then
\[
x_0L_1^{a+b-4} = L_1^{a+b-3} = F_1L_1L_2 + F_2L_1M_1 + F_3M_1M_2 + \beta_1L_2L_3 \cdots L_bM_3 \cdots M_a
\]
for some \( F_i \in R_{a+b-5} \) and \( \beta_j \in k \). Since two linear forms \( L_1 \) and \( M_2 \) vanish on a point \( \wp \), we get that \( \beta_1 = 0 \). Similarly, we have \( \beta_2 = 0 \) as well. This means that
\[
x_0L_1^{a+b-4} = L_1^{a+b-3} = F_1L_1L_2 + F_2L_1M_1 + F_3M_1M_2 \in [I_X]_{a+b-3},
\]
which is a contradiction (see Lemma 3.2). Hence the Jordan type of \( H_{R/(I_X+I_Y)} \) is of the form
\[
J_{L_1} = (a+b-2, \ldots).
\]
(ii) Similarly, it is from Lemma 3.2 that the following 3-forms
\[
x_0L_1^{a+b-5}, x_1L_1^{a+b-5}, x_2L_1^{a+b-5}
\]
are linearly independent. In particular, the following 2-forms
\[
x_1L_1^{a+b-5}, x_2L_1^{a+b-5}
\]
are linearly independent. Hence the Jordan type of \( H_{R/(I_X+I_Y)} \) is
\[
J_{L_1} = H_{R/(I_X+I_Y)} = (a+b-2, a+b-4, a+b-4).
\]
It is from (i) and (ii) with Lemma 2.2 that an Artinian \( k \)-configuration quotient \( R/(I_X+I_Y) \) has the SLP, which completes the proof. \( \Box \)

We now slightly extend the previous result.

**Lemma 3.5.** Let \( X \) be a \( k \)-configuration of type \( (1, 2, 3) \) in a basic configuration \( Z \) in \( \mathbb{P}^2 \) of type \( (a, 3) \) with \( a \geq 3 \) such that \( Y := Z - X \). (\( X \) is a set of solid 6-points in Figure 4.) Then an Artinian \( k \)-configuration quotient \( R/(I_X + I_Y) \) has the SLP.

![Figure 4](image-url)

**Proof.** If \( a = 3 \), then in Proposition 3.4, \( Z \) is a basic configuration of type \( (3, 3) \) and hence, \( Y \) is a set of 6 points, lemma holds. So we suppose that \( a > 3 \). First note that the Hilbert function of \( R/(I_X + I_Y) \) is
\[
H_{R/(I_X+I_Y)} : 1 \ 3 \ 6 \ \cdots \ (a-2)-nd \ 6 \ 3 \ 1 \ 0.
\]
We assume that $L_i$ is a line defined by a linear form $L_i = x_0 - (i - 1)x_2$ and $M_j$ is a line defined by a linear form $M_j = x_1 - (j - 1)x_2$ for $i$ and $j$. Let $\varphi_{i,j}$ be a point defined by two linear forms $L_i$ and $M_j$. Then

$$I_X = (L_1L_2L_3, L_1L_2M_1, L_1M_1M_2, M_1M_2M_3), \quad \text{and}\quad I_Y = (L_1L_2L_3, L_2L_3M_4 \cdots M_a, L_3M_4 \cdots M_a, M_2M_3 \cdots M_a).$$

So an ideal $I_X + I_Y$ has 7-minimal generators, i.e.,

$$I_X + I_Y = (L_1L_2L_3, L_1L_2M_1, L_1M_1M_2, M_1M_2M_3, L_2L_3M_4 \cdots M_a, L_3M_4 \cdots M_a, M_2M_3 \cdots M_a).$$

Note that

$$H_{R/\langle t_k + t_i \rangle}(i) = H_{R/t_i}(i)$$

for $0 \leq i \leq a - 2$.

(i) Assume $x_0L_1^{a-1} = L_1^a \in [I_X + I_Y]_a$. Then

$$x_0L_1^{a-1} = L_1^a = F_1L_2L_3 + F_2L_1L_2M_1 + F_3L_1M_1M_2 + F_4M_1M_2M_3 + \beta_1L_2L_3M_4 \cdots M_a + \beta_3L_3M_4 \cdots M_a + \beta_3M_2M_3 \cdots M_a$$

for some $F_i \in R_{a-3}$ and $\beta_j \in k$. Since two linear forms $L_1$ and $M_3$ vanish on a point $\varphi_{1,3}$, we get that $\beta_1 = 0$. Similarly, we have $\beta_2 = \beta_3 = 0$ as well. This means that

$$x_0L_1^{a-1} = L_1^a = F_1L_2L_3 + F_2L_1L_2M_1 + F_3L_1M_1M_2 + F_4M_1M_2M_3 \in [I_X]_a,$$

which is a contradiction (see Lemma 3.2). Hence the Jordan type of $H_{R/\langle t_k + t_i \rangle}$ is of the form

$$J_{L_1} = (a + 1, \ldots).$$

(ii) By the analogous argument as in (i), one can show that

$$x_1L_1^{a-2}, x_2L_1^{a-2} \notin [I_X + I_Y]_{a-1}.$$

We now suppose that

$$\alpha x_1L_1^{a-2} + \beta x_2L_1^{a-2} \in [I_X + I_Y]_{a-1}$$

for some $\alpha, \beta \in k$. Then

$$\alpha x_1L_1^{a-2} + \beta x_2L_1^{a-2} = F_1L_2L_3 + F_2L_1L_2M_1 + F_3L_1M_1M_2 + F_4M_1M_2M_3 + \beta_1L_2L_3M_4 \cdots M_a + \beta_3L_3M_4 \cdots M_a + \beta_3M_2M_3 \cdots M_a$$

for some $F_i \in R_{a-3}$ and $\beta_j \in k$. Since two linear forms $L_1$ and $M_3$ vanish on a point $\varphi_{1,3}$, we get that $\beta_1 = 0$. Similarly, we have $\beta_2 = \beta_3 = 0$ as well. This means that

$$\alpha x_1L_1^{a-2} + \beta x_2L_1^{a-2} = F_1L_2L_3 + F_2L_1L_2M_1 + F_3L_1M_1M_2 + F_4M_1M_2M_3 \in [I_X]_{a-1}.$$
By Lemma 3.2, we get that
\[ \alpha x_1 + \beta x_2 = 0, \quad \text{i.e.,} \quad \alpha = \beta = 0, \]
which implies that two forms
\[ x_1 L_1^{a_1 - 2}, x_2 L_1^{a_2 - 2} \]
are linearly independent. Hence the Jordan type of \( H_{R/(I_X + I_Y)} \) is of the form
\[ J_{L_1} = (a + 1, a - 1, a - 1, \ldots). \]

(iii) It is from Lemma 3.2 that
\[ x_2^2 L_1^{a_1 - 4}, x_1 x_2 L_1^{a_2 - 4}, x_2^2 L_1^{a_3 - 4} \notin [I_X]_{a - 2} = [I_X + I_Y]_{a - 2} \]
and the following set of 6-forms
\[ \{x_0 L_1^{a_3 - 3}, x_1 L_1^{a_3 - 3}, x_2 L_1^{a_3 - 3}, x_1^2 L_1^{a_3 - 4}, x_1 x_2 L_1^{a_3 - 4}, x_2^2 L_1^{a_3 - 4}\} \]
is linearly independent. In particular, the 3-forms
\[ x_1^2 L_1^{a_3 - 4}, x_1 x_2 L_1^{a_3 - 4}, x_2^2 L_1^{a_3 - 4} \]
are linearly independent. Hence the Jordan type of \( H_{R/(I_X + I_Y)} \) is of the form
\[ J_{L_1} = (a + 1, a - 1, a - 1, a - 3, a - 3, a - 3). \]

It is from (i) ∼ (iii) that the Jordan type \( J_{L_1} \) is
\[ J_{L_1} = H_{R/(I_X + I_Y)}' = (a + 1, a - 1, a - 1, a - 3, a - 3, a - 3). \]

Therefore, by Lemma 2.2, an Artinian \( k \)-configuration quotient \( R/(I_X + I_Y) \) has the SLP.

\[ \square \]

**Theorem 3.6.** Let \( X \) be a \( k \)-configuration of type \((1, 2, 3)\) in a basic configuration \( Z \) in \( \mathbb{P}^2 \) of type \((a, b)\) with \( a \geq 4 \) and \( b \geq 3 \), and let \( Y := Z - X \). \( (X \) is a set of solid 6-points in Figure 5.) Then an Artinian ring \( R/(I_X + I_Y) \) has the SLP.

\[ \begin{array}{ccccccc}
\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\bullet & \circ & \circ & \circ & \circ & \circ & \circ \\
\bullet & \bullet & \circ & \circ & \circ & \circ & \circ \\
M_1 & M_2 & M_3 & M_4 & \cdots & M_a
\end{array} \]

**Figure 5**
Proof. If $b = 3$, then, by Lemma 3.5, it holds. So we suppose that $b > 3$. Note that, by [12, Theorem 2.1], the Hilbert function of $R/(I_X + I_Y)$ is

$$H_{R/(I_X+I_Y)} : 1 \quad 3 \quad 6 \quad \cdots \quad (a+b-5)\cdot \cdots \quad 6 \quad 3 \quad 1 \quad 0.$$ 

We assume that $L_i$ is a line defined by a linear form $L_i = x_0 - (i-1)x_2$ and $M_j$ is a line defined by a linear form $M_j = x_1 - (j-1)x_2$ for $i$ and $j$. Let $\varphi_{i,j}$ be a point defined by two linear forms $L_i$ and $M_j$. Then

$$I_X = (L_1L_2L_3, L_1L_2M_1, L_1M_1M_2, M_1M_2M_3), \quad \text{and}$$

$$I_Y = (L_1L_2 \cdots L_b, L_2 \cdots L_bM_4 \cdots M_a, L_3 \cdots L_bM_4 \cdots M_a, L_4 \cdots L_bM_2 \cdots M_a).$$

So an ideal $I_X + I_Y$ has 7-minimal generators, i.e.,

$$I_X + I_Y = (L_1L_2L_3, L_1L_2M_1, L_1M_1M_2, M_1M_2M_3, L_2 \cdots L_bM_4 \cdots M_a, L_3 \cdots L_bM_3 \cdots M_a, L_4 \cdots L_bM_2 \cdots M_a).$$

Note that

$$H_{R/(I_X+I_Y)}(i) = H_{R/I_X}(i)$$

for $0 \leq i \leq a + b - 5$.

(i) Assume $x_0L_1^{a+b-4} = L_1^{a+b-3} \in [I_X + I_Y]_{a+b-3}$. Then

$$x_0L_1^{a+b-4} = L_1^{a+b-3} = F_1L_1L_2L_3 + F_2L_1L_2M_1 + F_3L_1M_1M_2 + F_4M_1M_2M_3$$

$$+ \beta_1L_2 \cdots L_bM_4 \cdots M_a + \beta_2L_3 \cdots L_bM_4 \cdots M_a$$

$$+ \beta_3L_4 \cdots L_bM_2 \cdots M_a$$

for some $F_i \in R_{a+b-6}$ and $\beta_j \in k$. Since two linear forms $L_1$ and $M_3$ vanish on a point $\varphi_{1,3}$, we get that $\beta_1 = 0$. Similarly, we have $\beta_2 = \beta_3 = 0$ as well. This means that

$$x_0L_1^{a+b-4} = L_1^{a+b-3}$$

$$= F_1L_1L_2L_3 + F_2L_1L_2M_1 + F_3L_1M_1M_2 + F_4M_1M_2M_3 \in [I_X]_{a+b-3},$$

which is a contradiction (see Lemma 3.2). Hence the Jordan type of $H_{R/(I_X+I_Y)}$ is of the form

$$J_{L_1} = (a + b - 2, \ldots).$$

(ii) By the analogous argument as in (i), one can show that

$$x_1L_1^{a+b-5}, x_2L_1^{a+b-5} \notin [I_X]_{a+b-4} = [I_X + I_Y]_{a+b-4}.$$

We now suppose that the following 3-forms

$$\alpha x_0L_1^{a+b-5} + \beta x_1L_1^{a+b-5} + \beta x_2L_1^{a+b-5} \in [I_X + I_Y]_{a+b-4}$$

for some $\alpha, \beta, \gamma \in k$, that is,

$$\alpha x_0L_1^{a+b-5} + \beta x_1L_1^{a+b-5} + \beta x_2L_1^{a+b-5}$$

$$= F_1L_1L_2L_3 + F_2L_1L_2M_1 + F_3L_1M_1M_2 + F_4M_1M_2M_3.$$
Remark has the SLP, which completes the proof of this theorem. Therefore, by Lemma 2.2, an Artinian, of type $(1, 2)$ or $(1, 2, 3)$ in a basic configuration in $\mathbb{P}^2$. However, if $X$ is a $k$-configuration in $\mathbb{P}^2$ of type $(1, 2, \ldots, d)$ in a basic configuration in $\mathbb{P}^2$ with $d \geq 4$, then it cannot be proved by the same method as in the proof of Theorem 3.6.

References

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