A ONE-PARAMETER FAMILY OF TOTALLY UMBILICAL HYPERSPRHERES IN THE NEARLY KÄHLER 6-SPHERE

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Abstract. We discuss two kinds of almost contact metric structures on a one-parameter family of totally umbilical hyperspheres in the nearly Kähler unit 6-sphere $S^6$.

1. Introduction

An odd dimensional smooth manifold $M$ with a quadruple $(\phi, \xi, \eta, g)$ of a $(1,1)$-tensor field $\phi$, a vector field $\xi$, a 1-form $\eta$ and a Riemannian metric $g$ satisfying the following conditions is called an almost contact metric manifold:

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1$$

$$\phi \xi = 0, \quad \eta \circ \phi = 0$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any $X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields on $M$. Further, an almost contact metric manifold $(M, \phi, \xi, \eta, g)$ is called a contact metric manifold if it satisfies the following condition:

$$d\eta(X, Y) = g(X, \phi Y)$$

for any $X, Y \in \mathfrak{X}(M)$. An almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$ is called a quasi contact metric manifold if the corresponding almost Hermitian cone $\bar{M} = (\bar{M}, \bar{J}, \bar{g})$ is a quasi Kähler manifold [4, 6, 7]. In [6], the quasi contact metric manifold is proved to be a generalization of a contact metric manifold. Further, the authors raised the following question based on the discussion.

**Question A.** Does there exist a $(2n+1)(\geq 5)$-dimensional quasi contact metric manifold which is not a contact metric manifold?
Concerning the above Question A, authors discussed oriented hypersurfaces in a quasi Kähler manifold which are quasi contact metric manifolds with respect to the naturally induced almost contact metric structure, and obtained the following results in [1].

**Theorem B.** Let \( (\bar{M}, J, \bar{g}) \) be a nearly Kähler manifold and \( M \) be a hypersurface of \( \bar{M} \) oriented by a unit normal vector field \( \nu \). Then \( M = (M, \phi, \xi, \eta, g) \) is a quasi contact metric manifold with respect to the naturally induced almost contact metric structure \( (\phi, \xi, \eta, g) \) if and only if it satisfies the equality

\[
g((A\phi + \phi A)X,Y) = -2g(\phi X,Y)
\]

for any \( X,Y \in \mathfrak{X}(M) \), where \( A \) is the shape operator with respect to the unit normal vector field \( \nu \), and hence, \( M \) is a contact metric manifold.

**Theorem C.** There does not exist oriented totally umbilical hypersurface in the nearly Kähler unit 6-sphere which is a quasi contact metric manifold with respect to the naturally induced almost contact metric structure.

In the present paper, we provide explicit examples of totally umbilical hyperspheres in the nearly Kähler unit 6-sphere which support Theorem B or Theorem C. We also discuss the properties from the viewpoint of almost contact metric geometry.

### 2. Preliminaries

First, we shall recall fundamental the nearly Kähler structure on a unit 6-sphere \( S^6 \). Let \( \mathfrak{C} \) be the Cayley algebra \( \mathfrak{C} = \{ x = x_0 + \sum_{i=1}^{7} x_i e_i \mid x_0, x_i \in \mathbb{R}, e_i^2 = -1 \ (1 \leq i \leq 7) \} \), and \( \mathfrak{C}_+ = \{ x = \sum_{i=1}^{7} x_i e_i \in \mathfrak{C} \mid x_i \in \mathbb{R} \ (1 \leq i \leq 7) \} \) both set of all pure imaginary Cayley numbers. Here, the multiplication operation on \( \mathfrak{C} \) is defined by the figure below;

![Figure 1]
We denote by $\langle, \rangle$ the canonical inner product on $\mathcal{E}$ and let $|x| = \sqrt{(x,x)}$ (the length of $x \in \mathcal{E}$). Then, $(\mathcal{E}, \langle, \rangle)$ (resp. $(\mathcal{E}_+ , \langle, \rangle))$ can be identified with 8-dimensional Euclidean space $\mathbb{E}^8$ (resp. 7-dimensional Euclidean space $\mathbb{E}^7$) in the natural way. We also define cross product $x \times y$ for $x, y \in \mathcal{E}_+$ by $x \times y = xy + \langle x, y \rangle 1 (\in \mathcal{E}_+)$. Here, we identify $e_i \in \mathcal{E}_+$ $(1 \leq i \leq 7)$ with the coordinate vector field $\frac{\partial}{\partial x_i}$ (denoted by $\partial_i$ briefly) in our arguments and adopt them alternatively in the forthcoming arguments. We denote by $\partial$ the Levi-Civita connection on $\mathbb{E}^7$ with respect to the Riemannian metric induced from the inner product $\langle, \rangle$. Let $S^6$ be a unit 6-sphere in $\mathbb{E}^7(\simeq \mathcal{E}_+)$ centered at the origin $o$. Then, $S^6$ is expressed as $S^6 = \{ x \in \mathcal{E}_+ \mid |x| = 1 \}$.

For any point $x \in S^6$, we denote by $N_x$ the outward oriented unit normal vector with initial point $x$, $N_x = 6\partial_{\theta}$. In this paper, we identify $N_x(x \in S^6)$ with the position vector $x(\in \mathcal{E}_+)$. The unit normal vector $N$ is also written as $N = \sum_{i=1}^7 x_i \partial_i$ in terms of the coordinate vector fields $\partial_i (1 \leq i \leq 7)$. Here we note that the tangent space $T_xS^6$ can be regarded as the subspace $\{ y \in \mathcal{E}_+ \mid \langle y, x \rangle = 0 \}$ of $\mathcal{E}_+$. Now, we define $(1, 1)$-tensor field $J$ on $S^6$ by

$$J_x y = N_x \times y = x \times y (=xy), \quad y \in T_xS^6.$$  

Then, we may easily check that $J$ is an almost complex structure on $S^6$ and $(J, g)$ is a nearly Kähler structure on $S^6$, namely, $(\nabla_X J)Y = -(\nabla_Y J)X$ holds for any vector fields $X, Y$ tangent to $S^6$, where $g$ and $\nabla$ are the Riemannian metric on $S^6$ induced from the inner product $\langle, \rangle$ on $\mathcal{E}_+$ and $\nabla$ is the Levi-Civita connection of $g$, respectively. We shall call the nearly Kähler structure $(J, g)$ given above on $S^6$ the standard one.

3. One parameter family of totally umbilical hyperspheres in $S^6$

First, for each real number $r \ (-1 < r < 1)$, we define hypersurface $M_r$ by

$$M_r = S^6 \cap \{ x = \sum_{i=1}^6 x_ie_i + re_7 \in \mathcal{E}_+ \mid x_i \in \mathbb{R} \ (1 \leq i \leq 6) \}$$

$$= \{ x = \sum_{i=1}^6 x_ie_i + re_7 \in \mathcal{E}_+ \mid \sum_{i=1}^6 x_i^2 = 1 - r^2 \}.$$  

We observe that $M_r$ is diffeomorphic to a 5-sphere $S^5$.

Now, let $x$ be any point of $M_r$ and $\gamma_x$ be the smooth curve in $M_r$ through $x = \gamma_x(\theta)$ $(0 < \theta < \pi)$ defined by

$$\gamma_x(t) = (\cos t)e_7 + \left( \frac{1}{\sqrt{1 - r^2}} \sin t \right) \sum_{i=1}^6 x_ie_i \ (0 \leq t \leq \pi),$$  

where $\cos \theta = r$, $\sin \theta = \sqrt{1 - r^2}$. We here define a vector field $\nu$ on $M_r$ by

$$\nu_x = \frac{d}{dt}|_{t=0} \gamma_x(t).$$  

Thus, from (6), the following equalities hold for any $x \in M_r$;

$\bar{g}(\nu_x, \nu_x) = \langle \nu_x, \nu_x \rangle = 1$,

\[
\langle \nu_x, N_x \rangle = \langle \nu_x, x \rangle = r \sqrt{1 - r^2} (1 - r^2) - r \sqrt{1 - r^2} = 0.
\]

On the other hand, for any $x = \sum_{i=1}^{6} x_i e_i + re_7 \in M_r$, we may find an integer $a$ ($1 \leq a \leq 6$) such that $x_a \neq 0$ and fix it. Now, we shall define a smooth curve $\alpha_{a,b}(s)$ ($1 \leq b \leq 6$, $b \neq a$) ($-\pi < s < \pi$) through the point $x = \alpha_{a,b}(0)$ by

\[
\alpha_{a,b}(s) = \left( \sqrt{x_a^2 + x_b^2 \cos (s + \theta_{a,b})} e_a + \sqrt{x_a^2 + x_b^2 \sin (s + \theta_{a,b})} e_b \right. + \sum_{1 \leq i \leq 6, i \neq a,b} x_i e_i + re_7,
\]

where $\cos \theta_{a,b} = \frac{x_a}{\sqrt{x_a^2 + x_b^2}}$, $\sin \theta_{a,b} = \frac{x_b}{\sqrt{x_a^2 + x_b^2}}$ $(0 \leq \theta_{a,b} < 2\pi)$. Then, from (8), we have

\[
\frac{d}{ds} |_{s=0} \alpha_{a,b}(s) = - x_b e_a + x_a e_b \quad (= -x_b \partial_a + x_a \partial_b)
\]

at $x$. We here set

\[
X_{a,b} = - x_b e_a + x_a e_b \quad (= -x_b \partial_a + x_a \partial_b).
\]

From (9) and (10), it follows that

\[
X_a X_{a,b} = \text{span}_\mathbb{R} \{ X_{a,b} \ (b \neq a, \ 1 \leq b \leq 6) \}
\]

and

\[
\bar{g}(X_{a,b}, \nu_x) = 0,
\]

at $x \in M_r$. Thus, from (11) and (12), we can see that $\nu_x$ is a unit normal vector at any $x \in M_r$ in $S^6$, namely the vector field $\nu$ is a unit normal vector field on $M_r$ in $S^6$.

Now, since $S^6$ is a totally umbilical hypersurface in $\mathbb{E}^7 \cong \mathbb{C}^+$ with respect to the unit normal vector field $N$, the corresponding shape operator $\bar{A}$ is given by $\bar{A} = -I$. Thus, taking account of the Gauss formula, we have

\[
D_{X_{a,b}} \nu = \nabla_{X_{a,b}} \nu.
\]
From (6), the unit normal vector field $\nu$ can be expressed by

$$\nu = \frac{r}{\sqrt{1-r^2}} \sum_{i=1}^{6} x_i \partial_i - \sqrt{1-r^2} \partial_r. \tag{14}$$

Thus, from (10), (13) and (14), we have

$$D_{X_{a,b}} \nu = \frac{r}{\sqrt{1-r^2}} (-x_b \partial_a + x_a \partial_b) = \frac{r}{\sqrt{1-r^2}} X_{a,b} \tag{15}$$

for any $X_{a,b}$ at any point $x \in M_r$. Therefore, from (15), we see that $(M_r, g)$ is a totally umbilical hypersurface of $(S^6, \bar{g})$ with the shape operator $A = -\frac{r}{\sqrt{1-r^2}} I$ with respect to the unit normal vector field $\nu$ on $(M_r, g)$ in $(S^6, \bar{g})$.

4. Almost contact metric structures on $(M_r, g)$

In this section, we define two kinds almost contact metric structures on $(M_r, g)$ and discuss their respective geometric properties. First, let $\xi$ be the unit vector field on $M_r$ defined by

$$\xi = -J\nu = -N \times \nu. \tag{16}$$

Then, from (16), it follows that the vector field $\xi$ is orthogonal to both of the vector fields $N$ and $\nu$ along $M_r$. Further, from Fig. 1, (6) and (16), we have

$$\xi = -\left(\sum_{i=1}^{6} x_i e_i + r e_7\right) \times \left(\frac{r}{\sqrt{1-r^2}} \sum_{j=1}^{6} x_j e_j - \sqrt{1-r^2} e_7\right)$$

$$= \sqrt{1-r^2} \left(\sum_{i=1}^{6} x_i e_i\right) \times e_7 - \frac{r^2}{\sqrt{1-r^2}} e_7 \times \left(\sum_{j=1}^{6} x_j e_j\right)$$

$$= \frac{1}{\sqrt{1-r^2}} (x_6 e_1 + x_5 e_2 + x_4 e_3 - x_3 e_4 - x_2 e_5 - x_1 e_6). \tag{17}$$

From (17), $\xi$ is also rewritten as

$$\xi = \frac{1}{\sqrt{1-r^2}} (x_6 \partial_1 + x_5 \partial_2 + x_4 \partial_3 - x_3 \partial_4 - x_2 \partial_5 - x_1 \partial_6). \tag{18}$$

Thus, the 1-form $\eta$ dual to the vector field $\xi$ is given by

$$\eta = \frac{1}{\sqrt{1-r^2}} (x_6 dx_1 + x_5 dx_2 + x_4 dx_3 - x_3 dx_4 - x_2 dx_5 - x_1 dx_6). \tag{19}$$

From (19), we also have

$$d\eta = -\frac{2}{\sqrt{1-r^2}} (dx_1 \wedge dx_6 + dx_2 \wedge dx_5 + dx_3 \wedge dx_4). \tag{20}$$
From (19) and (20), we have further
\[
\eta \wedge (d\eta)^2 = - \frac{8}{(\sqrt{1 - r^2})^3} \left\{ -x_1 dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5 \wedge dx_6 \\
+x_2 dx_1 \wedge dx_3 \wedge dx_4 \wedge dx_5 \wedge dx_6 \\
-x_3 dx_1 \wedge dx_2 \wedge dx_4 \wedge dx_5 \wedge dx_6 \\
+x_4 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_5 \wedge dx_6 \\
-x_5 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_6 \\
+x_6 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5 \right\} \neq 0.
\]
(21)

Therefore, \( \eta \) is a contact form on \( M_r \). Now, we shall show \( \nabla_\xi \xi = 0 \). From (6), we have
\[
D_\xi \xi = - \frac{1}{1 - r^2} \sum_{i=1}^{6} x_i \partial_i.
\]
Taking account of the Gauss formula for \((S^6, \bar{g})\) and \((\mathbb{E}^7, \langle \cdot, \cdot \rangle)\), we have
\[
D_\xi \xi = \bar{\nabla}_\xi \xi - N = \bar{\nabla}_\xi \xi - \sum_{i=1}^{6} x_i \partial_i - r \partial_7.
\]
(23)

On the other hand, since \((M_r, g)\) is a totally umbilical hypersurface of \((S^6, \bar{g})\) with the shape operator \( A = - \frac{r}{\sqrt{1 - r^2}} I \) with respect to the unit normal vector field \( \nu \), from (14), taking account of the Gauss formula, we get
\[
\bar{\nabla}_\xi \xi = \nabla_\xi \xi - \frac{r}{\sqrt{1 - r^2}} \nu
\]
\[
= \nabla_\xi \xi - \frac{r}{\sqrt{1 - r^2}} \left( \sum_{i=1}^{6} x_i \partial_i - \sqrt{1 - r^2} \partial_7 \right)
\]
\[
= \nabla_\xi \xi - \frac{r^2}{1 - r^2} \sum_{i=1}^{6} x_i \partial_i + r \partial_7.
\]
(24)

Then, from (22)~(24), we have
\[
- \frac{1}{1 - r^2} \sum_{i=1}^{6} x_i \partial_i = \nabla_\xi \xi - (1 + \frac{r^2}{1 - r^2}) \sum_{i=1}^{6} x_i \partial_i,
\]
and hence
\[
\nabla_\xi \xi = 0.
\]
(25)

From (25), it follows that each integral curve of the vector field \( \xi \) is a geodesic of \((M_r, g)\). Thus, taking account of the definition of the vector field \( \xi \) in (16), we see that \((M_r, g, \xi)\) is a Hopf hypersurface in \((S^6, J, \bar{g})\). Further, since \((M_r, g)\) is a totally umbilical hypersurface in \((S^6, \bar{g})\) with the shape operator \( A = - \frac{r}{\sqrt{1 - r^2}} I \),
from the Gauss equation for \((M_r, g)\), we see that the curvature tensor \(R\) of \((M_r, g)\) is given
\[
R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + \frac{r^2}{1 - r^2}(g(Y, Z)X - g(X, Z)Y)
\]
for any \(X, Y, Z \in T_xM_r\). From (26), it follows that \((M_r, g)\) is a hypersurface of \((S^6, \bar{g})\) of constant sectional curvature \(\frac{1}{1 - r^2}\). We define \((1, 1)\)-tensor field \(\phi\) on \(M_r\) by
\[
\phi X = JX - \eta(X)\nu
\]
for any \(X \in T_xM_r\). Then, from (16), (19) and (27), we see that \((\phi, \xi, \eta, g)\) is the naturally induced almost contact metric structure on \(M_r\). Now, choose \(x = \sum_{i=1}^{6} x_i e_i + re_7 \in M_r\) arbitrary. Without loss of essentiality, we may suppose \(x_1 \neq 0\), for example. Then, from (4), (10) and (27), taking account of Fig. 1, we have
\[
X_{1,2} = -x_2 \partial_1 + x_1 \partial_2, \quad X_{1,3} = -x_3 \partial_1 + x_1 \partial_3,
\]
and
\[
\phi X_{1,3} = JX_{1,3} - \eta(X_{1,3})\nu
\]
\[
= (x_1 x_2 - \frac{r}{1 - r^2}(x_1^2 x_4 - x_1 x_3 x_6))\partial_1
\]
\[
+ (-x_1^2 + x_4^2) - \frac{r}{1 - r^2}(x_1 x_2 x_4 - x_2 x_3 x_6))\partial_2
\]
\[
+ (x_2 x_3 - \frac{r}{1 - r^2}(x_2 x_3 x_4 - x_3^2 x_6))\partial_3
\]
\[
+ ((-x_3 x_5 + x_1 x_1) - \frac{r}{1 - r^2}(x_1 x_2^2 - x_2 x_4 x_6))\partial_4
\]
\[
+ (x_1 x_6 + x_3 x_4 - \frac{r}{1 - r^2}(x_1 x_4 x_5 - x_3 x_5 x_6))\partial_5
\]
\[
+ (-r x_3 + x_1 x_5) - \frac{r}{1 - r^2}(x_1 x_4 x_6 - x_3 x_6^2))\partial_6.
\]
Thus, from (28) and (29), we have
\[
g(X_{1,2}, \phi X_{1,3}) = -x_1(x_1^2 + x_2^2 + x_3^2)(\neq 0).
\]
On the other hand, from (20) and (28), we have
\[
d\eta(X_{1,2}, X_{1,3}) = -\frac{2}{\sqrt{1 - r^2}}(dx_1 \wedge dx_6 + dx_2 \wedge dx_5 + dx_3 \wedge dx_4)
\]
\[
(-x_2 \partial_1 + x_1 \partial_2, -x_3 \partial_1 + x_1 \partial_3)
\]
\[
= 0.
\]
Thus, from (30) and (31), we have
\[
d\eta(X_{1,2},X_{1,3}) \neq g(X_{1,2},\phi X_{1,3}).
\]
We may also derive the similar conclusion as (32) for the other cases \(x_b \neq 0\) \((3 \leq b \leq 6)\). Thus, from (32), the almost contact metric manifold \((M_r, \phi, \xi, \eta, g)\) is not a contact metric manifold for any \(r (−1 < r < 1)\). This supports Theorem C, since the quasi contact metric structure is a generalization of a contact metric structure.

Now, let \(\tau\) be the scalar curvature of \((M_r, g)\) and define the smooth functions \(f\) on \(M_r\) and the mean curvature \(\alpha\) respectively by
\[
f = g(A\xi, \xi)
\]
and
\[
\alpha = \frac{1}{5} \text{tr} A.
\]
Then, from (26), since \(A = -\frac{r}{\sqrt{1-r^2}} I\) and \(g(\xi, \xi) = 1\), we have
\[
\tau = \frac{20}{1-r^2}, \quad f = -\frac{r}{\sqrt{1-r^2}}, \quad \alpha = -\frac{r}{\sqrt{1-r^2}}.
\]
Then, from (35), we may check that the hypersurface \((M_r, \phi, \xi, \eta, g)\) satisfies the following equality;
\[
\tau = 20 + 5\alpha (5\alpha - f)
\]
for any real number \(r (−1 < r < 1)\). Here, we note that \((M_r, \phi, \xi, \eta, g)\) is totally geodesic in \((S^6, g)\) if and only if \(r = 0\). Now, let \(M\) be an orientable compact and connected hypersurface of the nearly Kähler unit 6-sphere \((S^6, J, \bar{g})\) endowed with the naturally induced almost contact metric structure \((\phi, \xi, \eta, g)\) and define the functions \(f\) and \(\alpha\) on \(M\) by (33) and (34) respectively, in terms of the hypersurface \(M\). Under the above setting, in [5], the authors asserted that if the scalar curvature \(\tau\) of \(M\) satisfies the inequality \(\tau \geq 20 + 5\alpha (5\alpha - f)\), then \((M, \phi, \xi, \eta, g)\) is a totally geodesic hypersphere \(S^5\) of \((S^6, J, \bar{g})\) ([5], Theorem 1.1). However, by taking account of the equality (36), we may check that their assertion is not appropriate, since every hyperspheres \((M_r, \xi, g)\) \((−1 < r < 1, r \neq 0)\) belonging to our one parameter family of totally umbilical hypersurfaces of the nearly Kähler unit 6-sphere \((S^6, J, \bar{g})\) is not totally geodesic for any \(r (−1 < r < 1, r \neq 0)\).

Next, we define another almost contact metric structure on the hypersurface \((M_r, g)\) and discuss on the geometric properties. Let \(\phi'\) be the \(\{1, 1\}\)-tensor field on \(M_r\) defined by
\[
\phi' \xi = 0
\]
and
\[
\phi' X = -\nu \times X = -\nu X
\]
for any $X \in \mathfrak{X}(M_r)$ with $X \perp \xi$. Then, from (16), (37) and (38), we may check

$$
\langle \phi' X, N \rangle = -\langle \nu X, N \rangle = -\langle \nu, \nu \rangle \langle \nu X, N \rangle \\
= -\langle \nu (\nu X), \nu N \rangle = -\langle \nu^2 X, \nu N \rangle \\
= \langle X, \nu N \rangle = -\langle X, \nu N \rangle \\
= \langle X, \xi \rangle = 0,
$$

(39)

$$
\langle \phi' X, \nu \rangle = \langle -\nu X, \nu \rangle \\
= -\langle \nu, \nu \rangle \langle \nu X, \nu \rangle \\
= -\langle \nu^2 X, \nu^2 \rangle = \langle X, 1 \rangle \\
= 0,
$$

(40)

$$
\langle \phi' X, \xi \rangle = \langle -\nu X, -\nu N \rangle \\
= \langle -\nu, \nu \rangle \langle X, N \rangle \\
= 0.
$$

(41)

Thus, from (38) \sim (41), we have finally

$$
\phi'^2 X = \phi' (\phi' X) = \nu (\nu X) = \nu^2 X = -X.
$$

(42)

Taking account of (37) \sim (42), we see that $(\phi', \xi, \eta, g)$ is an almost contact metric structure on $M_r$. Further, we may note that the almost contact metric manifold $(M_0, \phi', \xi, \eta, g)$ coincides with the almost contact metric manifold introduced in ([3], p. 64) which is different from the naturally induced cone from the nearly Kähler structure $(J, \bar{g})$ on $S^6$ with respect to the unit normal vector field $\nu$. Later, we shall show that $(\phi', \xi, \eta, g)$ is a contact metric structure on the hypersurface $M_0$.

For any $X \in \mathfrak{X}(M_r)$, we set

$$
Y = X - \eta(X) \xi.
$$

(43)

Then, $Y \in \mathfrak{X}(M_r)$ and $Y \perp \xi$. From (16), (37) and (38), we have

$$
\phi' Y = -\nu \times Y = -\nu \times (X - \eta(X) \xi) \\
= -\nu \times X + \eta(X)(\nu \times \xi) = -\nu X + \eta(X) \nu \xi \\
= -\nu X - \eta(X) \nu (\nu N) = -\nu X + \eta(X) \nu^2 N \\
= -\nu X - \eta(X) N
$$

(44)

for any $X \in \mathfrak{X}(M_r)$. Comparing (27) and (44), we see that $\phi X \neq \phi' X$ for $X \in \mathfrak{X}(M_r)$ with $X \perp \xi$. It is known that the almost contact metric structure $(\phi', \xi, \eta, g)$ on the hypersphere $M_0$ in the nearly Kähler 6-sphere $S^6$ is a contact metric structure by ([3], p. 64). Here, we shall provide an exact proof for this
fact, now, we choose a point $x = \sum_{i=1}^{6} x_i e_i \in M$ arbitrary and fix it. Here, for our purpose without also discuss in the case where $x_i \neq 0$, now, we set

$$Y_{1,b} = X_{1,b} - \eta(X_{1,b})\xi \quad (1 < b \leq 6)$$

for any $X \in \mathfrak{X}(M_{\varepsilon})$. Then, from (45), taking account of (10), (17) with $r = 0$, (18) and Fig. 1, we have

$$Y_{1,2} = (-x_2 + x_2 x_5^2 - x_1 x_3 x_6)\partial_1 + (x_1 + x_2 x_5 x_6 - x_1 x_3^2)\partial_2 + (x_2 x_4 x_6 - x_1 x_4 x_5)\partial_3 + (-x_2 x_3 x_6 + x_1 x_3 x_5)\partial_4 + (-x_2 x_6 + x_1 x_2 x_3)\partial_5 + (-x_1 x_2 x_6 + x_1^2 x_5)\partial_6,$$

$$Y_{1,3} = (-x_3 + x_3 x_2^2 - x_1 x_4 x_6)\partial_1 + (x_3 x_5 x_6 - x_1 x_4 x_3)\partial_2 + (x_1 + x_3 x_4 x_6 - x_1 x_4^2)\partial_3 + (-x_2 x_3 x_6 + x_1 x_3 x_4)\partial_4 + (-x_2 x_3 x_6 + x_1 x_2 x_4)\partial_5 + (-x_1 x_3 x_6 + x_1^2 x_4)\partial_6,$$

$$Y_{1,4} = (-x_4 + x_4 x_3^2 + x_1 x_3 x_6)\partial_1 + (x_4 x_5 x_6 + x_1 x_3 x_5)\partial_2 + (x_1 x_4 x_6 + x_4 x_5 x_6 - x_1 x_4 x_3)\partial_3 + (x_1 - x_3 x_4 x_6 - x_1 x_4^2)\partial_4 + (-x_2 x_4 x_6 - x_1 x_2 x_3)\partial_5 + (-x_1 x_4 x_6 - x_1^2 x_3)\partial_6,$$

$$Y_{1,5} = (-x_5 + x_5 x_2^2 + x_1 x_2 x_6)\partial_1 + (x_5 x_6 + x_1 x_2 x_5)\partial_2 + (x_4 x_5 x_6 + x_1 x_2 x_4)\partial_3 + (x_3 x_5 x_6 - x_1 x_2 x_3)\partial_4 + (x_1 - x_2 x_5 x_6 - x_1 x_2^2)\partial_5 + (-x_1 x_5 x_6 - x_1^2 x_2)\partial_6,$$

$$Y_{1,6} = (-x_6 + x_6 x_3^2 + x_2^3)\partial_1 + (x_1 x_3 x_6 - x_1^2 x_4)\partial_2 + (x_1 + x_4 x_6 - x_1 x_3^2)\partial_3 + (x_3 x_5 x_6 - x_1 x_4 x_5)\partial_4 + (-x_3 + x_3 x_6^2 - x_1 x_4 x_6)\partial_5,$$

$$Y_{1,7} = (-x_7 + x_7 x_2^2 + x_3^2 x_4)\partial_1 + (x_4 x_5 x_6 + x_1 x_2 x_3)\partial_2 + (x_1 + x_3 x_4 x_6 - x_1 x_4 x_3)\partial_3 + (x_2 x_5 x_6 + x_1 x_3 x_5)\partial_4 + (-x_4 + x_4 x_6^2 + x_1 x_3 x_6)\partial_5,$$

$$Y_{1,8} = (-x_8 + x_8 x_3^2 + x_4^2 x_5)\partial_1 + (x_1 + x_4 x_5 x_6 + x_1 x_2 x_3)\partial_2 + (x_3 x_5 x_6 + x_1 x_4 x_5)\partial_3 + (x_4 x_5 x_6 + x_1 x_2 x_4)\partial_4 + (x_2 x_5 x_6 + x_1 x_2 x_4)\partial_5 + (x_3 x_5 x_6 + x_1 x_2 x_3)\partial_6.$$
\[ \phi'Y_{1,6} = (-x_1 + x_3^2 + x_6^2)x_1 \partial_1 + (x_1^2 x_2 + x_3^2 x_2) \partial_2 \\
+ (x_1^2 x_3 + x_6^2 x_3) \partial_3 + (x_1^2 x_4 + x_6^2 x_4) \partial_4 \\
+ (x_1^2 x_5 + x_6^2 x_5) \partial_5 + (-x_6 + x_1^2 x_6 + x_3^2) \partial_6. \]

Thus, from (17), (20) and (46), we have
\[ d\eta(Y_{1,2}, Y_{1,3}) = 0, \quad d\eta(Y_{1,2}, Y_{1,4}) = 0, \tag{48} \]
\[ d\eta(Y_{1,2}, Y_{1,5}) = -x_1^2, \quad d\eta(Y_{1,2}, Y_{1,6}) = x_1 x_2, \quad d\eta(Y_{1,6}, \xi) = 0 \]
for any \( b \ (1 < b \leq 6). \) Similarly, from (37), (46) and (47), we have
\[ g(Y_{1,2}, \phi'Y_{1,3}) = 0, \quad g(Y_{1,2}, \phi'Y_{1,4}) = 0, \tag{49} \]
\[ g(Y_{1,2}, \phi'Y_{1,5}) = -x_1^2, \quad g(Y_{1,2}, \phi'Y_{1,6}) = x_1 x_2, \quad g(Y_{1,6}, \phi'\xi) = 0 \]
for any \( b \ (1 < b \leq 6). \) Therefore, from (3), (48) and (49), we can see that \((M_0, \phi', \xi, \eta, g)\) is a contact metric manifold. This support both Theorem B and Theorem C, since the contact metric structure \((\phi', \xi, \eta, g)\) is different from the naturally induced one \((\phi, \xi, \eta, g)\) on \( M_0. \) On the other hand, from (26) with \( r = 0, \) \((M_0, \phi', \xi, \eta, g)\) is a space of constant sectional curvature 1. Therefore, from the fact (3), Theorem 7.3, we see finally that \((M_0, \phi', \xi, \eta, g)\) is a Sasakian manifold. Further, taking account of (5), we may check that, for each \( r(-1 < r < 1), \) the map \( F_r : M_r \to M_0 \) defined by
\[ F_r(\sum_{i=1}^{6} x_i e_i + r \gamma) = \frac{1}{\sqrt{r}} \left( \sum_{i=1}^{6} x_i e_i \right), \left( \sum_{i=1}^{6} x_i^2 = 1 - r^2 \right) \]
on \( M_0 \) is a diffeomorphism from \( M_r \) to \( M_0. \) Thus, the pullback of the Sasakian structure \((\phi', \xi_0, \eta_0, g_0)\) on \( M_0 \) to \( M_r \) by the diffeomorphism \( F_r \) is also a Sasakian structure on \( M_r \) of constant sectional curvature 1 for each \( r(-1 < r < 1). \) We here note that the pullback Sasakian structure is given by \( \phi = (F_r^{-1})_* \circ \phi_0 \circ (F_r)_*, \xi = (F_r^{-1})_* \xi_0 = \xi, \eta = (F_r^{-1})_*(\eta_0) = \eta, g = (F_r^{-1})_*(g_0) = g. \) On the other hand, by modifying the above arguments suitably, we may also check that \((M_r, \phi', \xi, \eta, g)\) is not a contact metric manifold for any \( r \) with \((-1 < r < 1, r \neq 0). \)

**Remark 1.** Let \( M \) be a hypersurface in the nearly Kähler unit 6-sphere \((S^6, J, \tilde{g})\) oriented by unit normal vector field \( \nu \) and \((\phi, \xi, \eta, g)\) be the corresponding naturally induced almost contact metric structure on \( M. \) Now, let \( G \) be the \((1,2)\)-tensor field on \((S^6, J, \tilde{g})\) given by \( G(X, Y) = (\nabla_X J)(Y) \) for any \( X, Y \in \mathfrak{X}(S^6), \) and \( \psi \) be the \((1,1)\)-tensor field on \( M \) defined by \( \psi X = G(X, \nu) \) for any \( X \in \mathfrak{X}(M) \) [5]. Here, specifying \((M, \nu)\) as the hypersurface \((M_0, \nu)\) introduced in §3, we can show that \( \psi = \phi' \) holds for \( M_0 \) by making use of the discussions in [5, 8].

**Remark 2.** J. Berndt, J. Bolton and L. M. Woodward have proved that a Hopf hypersurface in the nearly Kähler 6-sphere \( S^6 \) is either an open part of (i) a geodesic hypersphere of \( S^6 \) or (ii) a tube around an almost complex curve in \( S^6 [(2], \) Theorem 2). Taking account of this result, it seems also meaningful to discuss the Hopf hypersurfaces of type (ii) in \( S^6 \) from the geometry of almost contact metric structures viewpoint.
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