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# ESTIMATION OF HURST PARAMETER AND MINIMUM VARIANCE SPECTRUM

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Abstract. Consider FARIMA time series with innovations that have infinite variances. We are interested in the estimation of self-similarities $H_n$ of FARIMA$(0, d, 0)$ by using modified $R/S$ statistic. We can confirm that the $H_n$ converges to Hurst parameter $H = d+\frac{1}{2}$. Finally, we figure out ARMA and minimum variance power spectrum density of FARIMA processes.

## 1. Introduction

Self-similar processes have been studied in the analysis of traffic load in high speed networks and financial mathematics( [3], [9], [10]). On the other hand, Self-similarity, long range dependence and heavy tailed process have been observed in many time series, i.e. signal processing and finance( [7], [11]). In particular, Fractional autoregressive integrated moving average (FARIMA) processes in packet network traffic has been the focus of much attention( [8], [12]). And, there has been a recent flood of literature and discussion on the tail behavior of queue-length distribution, motivated by potential applications to the design and control by high-speed telecommunication networks( [1], [2], [5], [6]).

In this paper we consider self-similar process and FARIMA processes with Gaussian innovations and study convergence of Hurst parameter

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$H_n$ by using modified $R/S$ statistic. On the other hand, we estimate autoregressive moving average (ARMA) spectrum and minimum variance power spectrum of FARIMA processes.

In section 2, we define self-similar process, Hurst parameter, long range dependence and FARIMA process.

The following $R/S$ method which was used by Taqqu( [11]) is one of the better known method. For a time series $X = \{X_i : i \leq 1\}$, with partial sum $Y(n) = \sum_{i=1}^n X_i$, and sample variance $S^2(n) = (1/n)\sum_{i=1}^n X_i^2 - (1/n)^2 Y(n)^2$, the $R/S$ statistic, or the rescaled adjusted range, is given by

$$\frac{R}{S}(n) = \frac{1}{S(n)}\left[\max_{0\leq t\leq n}(Y(t) - \frac{t}{n}Y(n)) - \min_{0\leq t\leq n}(Y(t) - \frac{t}{n}Y(n))\right].$$

For fractional Gaussian noise

$$E[R/S(n)] \sim C_H n^H, \text{ as } n \to \infty,$$

where $C_H$ is positive, finite constant not depend on $n$.

B. Sikdar and K.S. Vastola( [7]) estimated self-similar parameter by above $R/S$ method and its computer simulation. H. Stark and J.W. Woods( [10]) figured out power spectrum density of signal process and J.M. Kim and Y.K. Kim( [4]) figured out power spectrum density of FARIMA processes.

In section 3, by using modified $R/S$ statistic which is normalized by a standard deviation which takes into account the covariances of the first $q$ lags , we estimate self-similarities $H_n$ of FARIMA$(0, d, 0)$ and show that self-similarity $H_n$ converges to Hurst parameter $H = d + \frac{1}{2}$. In section 4, we figure out ARMA and minimum variance power spectrum density of FARIMA$(0, d, 0)$ processes.

## 2. Definition and Preliminary

In this section we first define self-similar processes and FARIMA processes.

Definition 2.1. A continuous process $X(t)$ is *self-similar with self-similarity parameter* $H \geq 0$ if it satisfies the condition:

$$X(t) \overset{d}{=} c^{-H}X(ct), \ \ \forall\, t \geq 0, \forall c > 0,$$

where the equality $\stackrel{d}{=}$ is in the sense of finite-dimensional distributions.

Self-similar processes are invariant in distribution under scaling of time and space. Brownian motion is a Gaussian process with mean zero and autocovariance function

$$E[X(t_1)X(t_2)] = \min(t_1, t_2).$$

It is $H$ self-similar with $H = 1/2$. And, Fractional Brownian motion is important example of self-similar process.

Let $X = \{X(i), i \geq 1\}$ be a stationary sequence and

$$X^{(m)}(k) = \frac{1}{m} \sum_{i=(k-1)m+1}^{km} X(i), \quad k = 1, 2, 3, \cdot$$

be corresponding aggregated sequence with the level of aggregation $m$ and averaging over each block. The index, $k$, labels the block. Now we define exact self-similar which is appropriate in the context of time series theory.

Definition 2.2. A stationary sequence $X = \{X(i), i \geq 1\}$ is exactly self-similar if it satisfies

$$X \stackrel{d}{=} m^{1-H} X^{(m)}$$

for all aggregation levels $m$.

Let $\tau_X(k)$ be a covariance of stationary sequence, i.e.

$$\tau_X(k) = Cov(X(i), X(i+k))$$

Definition 2.3. A stationary sequence $X(i)$ exhibits *long range dependence* if

$$\sum_{k=-\infty}^{\infty} |\tau_X(k)| = \infty.$$

A standard example of a long range dependent process is fractional Brownian motion, with Hurst parameter $H > \frac{1}{2}$.

Since fractional Brownian motion $\{B_H(t) : t \in R\}$ has stationary increments, its increments $G_j$ form a stationary sequence. Fractional

Gaussian noise is a mean zero and stationary Gaussian time series whose autocovariance function $\tau(h) = EG_iG_{i+h}$ is given by

$$\tau(h) = 2^{-1}\{(h+1)^{2H} - 2h^{2H} + |h-1|^{2H}\},$$

$h \geq 0$. As $h \to \infty$,

$$\tau(h) \sim H(2H-1)h^{2H-2}.$$

Since $\tau(h) = 0$ for $h \geq 1$ when $H = 1/2$. the $G_i$ are white noise. When $1/2 < H < 1$, they display long-range dependence.

We introduce a FARIMA$(p, d, q)$ which is both long range dependent and has heavy tails. FARIMA$(p, d, q)$ processes are capable of modeling both short and long range dependence in traffic models since the effect of $d$ on distant samples decays hyperbolically while the effects of $p$ and $q$ decay exponentially.

Definition 2.4. A series $X_i$ is called a *FARIMA*$(p, d, q)$ if

$$\Phi(B)\nabla^d X_i = \Theta(B)\varepsilon_i$$

where the $\varepsilon_i$ are independent, identically distributed normal random variables with mean 0 and variance 1, $\Phi(B) = 1 - \Phi_1 B - \cdots - \Phi_p B^p$, $\Theta(B) = 1 - \Theta_1 B - \cdots - \Theta_q B^q$ and the coefficients $\Phi_1, \cdots, \Phi_p$ and $\Theta_1, \cdots, \Theta_q$ are constants,

$$\nabla^d = (1-B)^d = \sum_{i=0}^{\infty} b_i(-d)B^i$$

and B is the backward shift operator defined as $B^i X_t = X_{t-i}$ and

$$b_i(-d) = \prod_{k=1}^{i} \frac{k+d-1}{k} = \frac{\Gamma(i+d)}{\Gamma(d)\Gamma(i+1)}.$$

For large lags $d$, the autocovariance function satisfies for $0 < d < 1/2$,

$$\tau(h) \sim \sigma^2 C h^{2d-1} \text{ as } h \to \infty$$

where $\sigma^2$=Var $X_i$ and $C$ depend on $d, \Phi$ and $\Theta$ . Thus, for large lags $d$, the autocovariance function has the same power decay as the autocovariance of fractional Gaussian noise. Relating the exponents gives

$$d = H - \frac{1}{2}.$$

## 3. Estimation of Hurst parameter $H$

First, we estimate Hurst parameter through the behavior of the absolute moments. Consider $q^{th}$ moment

$$\mu^{(m)}(q) = E|X^{(m)}|^q = E\left|\frac{1}{m}\sum_{i=1}^{m} X(i)\right|^q .$$

Definition 3.1. A stationary sequence $X = \{X(i), i \geq 1\}$ is called multi-fractal if the logarithms of the absolute moments scale linearly with the logarithms of the aggregation level $m$, that is,

$$\log \mu^{(m)}(q) = \beta(q) \log m + c(q).$$

Definition 3.2. A multi-fractal sequence $X = \{X(i), i \geq 1\}$ is called self-similar if the exponent $\beta(q)$ is linear with respect to $q$.

Theorem 3.1. *Let $\{X(t), t \geq 0\}$ be a self-similar. Then $\{X(i), i \geq 1\}$ is multi-fractal self-similar sequence with $\beta(q) = q(H-1)$ and $c(q) = logE|X(1)|^q$. Therefore, we can estimate Hurst parameter $H$ as*

$$\hat{H} = 1 + \frac{\hat{\beta}(q)}{q}.$$

*Proof.* By the self-similarity of $X$, we know

$$X(1) \stackrel{d}{=} m^{1-H} X^{(m)}(1).$$

Therefore, we get

$$\begin{aligned} \mu^{(m)}(q) &= E|X^{(m)}|^q = E|X^{(m)}(1)|^q \\ &= E|m^{H-1} X(1)|^q \\ &= m^{q(H-1)} E|X(1)|^q. \end{aligned}$$

Thus,

$$\log \mu^{(m)}(q) = q(H-1)\log m + \log E|X(1)|^q.$$

□

To estimate the $q^{th}$ moment of $X$, $\mu^{(m)}(q)$, we can use the $q^{th}$ sample absolute moment $\hat{\mu}^{(m)}(q)$ of the aggregated series $X^{(m)}$, that is,

$$\hat{\mu}^{(m)}(q) = \frac{1}{N/m} \sum_{k=1}^{N/m} \left| X^{(m)}(k) \right|^q .$$

If $\hat{\mu}^{(m)}(q)$ scales linearly with $\log m$, then a multi-fractal model can be applied. If, in addition, $\beta(q)$ is linear in $q$, then a self-similar model is adequate and we can estimate Hurst parameter $H$ for fractional ARIMA sequences.

Now, we consider FARIMA time series with $d = 0.35$. $d$ can be $0 < d < 0.5$ because FARIMA displays long range dependence. For each $q = 1, 2, 3, 4$, we calculate the $q^{th}$ sample absolute moments of aggregated series $X^{(m)}$. And, we can know that $\hat{\mu}^{(m)}(q)$ scales linearly with $\log m$. Therefore, a multi-fractal model can be applied.

To know a self-similar model is adequate, we need figure out whether the slope $\hat{\beta}(q)$ is linear in $q$ or not. The following two tables show that the correlation of $\hat{\beta}(q)$ and $q$ is $-0.9982$ and the slope $\hat{\beta}(q)$ is linear in $q$.

| | q | slope |
|---|---|---|
| q | 1.00/0.0 | −0.9982/0.0018 |
| slope | −0.9982/0.0018 | 1.00/0.0 |

Table 1. Correlation Coefficients and $Prob > |R|$ under $H_0 : \rho = 0$

| | $q = 1$ | $q = 2$ | $q = 3$ | $q = 4$ |
|---|---|---|---|---|
| slope | −0.1557 | −0.3745 | −0.5537 | −0.7941 |
| intercept | −0.0161 | 0.2167 | 0.5866 | 1.0533 |

Table 2. Slope and intercept of $\hat{\mu}^{(m)}(q)$ for each $q$

Therefore, we can conclude that FARIMA time series with $d = 0.35$ is self-similar process with Hurst parameter $\hat{H}$, where,

$$\hat{H} = 1 + \hat{\beta}(q) = 0.8434 \pm 0.036.$$

Now, let us estimate Hurst parameter by another method.

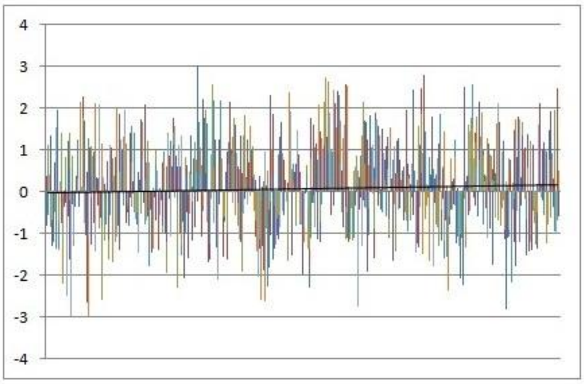


FIGURE 1. Sample traces of FARIMA$(0, 0.35, 0)$, $n = 1,000$

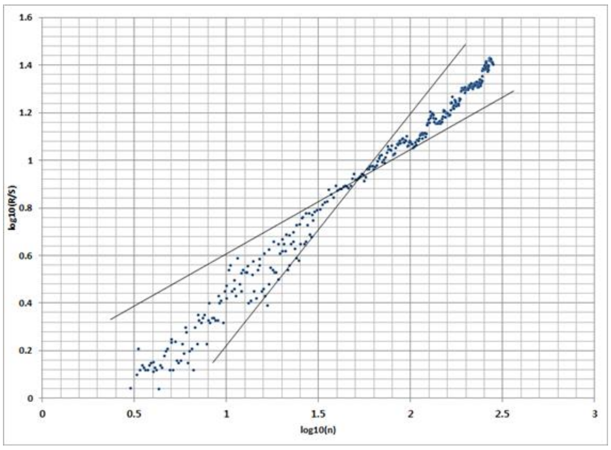


FIGURE 2. Graphical $R/S$ output and Hurst parameter $H$

To define modified $R/S$ statistic $V_q(N)$( [13]), instead of using the sample standard deviation, S, we use a weight sum of autocovariances, namely, $S_q(N)$ as

$$\left(\frac{1}{N}\sum_{i=1}^{N}(X_i - \bar{X_N})^2 + \frac{2}{N}\sum_{i=1}^{q}\omega_i(q)\left[\sum_{j=i+1}^{N}(X_j - \bar{X_N})(X_{j-i}\bar{X_N})\right]\right)^{1/2},$$

where $\bar{X_N}$ denotes the sample mean of the times series, and the weights $\omega_i(q)$ are given by

$$\omega_i(q) = 1 - \frac{i}{q+1}, \quad q < N.$$

We calculate Lo's modified $R/S$ statistic, $V_q(N)$, defined by

$$V_q(N) = N^{-1/2} R(N)/S_q(N).$$

Hence, we consider sample traces, graphical $R/S$ output and Hurst parameter $H$ of FARIMA$(0, d, 0)$ in terms of $d = 0.35$.

The following Figure 1 and Figure 3 illustrate FARIMA processes in the case $n = 1,000$ and $n = 10,000$.

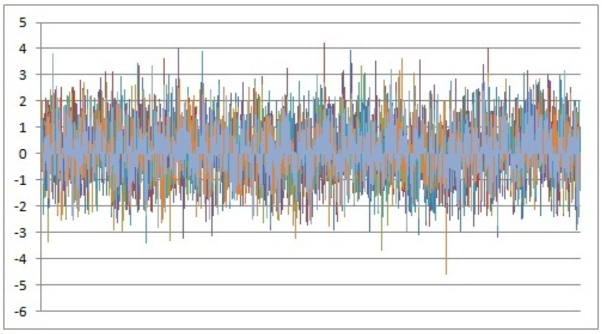


FIGURE 3. Sample traces of FARIMA$(0, 0.35, 0)$, $n = 10,000$

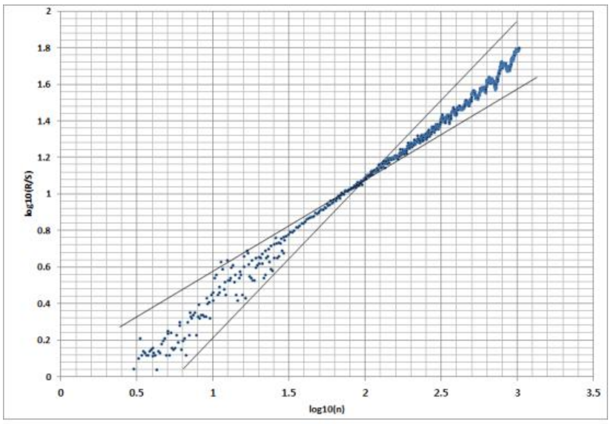


FIGURE 4. Graphical $R/S$ output and Hurst parameter $H$

We consider FARIMA$(0, d, 0)$ in terms of $d = 0.35$ and estimate the self-similarity $H_n$ and figure out convergence of $H_n$ into $H = d + \frac{1}{2}$.

We can estimate $H_{1,000} = 0.883$ as shown in Figure 2 and $H_{10,000} = 0.857$ in Figure 4. Therefore, we can figure out convergence of $H_n$ into $H = d + \frac{1}{2}$.

## 4. ARMA and Minimum Variance Power Spectrum Density

Let $X_T(t)$ be a sample function $X(t)$ in $-T < t < T$ . If $T$ is finite, then we assume $X_T(t)$ is satisfied the followimg

$$\int_T^T |X_T(t)|dt < \infty.$$

Then we obtain the Fourier Transform

$$X_T(\omega) = \int_T^T X(t)e^{-i\omega t}dt.$$

Power spectrum density is

$$P_{XX}(\omega) = \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

and the following is satisfied

$$P_{XX}\left(e^{i\omega}\right) = \sum_{k=-\infty}^{\infty} R_{XX}(k)e^{-i\omega k}$$

where, $R_{XX}$ is approximated with estimated correlation sequence $\hat{R}_{XX}$.

Recall biased correlation estimate of $X(n), \quad n = 0, 1, 2, \cdots, N-1$, defined as

$$\hat{R}_{XX}(k) = \frac{1}{N} \sum_{k=0}^{N-1-k} X(l+k)X(l), \;\; 0 \le k < N-1.$$

The power spectrum estimate is

$$\hat{P}_{XX}\left(e^{i\omega}\right) = \sum_{k=-L}^{L} \hat{R}_{XX}(k)e^{-i\omega k}, \;\; L \le N$$

Let us estimate ARMA power spectrum and minimum variance power spectrum. First, ARMA spectrum density of a FARIMA process is equal to

$$P(\omega) = \frac{\sigma^2|\Theta(\exp(-i\omega))|^2}{2\pi|\Phi(\exp(-i\omega))|^2}|1-(\exp(-i\omega))|^{-2d}.$$

In Figure 5, we display ARMA spectrum of FARIMA process.

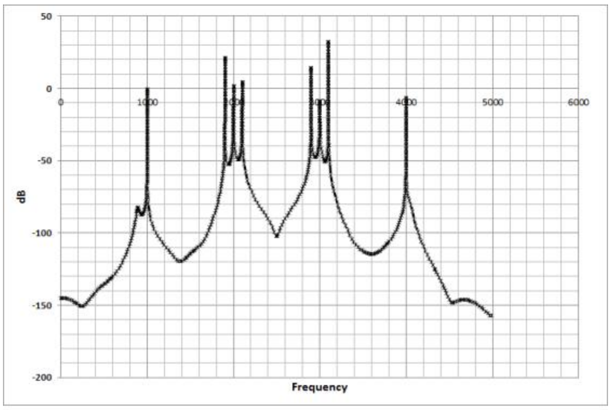


FIGURE 5. Autoregressive Moving Average Spectrum

Now, we consider minimum variance spectrum estimation. We take overlapping segments of the data, each of length $L$(total will be $K = N - L + 1$ segments), so that the spectrum is evaluated at distinct frequences $\omega_l = \frac{2\pi}{L}l$, where, $l = 0, 1, 2, \cdots, L-1$.

The the average periodogram is

$$\hat{P}\left(e^{i\omega_l}\right) = \frac{1}{KL}\sum_{j=0}^{K-1}|Y_l(j)|^2,$$

where, $Y_l(j) = \sum_{n=0}^{L-1} X(n+j)e^{-in\omega_l}$.

Minimum Variance is to limit the total output energy $\frac{1}{K}\sum_{j=0}^{K}|Y_l(j)|^2$ of the filter. For the sample covariance matrix $\hat{\mathbf{R}}_X$, we get minimum variance power spectrum estimate as the following,

$$\hat{P}_{MV}\left(e^{i\omega}\right) = \frac{L}{\bar{\mathbf{e}}\hat{\mathbf{R}}_X^{-1}\mathbf{e}},$$

where, $\mathbf{e} = [1 \;\; e^{i\omega} \;\; e^{i2\omega} \;\; \cdots \;\; e^{i(L-1)\omega}]^T$. By simulating closely spaced frequencies in noise, we can estimate minimum variance spectrum as shown in Figure 6. Since minimum variance spectrum density is very sensitive to frequency partitioning selected, we need very fine frequency spacing to accurately measure power spectrum density.

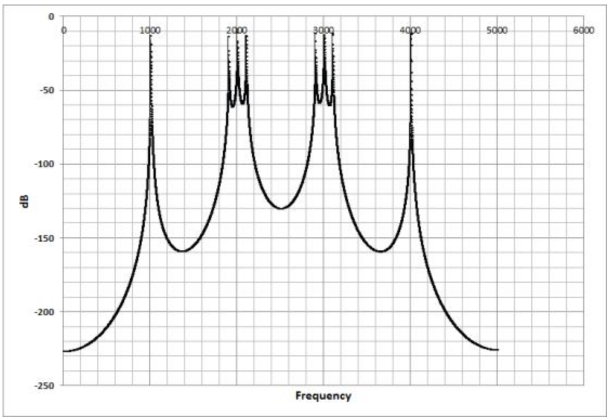


FIGURE 6. Minimum Variance Spectrum

In section 3 of this paper, we estimate Hurst parameter $H$ by

$$\hat{H} = 1 + \frac{\hat{\beta}(q)}{q}$$

of Theorem 3.1 and modified $R/S$ statistic. Finally, we figure out ARMA power spectrum and minimum variance power spectrum density in section 4.

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