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A STUDY ON $(k,\mu)'$-ALMOST KENMOTSU MANIFOLDS

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Abstract. Let $\mathcal{C}$, $\mathcal{M}$, $\mathcal{L}$ be concircular curvature tensor, M-projective curvature tensor and conharmonic curvature tensor, respectively. We obtain that if a non-Kenmotsu $(k,\mu)'$-almost Kenmotsu manifold satisfies $\mathcal{C}\cdot S=0$, $R\cdot\mathcal{M}=0$ or $R\cdot\mathcal{L}=0$, then it is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$.

1. Introduction

A $(2n+1)$-dimensional smooth differentiable manifold M^{2n+1} is said to be an *almost contact metric manifold* if on M^{2n+1} there exists a structure (ϕ,ξ,η,g) satisfying

$$\phi^2=-id+\eta\otimes\xi,\quad \eta(\xi)=1, \tag{1.1}$$

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y) \tag{1.2}$$

for any vector fields $X,Y\in\mathfrak{X}(M)$, where ϕ is a $(1,1)$-type tensor field, ξ a global vector field, η a 1-form and g a Riemannian metric [7]. An almost Kenmotsu manifold is defined as an *almost contact metric manifold* such that η is closed, i.e., $d\eta=0$, and $d\Phi=2\eta\wedge\Phi$, where Φ is a 2-form on M^{2n+1} defined by $\Phi(X,Y)=g(X,\phi Y)$ for any $X,Y\in\mathfrak{X}(M)$. An *almost contact metric manifold* is said to be *normal* when the Nijenhuis tensor ϕ is given by

$$[\phi,\phi]=-2d\eta\otimes\xi,$$

where

$$[\phi,\phi](X,Y)=\phi^2[X,Y]+[\phi X,\phi Y]-\phi[\phi X,Y]-\phi[X,\phi Y],\forall\, X,Y\in\mathfrak{X}(M).$$

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A normal almost Kenmotsu manifold is called a *Kenmotsu manifold* [4]. From [10], the normality is equivalent to the following

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \quad \forall\, X, Y \in \mathfrak{X}(M).$$

On an almost Kenmotsu manifold M^{2n+1}, we consider two (1,1)-type tensor fields $h = \frac{1}{2}L_\xi \phi$ and $h' = h \circ \phi$, where L is the Lie differentiation. According to [9], we see that the tensor fields h and h' are symmetric operators and satisfy the following conditions

$$h\xi = h'\xi = 0, \quad tr(h) = tr(h') = 0, \quad h\phi + \phi h = 0. \tag{1.3}$$

For more details and results on almost Kenmotsu manifolds, we refer the reader to [2, 8].

In this paper, we aim to investigate the concircular curvature tensor $\mathcal{C}$, the M-projective curvature tensor $\mathcal{M}$ and the conharmonic curvature tensor $\mathcal{L}$ on $(k,\mu)'$-almost Kenmotsu manifolds. Let $\mathcal{S}$ be the Ricci tensor, R the Riemannian curvature tensor and Q the Ricci operator defined by $\mathcal{S}(X,Y) = g(QX,Y)$. In preliminaries Section, we provide some properties to prove our main results. In Section 3, we prove that a $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} satisfying $\mathcal{C} \cdot \mathcal{S} = 0$ is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$ or it is an Einstein manifold. In Section 4 and 5, we study a non-Kenmotsu $(k,\mu)'$-almost Kenmotsu manifold satisfying $R \cdot \mathcal{M} = 0$ or $R \cdot \mathcal{L} = 0$, then it is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

2. Preliminaries

An almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is said to be a $(k,\mu)'$-*almost Kenmotsu manifold* if the vector field ξ satisfies the $(k,\mu)'$-nullity condition, i.e.,

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)h'X - \eta(X)h'Y) \tag{2.1}$$

where $X, Y \in \mathfrak{X}(M)$, $k, \mu \in \mathbb{R}$. Similarly, if ξ satisfies the (k,μ)-nullity condition, i.e.,

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \tag{2.2}$$

where $X, Y \in \mathfrak{X}(M)$, $k, \mu \in \mathbb{R}$, then M^{2n+1} is said to be a (k,μ)-*almost Kenmotsu manifold* [3].

From [3], we know that on $(k,\mu)'$-almost Kenmotsu manifolds with $h' \neq 0$, then $k < -1$, $\mu = -2$, and

$$h'^2 X = -(k+1)(X - \eta(X)\xi) \tag{2.3}$$

for any $X \in \mathfrak{X}(M)$. Moreover, $h' = 0$ if and only if $k = -1$. According to (2.1), we have

$$R(\xi, X)Y = k(g(X,Y)\xi - \eta(Y)X) - 2(g(h'X,Y)\xi - \eta(Y)h'X) \tag{2.4}$$

for any $X, Y \in \mathfrak{X}(M)$. When $k < -1$, we denote by $[\lambda]'$ and $[-\lambda]'$ the eigenspaces of h' associated with eigenvalues λ, $-\lambda$, respectively. From (2.3), we have $\lambda^2 = -(k+1)$ [6].

We give an important lemma used to proof our main results:

Lemma 2.1 ([10, Lemma 3.2]). *Let M^{2n+1} be a $(k,\mu)'$-almost Kenmotsu manifold such that $h' \neq 0$. Then the Ricci operator of M^{2n+1} is given by*

$$QX = -2nX + 2n(k+1)\eta(X)\xi - 2nh'X. \tag{2.5}$$

Moreover, the scalar curvature of M^{2n+1} is $2n(k-2n)$.

3. Concircular curvature tensor

The concircular curvature tensor $\mathcal{C}$ on a $(2n+1)$-dimensional Riemannian manifold (M^{2n+1}, g) is defined by

$$\mathcal{C}(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}(g(Y,Z)X - g(X,Z)Y) \tag{3.1}$$

for any vector fields $X, Y, Z \in \mathfrak{X}(M)$, where r is the scalar curvature of M^{2n+1}.

Let M^{2n+1} be a $(k,\mu)'$-almost Kenmotsu manifold satisfying

$$\mathcal{C}(\xi, Y) \cdot \mathcal{S}(Z, W) = 0. \tag{3.2}$$

Therefore, it implies that

$$\mathcal{S}(\mathcal{C}(\xi,Y)Z, W) + \mathcal{S}(Z, \mathcal{C}(\xi,Y)W) = 0. \tag{3.3}$$

Firstly, we consider the case $h' \neq 0$, or equivalently, $k < -1$. Putting $W = \xi$ in (3.3), using (2.4) and Lemma 2.1, we obtain

$$\begin{aligned} &\mathcal{S}(\mathcal{C}(\xi,Y)Z,\xi) \\ =&2nkg(R(\xi,Y)Z,\xi) - \frac{r \cdot 2nk}{2n(2n+1)}g(Y,Z) + \frac{r \cdot 2nk}{2n(2n+1)}\eta(Z)\eta(Y) \\ =&\left(2nk^2 - \frac{k-2n}{2n+1}\cdot 2nk\right)(g(Y,Z) - \eta(Z)\eta(Y)) - 4nkg(h'Y,Z) \end{aligned} \tag{3.4}$$

for any vector fields $Y, Z \in \mathfrak{X}(M)$. Similarly, we also have

$$\begin{aligned} &\mathcal{S}(Z, \mathcal{C}(\xi,Y)\xi) \\ =&g(R(\xi,Y)\xi, QZ) - \frac{r}{2n(2n+1)}\eta(Y)g(QZ,\xi) + \frac{r}{2n(2n+1)}g(Y,QZ) \end{aligned} \tag{3.5}$$

for any vector fields $Y, Z \in \mathfrak{X}(M)$. Further, according to $\mathcal{S}(X,Y) = g(QX,Y)$ and (2.3), we have

$$\begin{aligned} & g(R(\xi,Y)\xi, QZ) \\ =& g(k\eta(Y)\xi - kY + 2h'Y, -2nZ + 2n(k+1)\eta(Z)\xi - 2nh'Z) \\ =& (4n+6nk)g(Y,Z) + (2nk-4n)g(h'Y,Z) - (6nk+4n)\eta(Y)\eta(Z). \end{aligned} \tag{3.6}$$

From Lemma 2.1, we get

$$\eta(Y)g(QZ,\xi) = 2nk\eta(Y)\eta(Z), \tag{3.7}$$

$$g(Y,QZ) = -2ng(Y,Z) - 2ng(h'Z,Y) + 2n(k+1)\eta(Y)\eta(Z). \tag{3.8}$$

Putting (3.6)–(3.8) into (3.5) and using Lemma 2.1, we obtain

$$\begin{aligned} \mathcal{S}(Z, \mathcal{C}(\xi,Y)\xi) =& \left(4n + 6nk - \frac{2n(k-2n)}{2n+1}\right)(g(Y,Z) - \eta(Y)\eta(Z)) \\ &+ \left(2nk - 4n - \frac{2n(k-2n)}{2n+1}\right) g(h'Z,Y). \end{aligned} \tag{3.9}$$

Putting (3.4) and (3.9) into (3.3), we get

$$\begin{aligned} &\left(2nk^2 - \frac{2n(k+1)(k-2n)}{2n+1} + 4n + 6nk\right)(g(Y,Z) - \eta(Y)\eta(Z)) \\ &+ \left(-2nk - 4n - \frac{2n(k-2n)}{2n+1}\right) g(h'Z,Y) = 0. \end{aligned} \tag{3.10}$$

Assume that $Y, Z \in [\lambda]'$, then we have

$$g(h'Y,Z) = \lambda g(Y,Z), \quad \eta(Y)\eta(Z) = 0.$$

Now (3.10) becomes

$$2n\left((k^2 + 2 + 3k - 2\lambda - k\lambda) - \frac{(k-2n)(k+1+\lambda)}{2n+1}\right) g(Y,Z) = 0. \tag{3.11}$$

Using $k = -\lambda^2 - 1$, we get

$$\lambda^2(\lambda - 1)(2n\lambda + 4n + 2) = 0.$$

Since $\lambda > 0$, it follows that $\lambda = 1$, and hence $k = -2$.

Otherwise, in case of $h' = 0 (\Leftrightarrow k = -1)$, equations (2.1) and (2.4) become

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{3.12}$$

$$R(\xi,X)Y = -g(X,Y)\xi + \eta(Y)X \tag{3.13}$$

for any $X, Y \in \mathfrak{X}(M)$ [5]. According to (3.12),

$$\mathcal{S}(X,\xi) = -2n\eta(X), \quad Q(\xi) = -2n\xi. \tag{3.14}$$

Then, we have

$$\begin{aligned}&\mathcal{S}(\mathcal{C}(\xi,Y)Z,\xi)\\ &=2n\left(1+\frac{r}{2n(2n+1)}\right)g(Y,Z)-2n\left(1+\frac{r}{2n(2n+1)}\right)\eta(Z)\eta(Y),\end{aligned}\tag{3.15}$$

$$\begin{aligned}&\mathcal{S}(Z,\mathcal{C}(\xi,Y)\xi)\\ &=\left(1+\frac{r}{2n(2n+1)}\right)\mathcal{S}(Y,Z)+2n\left(1+\frac{r}{2n(2n+1)}\right)\eta(Z)\eta(Y).\end{aligned}\tag{3.16}$$

Using (3.3), (3.15) and (3.16), we get

$$\mathcal{S}(Y,Z)=-2ng(Y,Z).\tag{3.17}$$

Therefore, we conclude the following:

Theorem 3.1. *If a $(k,\mu)'$-almost Kenmotsu manifold satisfies $\mathcal{C}\cdot S=0$, either it is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$ or it is an Einstein manifold.*

4. M-projective curvature tensor

The M-projective curvature tensor $\mathcal{M}$ on a $(2n+1)$-dimensional Riemannian manifold (M^{2n+1},g) is given by

$$\begin{aligned}\mathcal{M}(X,Y)Z=&R(X,Y)Z-\frac{1}{4n}[\mathcal{S}(Y,Z)X\\ &-\mathcal{S}(X,Z)Y+g(Y,Z)QX-g(X,Z)QY]\end{aligned}\tag{4.1}$$

for any vector fields $X,Y,Z\in\mathfrak{X}(M)$.

Let M be a $(k,\mu)'$-almost Kenmotsu manifold satisfying $R(\xi,X)\cdot\mathcal{M}=0$. It implies that

$$\begin{aligned}&g(R(\xi,e_i)(\mathcal{M}(e_i,Z)W),\xi)-g(\mathcal{M}(R(\xi,e_i)e_i,Z)W,\xi)\\ &-g(\mathcal{M}(e_i,R(\xi,e_i)Z)W,\xi)-g(\mathcal{M}(e_i,Z)R(\xi,e_i)W,\xi)=0.\end{aligned}\tag{4.2}$$

By simple calculation, we get

$$\begin{aligned}&g(R(\xi,e_i)(\mathcal{M}(e_i,Z)W),\xi)\\ &=\frac{(2n+1)k}{4n}\mathcal{S}(Z,W)-\frac{(k-2n)k}{2}g(Z,W)-kg(\mathcal{M}(\xi,Z)W,\xi)\\ &-2g(\mathcal{M}(e_i,Z)W,h'e_i).\end{aligned}\tag{4.3}$$

Using (2.4), the second term in (4.2) becomes

$$g(\mathcal{M}(R(\xi,e_i)e_i,Z)W,\xi)=2nkg(\mathcal{M}(\xi,Z)W,\xi).\tag{4.4}$$

In view of the anti-symmetrization of X, Y in $\mathcal{M}(X,Y)Z$, we obtain

$$g(\mathcal{M}(e_i, R(\xi, e_i)Z)W, \xi) = g(\mathcal{M}(\xi, W)\xi, kZ - 2h'Z) + 2\eta(Z)g(\mathcal{M}(e_i, h'e_i)W, \xi). \tag{4.5}$$

Because of the anti-symmetrization of Z, W in $g(\mathcal{M}(X,Y)Z, W)$, the last term in (4.2) becomes

$$g(\mathcal{M}(e_i, Z)R(\xi, e_i)W, \xi) = -k\eta(W)g(\mathcal{M}(e_i, Z)e_i, \xi) + 2\eta(W)g(\mathcal{M}(e_i, Z)h'e_i, \xi). \tag{4.6}$$

Putting (4.3)–(4.6) into (4.2), we get

$$\begin{aligned} &\frac{(2n+1)k}{4n}\mathcal{S}(Z,W) - \frac{(k-2n)k}{2}g(Z,W) + g(\mathcal{M}(\xi, W)\xi, 2nkZ + 2h'Z) \\ &+ g(\mathcal{M}(e_i, Z)h'e_i, 2W) + 2\eta(Z)g(\mathcal{M}(e_i, h'e_i)\xi, W) \\ &+ \eta(W)g(\mathcal{M}(e_i, Z)\xi, 2h'e_i - ke_i) = 0. \end{aligned} \tag{4.7}$$

By using the Lemma 4.2 in [1], we have the following formulas

$$g(\mathcal{M}(\xi, W)\xi, 2nkZ + 2h'Z) = (nk+3)(1+k)[\eta(W)\eta(Z) - g(W,Z)] + (3nk - 1 - k)g(W, h'Z), \tag{4.8}$$

$$2\eta(Z)g(\mathcal{M}(e_i, h'e_i)\xi, W) = 0, \tag{4.9}$$

$$\eta(W)g(\mathcal{M}(e_i, Z)\xi, 2h'e_i - ke_i) = n(k+1)(6-k)\eta(Z)\eta(W), \tag{4.10}$$

$$g(\mathcal{M}(e_i, Z)h'e_i, 2W) = 2(2n+1)(k+1)[\eta(Z)\eta(W) - g(h'Z, W) - 2(n+1)(k+1)g(Z,W). \tag{4.11}$$

Assume that $Z \in [-\lambda]'$, $W \in [\lambda]'$, then we obtain

$$[4nk + 7k + 8n + 6]g(h'Z, W) + (k+1)(k + 2nk + 10 + 4n)g(Z,W) = 0. \tag{4.12}$$

So we have

$$\lambda(\lambda - 1)[-(2n+1)\lambda^2 - (6n+8)\lambda + (1-4n)] = 0. \tag{4.13}$$

We obtain $\lambda = 1$, then $k = -2$. Therefore, we conclude the following:

Theorem 4.1. *If a $(k,\mu)'$-almost Kenmotsu manifold satisfies $R \cdot \mathcal{M} = 0$ and $h' \neq 0$, then it is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.*

5. Conharmonic curvature tensor

The conharmonic curvature tensor $\mathcal{L}$ on a $(2n+1)$-dimensional Riemannian manifold (M^{2n+1}, g) is given by

$$\begin{aligned}\mathcal{L}(X,Y)Z =& R(X,Y)Z - \frac{1}{2n-1}[\mathcal{S}(Y,Z)X \\ &- \mathcal{S}(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]\end{aligned} \tag{5.1}$$

for any vector fields $X, Y, Z \in \mathfrak{X}(M)$.

Let M be a $(k,\mu)'$-almost Kenmotsu manifold satisfying $R(\xi, X)\cdot\mathcal{L} = 0$. It implies that

$$\begin{aligned}&g(R(\xi, e_i)(\mathcal{L}(e_i, Z)W), \xi) - g(\mathcal{L}(R(\xi, e_i)e_i, Z)W, \xi) \\ &- g(\mathcal{L}(e_i, R(\xi, e_i)Z)W, \xi) - g(\mathcal{L}(e_i, Z)R(\xi, e_i)W, \xi) = 0.\end{aligned} \tag{5.2}$$

Then we have

$$\begin{aligned}&\frac{2nk(2n-k)}{2n-1}g(Z,W) + 2g(\mathcal{L}(\xi, W)\xi, nkZ + h'Z) + g(\mathcal{L}(e_i, Z)h'e_i, 2W) \\ &+ 2\eta(Z)g(\mathcal{L}(e_i, h'e_i)\xi, W) + \eta(W)g(\mathcal{L}(e_i, Z)\xi, 2h'e_i - ke_i) = 0.\end{aligned} \tag{5.3}$$

By further calculation, assuming that $Z \in [-\lambda]'$, $W \in [\lambda]'$, we have

$$4\lambda(\lambda-1)[(-n^2 - n + 1)\lambda + n - n^2] = 0. \tag{5.4}$$

We also get that $\lambda = 1$ and $k = -2$. Therefore, we conclude the following:

Theorem 5.1. *If a $(k,\mu)'$-almost Kenmotsu manifold satisfies $R\cdot\mathcal{L} = 0$ and $h' \neq 0$, then it is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.*

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