Korean J. Math. **26** (2018), No. 2, pp. 175–189
https://doi.org/10.11568/kjm.2018.26.2.175

CONSTRUCTION OF Γ-ALGEBRA AND Γ-LIE ADMISSIBLE ALGEBRAS

A. H. Rezaei and Bijan Davvaz*

Abstract. In this paper, at first we generalize the notion of algebra over a field. A Γ-algebra is an algebraic structure consisting of a vector space V, a groupoid Γ together with a map from $V \times \Gamma \times V$ to V. Then, on every associative Γ-algebra V and for every $\alpha \in \Gamma$ we construct an α-Lie algebra. Also, we discuss some properties about Γ-Lie algebras when V and Γ are the sets of $m \times n$ and $n \times m$ matrices over a field F respectively. Finally, we define the notions of α-derivation, α-representation, α-nilpotency and prove Engel theorem in this case.

1. Γ-Lie algebras

In [7], Nobusawa introduced the notion of Γ-rings, as a generalization of rings. Barnes [2] weakened slightly the conditions in the definition of the Γ-ring in the sense of Nobusawa. After these two papers are published, many mathematicians made good works on Γ-ring in the sense of Barnes and Nobusawa. Luh [6] and Kyuno [5] studied the structure of Γ-rings and obtained various generalization analogous to corresponding parts in ring theory. In [1], Chakraborty and Pau defined an isomorphism, an anti-isomorphism and a Jordan isomorphism in a Γ-ring

Received December 1, 2017. Revised March 27, 2018. Accepted April 3, 2018.
2010 Mathematics Subject Classification: 17B60.
Key words and phrases: Γ-algebra, α-Lie algebra, α-ideal, α-derivation, α-representation, α-nilpotent.
∗ Corresponding author.

and developed some important results relating to these concepts, also see [8, 9].

Definition 1.1. Let Γ be a groupoid and V be a vector space over a field F. Then, V is called a Γ-*algebra* over the field F if there exists a mapping $V \times \Gamma \times V \longrightarrow V$ (the image is denoted by $x\alpha y$ for $x, y \in V$ and $\alpha \in \Gamma$) such that the following conditions hold:

(1) $(x+y)\alpha z = x\alpha z + y\alpha z$, $x\alpha(y+z) = x\alpha y + x\alpha z$,
(2) $x(\alpha+\beta)y = x\alpha y + x\beta y$,
(3) $(cx)\alpha y = c(x\alpha y) = x\alpha(cy)$,
for all $x, y, z \in V$, $c \in F$ and $\alpha, \beta \in \Gamma$.
Moreover, a Γ- algebra is called *associative* if
(4) $(x\alpha y)\beta z = x\alpha(y\beta z)$,

and *unital* if for every $\alpha \in \Gamma$, there is an element 1_α in V such that $1_\alpha \alpha v = v = v\alpha 1_\alpha$ for all non-zero elements of V. From part (1) of the definition we have $0\alpha x = x\alpha 0 = 0$ for all $x \in V, \alpha \in \Gamma$.

A non-empty subset V' of a Γ-algebra V is called a Γ-*subalgebra* if it is a subspace of V and for all $x, y \in V'$ and $\alpha \in \Gamma$ we have $x\alpha y \in V'$. A subset I of a Γ- algebra V is called a *left (right) ideal* if it is a Γ-subalgebra of V and for all $a \in I$ and $v \in V$ and $\alpha \in \Gamma$ we have $v\alpha a \in I$ ($a\alpha v \in I$) and is a *(two-sided) ideal* if it is both a left and right ideal. It easy to see that V and $\{0\}$ are ideals of V. An ideal I such that $\{0\} \subset I \subset V$ is called *proper*.

Let X be a subset of Γ-algebra V. Then, the smallest left (right, two-sided) ideal of V containing X exists and we shall call it the left (right or two-sided) ideal generated by X, and will be denoted by $< X >_l$ ($< X >_r$ or $< X >$). If $X = \{x\}$, then we also write $< x >$ instead of $< \{x\} >$.

Example 1.2. Let V be a vector space and Γ be a groupoid. For every $x, y \in V$ and $\alpha \in \Gamma$ we define $x\alpha y = 0$. Then, V is a Γ-algebra.

Example 1.3. Let F be a field, V and W be two vector spaces and $A = Hom_F(V, W)$, $\Gamma = Hom_F(W, V)$. For every $f, g \in A$ and $\alpha \in \Gamma$ we define $f\alpha g = f \circ \alpha \circ g$, where $\circ$ is the combination operation. Then, A is an associative Γ-algebra. Equivalently, let A and Γ be the sets of $n \times m$ and $m \times n$ matrices over the field F, respectively. Then, A is an associative Γ-algebra.

Example 1.4. Consider the pervious example. Let A be the set of 3×2 matrices over the field of real numbers $\mathbb{R}$ and

$$\Gamma = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \end{pmatrix} \ : \ a, b \in \mathbb{R} \right\}.$$

Then, A is an associative Γ-algebra and

$$B = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \\ 0 & 0 \end{pmatrix} \ : \ x, y \in \mathbb{R} \right\},$$

is a Γ-subalgebra of A.

The central role of Lie algebras in particle physics is well known. Starting from fundamental mathematical tools such as the Lie algebras of the Poincare group, the rotational group and the isospin group, the interest for Lie algebras received a determining impulse with the celebrated Racah lectures at Princeton in 1951. Finally in 1959 there has been the beginning of the use of the Lie algebras of the unitary compact groups whose importance for hadron physics is today well known [10].

Lie algebras are generally introduced in particle physics in terms of the product $[x\ y] = x \cdot y - y \cdot x$, where $a \cdot b$ is an associative product. However, according to Albert, a Lie algebra can also be introduced in terms of the product $[x\ y] = x \cdot y - y \cdot x$, where $x \cdot y$ is the product of a non-associative algebra. More exactly, any algebra A with the product $x \cdot y$ is called a *Lie admissible algebra* if the attached algebra A^-, which is the same vector space as A but with the new product $[x\ y] = x \cdot y - y \cdot x$, is a Lie algebra [3, 10].

The present paper is devoted to an elementary introduction to the theory of Γ-Lie admissible algebras.

Definition 1.5. Let V be an associative Γ-algebra over a field F. Then, for every $\alpha \in \Gamma$ one can construct an α- *Lie algebra* $L_\alpha(V)$. As a vector space, $L_\alpha(V)$ is the same as V. The Lie bracket of two elements of $L_\alpha(V)$ is defined to be their commutator in V:

$$[x\ y]_\alpha = x\alpha y - y\alpha x$$

Note that $[x\ y]_\alpha = -[y\ x]_\alpha$ for every $x, y \in V$ and $\alpha \in \Gamma$. Also, $L_\alpha(V)$ is abelian if $char(F) = 2$ or if $char(F) \neq 2$ then, $[x\ y]_\alpha = 0$ for every $x, y \in V$. From now on we suppose that $char(F) \neq 2$ and V is a finite dimensional vector space .

Let W be a non-empty subset of an associative Γ-algebra V and $\alpha \in \Gamma$. Then, we say that W is an *α-Lie subalgebra* of $L_\alpha(V)$ if it is a subspace of V and $[x\ y]_\alpha \in W$ for all $x, y \in W$. A subset I of $L_\alpha(V)$ is called a *left (right) α-ideal* if it is an α-Lie subalgebra and for all $a \in I$ and $v \in V$ we have $[v\ a]_\alpha \in I$ ($[a\ v]_\alpha \in I$) and is a *(two-sided) α-ideal* if it is both a left and right α-ideal. Since $[x\ y]_\alpha = -[y\ x]_\alpha$ for every $x, y \in V$ so every left (right) α-ideal is an α-ideal. Clearly 0 and V are α-ideals of V.

Example 1.6. Let W be a Γ-subalgebra of Γ-algebra V. Then, for every $\alpha \in \Gamma$, $L_\alpha(W)$ is an α-Lie subalgebra of $L_\alpha(V)$. If I is an ideal of V, then for every $\alpha \in \Gamma$, I is an α-ideal of $L_\alpha(V)$.

Note that if I and J be α-ideals of $L_\alpha(V)$, then also $I + J = \{x + y : x \in I, y \in J\}$ and $[I\ J]_\alpha = \{\sum_{i=1}^{i=n} x_i \alpha y_i :\ x_i \in I, y_i \in J, n \in \mathbb{N}\}$ are α-ideals of $L_\alpha(V)$.

Definition 1.7. Let V be an associative Γ algebra over a field F, $\alpha \in \Gamma$ and I be a non-zero α-ideal of $L_\alpha(V)$. Then the construction of a *quotient algebra* $L_\alpha(V/I)$ is formally the same as the construction of a quotient ring: as vector space V/I is just the quotient space, while its Lie multiplication is defined by $[x + I\ y + I]_\alpha = [x\ y]_\alpha + I$ for all $x, y \in V$. This is unambiguous, since if $x + I = x' + I$, $y + I = y' + I$, then we have $x' = x + u (u \in I)$, $y' = y + v (v \in I)$, whence $[x'\ y']_\alpha = [x\ y]_\alpha + ([u\ y]_\alpha + [x\ v]_\alpha + [u\ v]_\alpha)$ and therefore $[x' + I\ y' + I]_\alpha = [x + I\ y + I]_\alpha$, since the terms in parentheses are all lie in I.

Note that since as a vector space $L_\alpha(V)$ is the same as V, for simplicity we use V instead of $L_\alpha(V)$.

Definition 1.8. Let V and V' be two associative Γ algebras over a field F and $\alpha \in \Gamma$. A linear transformation $\phi^\alpha : V \longrightarrow V'$ is called an *α-homomorphism* if $\phi^\alpha([x\ y]_\alpha) = [\phi^\alpha(x)\ \phi^\alpha(y)]_\alpha$, for all $x, y \in V$. ϕ^α is called an *α-monomorphism* if $Ker\phi^\alpha = 0$, an *α-epimorphism* if $Im\phi^\alpha = V'$, an *α-isomorphism* if it is both α-monomorphism and α-epimorphism.

The first interesting observation to make is that $Ker\phi^\alpha$ is an α-ideal of $L_\alpha(V)$: indeed, if $\phi^\alpha(x) = 0$ and if $y \in V$ is arbitrary, then $\phi^\alpha([x\ y]_\alpha) = [\phi^\alpha(x)\ \phi^\alpha(y)]_\alpha = 0$. It is also apparent that $Im\phi^\alpha$ is an α-Lie subalgebra of $L_\alpha(V')$.

Definition 1.9. Let V be an associative Γ-algebra. Then, the ordinary dimension of V as a vector space is called the *dimension* and for every $\alpha \in \Gamma$ the dimension of the subspace of V generated by all products of the form $[x\ y]_\alpha$ is called the *α-dimension*.

Lemma 1.10. *Let V be an associative Γ-algebra and $\alpha = \alpha_1 + ... + \alpha_k \in \Gamma$. Then, for every $x, y \in V$ we have $[x\ y]_\alpha = [x\ y]_{\alpha_1} + ... + [x\ y]_{\alpha_k}$.*

Proof. $[x\ y]_\alpha = x\alpha y - y\alpha x = x(\alpha_1 + ... + \alpha_k)y - y(\alpha_1 + ... + \alpha_k)x = [x\ y]_{\alpha_1} + ... + [x\ y]_{\alpha_k}$. □

Theorem 1.11. *Let V be an associative Γ-algebra over a field F such that Γ be a finite dimensional vector space over the field F and $\{e_1, ..., e_k\}$ be a basis for Γ. Now, suppose that $\alpha = e_i + ... + e_j$, where $1 \leq i < j \leq k$. Then, for every $\beta = c_i e_i + ... + c_j e_j$ $(c_i, ..., c_j \in F)$, $L_\alpha(V)$ and $L_\beta(V)$ are isomorphic.*

Proof. The proof is straightforward by Lemma 1.10 and that Γ is a vector space over the field F □

2. Matrix forms

In this section, we are going to discuss some properties about Γ-Lie algebras when, V and Γ are the set of $m \times n$ and $n \times m$ matrices over a field F, respectively.

Theorem 2.1. *Let V be the set of $m \times n$ matrices over a field F where, $m < n$ $(m > n)$ and the columns $j_1, ..., j_k$ (the rows $i_1, ..., i_k$) are zero and Γ be the set of all $n \times m$ matrices over the field F. Then, for every $\alpha \in \Gamma$ such that all entries are zero other than the entries in rows $j_1, ..., j_k$ (the columns $i_1, ..., i_k$) that could be zero or not and for every $A, B \in V$, $[A\ B]_\alpha = 0$. So $L_\alpha(V)$ is abelian and the α-dimension of V is zero.*

Proof. Direct calculation. □

Corollary 2.2. *Let V be the set of $m \times n$ matrices over a field F, where $m < n$ $(m > n)$ and the ij entry is zero for all $j < m$ $(i < n)$ and Γ be the set of all $n \times m$ matrices over the field F. Then, for every $\alpha \in \Gamma$ such that the ij entry is zero for all $i \geq m$ $(j \geq n)$ and for every $A, B \in V$, $[A\ B]_\alpha = 0$.*

Proof. By Theorem 2.1, it is straightforward. □

EXAMPLE 2.3. Let V be the set of all real 5×3 matrices of the form

$$\begin{pmatrix} a & b & c \\ 0 & 0 & 0 \\ d & e & f \\ g & h & i \\ 0 & 0 & 0 \end{pmatrix}$$

and Γ be the set of all real 3×5 matrices. Then, for every $\alpha \in \Gamma$ of the form

$$\begin{pmatrix} 0 & \beta & 0 & 0 & \delta \\ 0 & \gamma & 0 & 0 & \theta \\ 0 & \mu & 0 & 0 & \lambda \end{pmatrix}$$

and for every $A, B \in V$, $[A\ B]_\alpha = 0$.

In the paper, we say that a matrix $(a_{ij})_{m\times n}$ is diagonal if all of entries are zero except entries $a_{ii},\ i = 1, ..., n$ when $n < m$ and $a_{ii},\ i = 1, ..., m$ when $m < n$ (these entries could be zero or not). A matrix $(a_{ij})_{m\times n}$ is upper triangular (strictly upper triangular) matrix if all of entries $a_{ij},\ i > j(a_{ij},\ i \geq j)$ are zero when $m > n$ and when $n > m$ all of entries $a_{ij},\ i > j\ or\ j > m(a_{ij},\ i \geq j\ or\ j > m)$ are zero. Similarly, we have lower triangular(strictly lower triangular) matrix.

PROPOSITION 2.4. *Let V be the set of all $m \times n$ matrices, Γ be the set of all $n \times m$ matrices and W be the set of all diagonal matrices over a field F respectively. Then, for every $\alpha = (a_{ij})_{n\times m} \in \Gamma$ where $a_{ij} = 0$ for $i > j$ or $j > i$ when $i, j \leq min\{m, n\}$, W is an α-Lie subalgebra of V where for every $A, B \in W$, $[A\ B]_\alpha = 0$. So $L_\alpha(W)$ is abelian and the α-dimension of W is zero.*

Proof. Direct calculation. □

EXAMPLE 2.5. In Proposition 2.4, let $m = 5$, $n = 3$,

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} d & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in W \text{ and}$$

$$\alpha = \begin{pmatrix} \beta & 0 & 0 & \eta & \theta \\ 0 & \gamma & 0 & \lambda & \mu \\ 0 & 0 & \delta & \phi & \psi \end{pmatrix} \in \Gamma.$$

Then, we can see that $[A\ B]_\alpha = 0$.

PROPOSITION 2.6. *Let V and Γ be the sets of $m \times n$ and $n \times m$ matrices over a field F respectively. If W be the subset of V such that the rows $i_1, ...i_k (i \in \{1, ..., m\})$(columns $j_1, ..., j_{k'} (k' \in \{1, ..., n\})$)are zero then, for every $\alpha \in \Gamma$, $L_\alpha(W)$ is an α-Lie subalgebra of V.*

Proof. Suppose that $A, B \in W$ and the rows $i_s (i \in \{1, ..., m\})$ of A, B are zero and $\alpha \in \Gamma$. Then the rows i_s in matrices $A\alpha$, $B\alpha$, $A\alpha B$, $B\alpha A$, $[A\ B]_\alpha = A\alpha B - B\alpha A$ are zero. □

LEMMA 2.7. *Let V and Γ be the sets of $m \times n$ and $n \times m$ matrices over a field F respectively. If W is the subset of V such that the rows $i_1, ...i_k (i \in \{1, ..., m\})$(columns $j_1, ..., j_{k'} (k' \in \{1, ..., n\})$) are zero, then W is a right (left) ideal of V but not necessarily an ideal or for $\alpha \in \Gamma$ an α-ideal of V.*

Proof. Suppose that $A \in W$ and the row $i_s (i \in \{1, ..., m\})$ of A is zero, $B \in V$ and $\alpha \in \Gamma$. Then, the row i_s in matrix $A\alpha B$ is zero. □

EXAMPLE 2.8. Let V and Γ be the sets of all real 5×3 and 3×5 matrices respectively. If we name the set of all matrices in V of the form

$$\begin{pmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ d & e & f \\ 0 & 0 & 0 \end{pmatrix}$$

W, then W is a right ideal of V. Now, let

$$A = \begin{pmatrix} 2 & 3 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 7 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix} \in W,\ B = \begin{pmatrix} 1 & -2 & 4 \\ 5 & 6 & -1 \\ -3 & 7 & 9 \\ -4 & -1 & 2 \\ 6 & 5 & 3 \end{pmatrix} \in V,$$

$$\alpha = \begin{pmatrix} 1 & 8 & 2 & 5 & -4 \\ 3 & -2 & -5 & 6 & -1 \\ 4 & -5 & 3 & 2 & -9 \end{pmatrix} \in \Gamma.$$

Then, we have

$$A\alpha B = \begin{pmatrix} 376 & 198 & -167 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -453 & -75 & 162 \\ 0 & 0 & 0 \end{pmatrix}, \ B\alpha A = \begin{pmatrix} 29 & 34 & -51 \\ 451 & 116 & 141 \\ 423 & 207 & -90 \\ -152 & -19 & -93 \\ 528 & 165 & 99 \end{pmatrix},$$

$$[A\ B]_\alpha = \begin{pmatrix} 374 & 164 & -116 \\ -451 & -116 & -141 \\ -423 & -207 & 90 \\ -301 & -65 & 255 \\ -528 & -165 & -99 \end{pmatrix}.$$

These shows that W is not a left ideal, ideal or α-ideal of V.

PROPOSITION 2.9. *Let V and Γ be the sets of $m \times n$ and $n \times m$ matrices over a field F respectively. If W is the set of all upper triangular matrices in V then, for every $\alpha = (a_{ij}) \in \Gamma$ such that $a_{ij} = 0$ for $i > j$ when $i, j \leq min\{m, n\}$, W is an α-Lie subalgebra of V. The dimension of W is $\frac{p(p+1)}{2}$, where $p = min\{m, n\}$ and the α-dimension of W is $\frac{p(p-1)}{2}$. (We have the same proposition for lower triangular matrices).*

Proof. By a direct calculation we can see for every $A, B \in W$, $[A\ B]_\alpha$ is an strictly upper triangular matrix. □

EXAMPLE 2.10. In Proposition 2.9, let $m = 5$, $n = 3$,

$$A = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} g & h & i \\ 0 & j & k \\ 0 & 0 & l \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in W \text{ and}$$

$$\alpha = \begin{pmatrix} \beta & 0 & 0 & \eta & \theta \\ 0 & \gamma & 0 & \lambda & \mu \\ 0 & 0 & \delta & \phi & \psi \end{pmatrix} \in \Gamma.$$

Then, we can see that

$$[A\ B]_\alpha = \begin{pmatrix} 0 & a\beta h + b\gamma j - g\beta b - h\gamma d & a\beta i + b\gamma k + c\delta l - g\beta c - h\gamma e - i\delta f \\ 0 & 0 & d\gamma k + e\delta l - j\gamma e - k\delta f \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We recall from ordinary Lie algebra that: $gl(V,F)$ or $gl(V)$ is the Lie algebra of linear transformations from finite dimensional vector space V over field F to itself. A_n, B_n, C_n, D_n are some of Lie subalgebras of $gl(V,F)$. For example, suppose that $dim(V) = 2n+1$ and $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}$. Then, $B_n = \{T \in gl(V,F) \mid T^tJ + JT = 0\}$. When V be the set of $n \times n$ matrices over a field F we use $gl(n,F)$ instead of $gl(V,F)$.

Theorem 2.11. *Let V be the set of all $m \times n$ $(m > n)$ matrices, Γ be the set of all $n \times m$ matrices over a field F and W be the set of all matrices in V of the form*

$$\begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $A, B, \cdots, D$ are matrices in ordinary Lie subalgebras of $gl(k_1,F)$, $gl(k_2,F), \cdots, gl(k_l,F)$ respectively such that $k_1 + k_2 + \cdots + k_l = n$. If $\alpha \in \Gamma$ is a matrix of the form

$$\begin{pmatrix} I_{k_1\times k_1} & 0 & \cdots & 0 & a_{1n+1} & a_{1n+2} & \cdots & a_{1m} \\ 0 & I_{k_2\times k_2} & \cdots & 0 & a_{2n+1} & a_{2n+2} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & I_{k_l\times k_l} & a_{nn+1} & a_{nn+2} & \cdots & a_{nm} \end{pmatrix},$$

where $a_{ij} \in F$. Then, $L_\alpha(W)$ is an α-Lie subalgebra of V. (We have the same lemma when $m < n$.)

Proof. Direct calculation. □

EXAMPLE 2.12. In Theorem 2.11, let W be the set of all 7×5 matrices, where $A \in B_2$. Then, for two elements

$$P = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ -3 & 5 & 6 & 0 & 3 \\ -4 & -2 & 3 & -3 & 0 \\ -1 & 0 & 4 & -5 & 2 \\ -2 & -4 & 0 & -6 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 5 & 2 & 7 & 8 \\ -7 & -1 & 2 & 0 & -2 \\ -8 & -3 & 4 & 2 & 0 \\ -5 & 0 & 3 & 1 & 3 \\ -2 & -3 & 0 & -2 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

in W and

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 4 & -3 \\ 0 & 0 & 0 & 0 & 1 & 2 & -7 \end{pmatrix} \in \Gamma,$$

we have

$$[P\ Q]_\alpha = \begin{pmatrix} 0 & -8 & -45 & 88 & -14 \\ -88 & -39 & 42 & 0 & -21 \\ 14 & 4 & 5 & 21 & 0 \\ 8 & 0 & -4 & 39 & -4 \\ 45 & 4 & 0 & -42 & -5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in W.$$

PROPOSITION 2.13. *Let V be the set of all $m \times n$ $(m > n)$ matrices , Γ be the set of all $n \times m$ matrices over a field F and I be the set of all matrices in V where ij entry is zero when $i > n$. Then, I is an α-ideal for every $\alpha \in \Gamma$ such that the ij entry is zero for all $j \leqslant n$. (we have the same proposition when $m < n$.)*

Proof. Direct calculation. □

EXAMPLE 2.14. In Proposition 2.13, let $m = 5$, $n = 3$, $F = \mathbb{R}$,

$$A = \begin{pmatrix} -3 & -2 & 4 \\ 5 & -6 & 1 \\ -3 & 7 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in I, \ B = \begin{pmatrix} 2 & 3 & -5 \\ 4 & -7 & 6 \\ -5 & 4 & 5 \\ 2 & 5 & -1 \\ 4 & -6 & 2 \end{pmatrix} \in V,$$

$$\alpha = \begin{pmatrix} 0 & 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & -7 & 1 \end{pmatrix} \in \Gamma.$$

Then, we can see that

$$[A\ B]_\alpha = \begin{pmatrix} -64 & -304 & 68 \\ -118 & 153 & -53 \\ 48 & -392 & 104 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in I.$$

THEOREM 2.15. *Let V be the set of all $m \times n$ matrices , Γ be the set of all $n \times m$ matrices over a field F and Δ be the set of all matrices in Γ such that the entries $a_{i_1 j_1}, ..., a_{i_k j_k}$ are non-zero. Then for every $\alpha, \beta \in \Delta$, $L_\alpha(V)$ and $L_\beta(V)$ are isomorphic.*

Proof. By Theorem 1.11. □

EXAMPLE 2.16. In Theorem 2.15 (Theorem 1.11), let $m = 2$, $n = 3$ and
$\Delta = \{(a_{ij}) | a_{12}, a_{31} \neq 0\}$. Suppose that

$$x = \begin{pmatrix} a_{11} & a_{12} & a_{12} \\ a_{21} & a_{22} & a_{23} \end{pmatrix},\ y = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} \in V,$$

$$\alpha = \begin{pmatrix} 0 & \beta \\ 0 & 0 \\ \gamma & 0 \end{pmatrix} = \beta \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} = \beta e_{12} + \gamma e_{31} \in \Delta.$$

Then,

$$\begin{aligned} [x\ y]_\alpha &= \beta \begin{pmatrix} a_{11}b_{21} - a_{21}b_{11} & a_{11}b_{22} - a_{22}b_{11} & a_{11}b_{23} - a_{23}b_{11} \\ 0 & a_{21}b_{22} - a_{22}b_{21} & a_{21}b_{23} - a_{23}b_{21} \end{pmatrix} \\ &\quad + \gamma \begin{pmatrix} a_{13}b_{11} - a_{11}b_{13} & a_{13}b_{12} - a_{12}b_{13} & 0 \\ a_{23}b_{11} - a_{11}b_{23} & a_{23}b_{12} - a_{12}b_{23} & a_{23}b_{13} - a_{13}b_{23} \end{pmatrix} \\ &= \beta [x\ y]_{e_{12}} + \gamma [x\ y]_{e_{31}}. \end{aligned}$$

3. Nilpotency

Let V be an associative Γ-algebra over a field F, $\alpha \in \Gamma$ and $L_\alpha(V)$ be the constructed α- Lie algebra on V. Then, we name the set of all linear transformations from V to V as an α-lie algebra by $gl_{(\alpha)}(V)$. It is clear that $gl_{(\alpha)}(V)$ is a Lie subalgebra of $gl(V)$.

Definition 3.1. Let V be an associative Γ-algebra over a field F, $\alpha \in \Gamma$ and δ^α be a linear transformation from V to V. We say that δ^α is an *α-derivation* if for every $x, y \in V$:

$$\delta^\alpha([x\ y]_\alpha) = [\delta^\alpha(x)\ y]_\alpha + [x\ \delta^\alpha(y)]_\alpha.$$

We name $Der_{(\alpha)}(V)$ as the set of all α-derivations on V.

Lemma 3.2. *$Der_{(\alpha)}(V)$ is a Lie subalgebra of $gl_{(\alpha)}(V)$.*

Proof. Let $\delta_1^\alpha, \delta_2^\alpha$ be arbitrary elements of $Der_{(\alpha)}(V)$ and $c \in F$. One can see that $\delta_1^\alpha + \delta_2^\alpha$ and $c\delta_1^\alpha$ are belongs to $Der_{(\alpha)}(V)$, these means that $Der_{(\alpha)}(V)$ is a vector space. Now, it is enough to show that $[\delta_1^\alpha\ \delta_2^\alpha] \in Der_{(\alpha)}(V)$. Indeed, we have

$$\begin{aligned}
[\delta_1^\alpha\ \delta_2^\alpha][x\ y]_\alpha &= (\delta_1^\alpha\delta_2^\alpha - \delta_2^\alpha\delta_1^\alpha)[x\ y]_\alpha \\
&= (\delta_1^\alpha\delta_2^\alpha)[x\ y]_\alpha - (\delta_2^\alpha\delta_1^\alpha)[x\ y]_\alpha \\
&= \delta_1^\alpha([\delta_2^\alpha x\ y]_\alpha + [x\ \delta_2^\alpha y]_\alpha) - \delta_2^\alpha([\delta_1^\alpha x\ y]_\alpha + [x\ \delta_1^\alpha y]_\alpha) \\
&= [\delta_1^\alpha\delta_2^\alpha x\ y]_\alpha + [\delta_2^\alpha x\ \delta_1^\alpha y]_\alpha + [\delta_1^\alpha x\ \delta_2^\alpha y]_\alpha + [x\ \delta_1^\alpha\delta_2^\alpha y]_\alpha \\
&\quad -[\delta_2^\alpha\delta_1^\alpha x\ y]_\alpha - [\delta_1^\alpha x\ \delta_2^\alpha y]_\alpha - [\delta_2^\alpha x\ \delta_1^\alpha y]_\alpha - [x\ \delta_2^\alpha\delta_1^\alpha y]_\alpha \\
&= [\delta_1^\alpha\delta_2^\alpha - \delta_2^\alpha\delta_1^\alpha x\ y]_\alpha + [x\ \delta_1^\alpha\delta_2^\alpha - \delta_2^\alpha\delta_1^\alpha]_\alpha \\
&= [[\delta_1^\alpha\ \delta_2^\alpha]x\ y]_\alpha + [x\ [\delta_1^\alpha\ \delta_2^\alpha]y]_\alpha.
\end{aligned}$$

□

Definition 3.3. Let V be an associative Γ-algebra over a field F, $\alpha \in \Gamma$ and $L_\alpha(V)$ be the constructed α- lie algebra on V. For every $x \in V$, we define $ad_x^\alpha : V \longrightarrow V$as follows:

$$ad_x^\alpha(y) = [x\ y]_\alpha.$$

Lemma 3.4. *ad_x^α belongs to $Der_{(\alpha)}(V)$.*

Proof. Suppose that $y_1, y_2 \in V$ and $c \in F$. One can check that

$$ad_x^\alpha(cy_1 + y_2) = cad_x^\alpha(y_1) + ad_x^\alpha(y_2),$$

this shows that ad_x^α is a linear transformation from V to V. Now, since

$$\begin{aligned}
ad_x^\alpha[y_1\ y_2]_\alpha &= [x\ [y_1\ y_2]_\alpha]_\alpha \\
&= [[x\ y_1]_\alpha\ y_2]_\alpha + [y_1\ [x\ y_2]_\alpha]_\alpha \\
&= [ad_x^\alpha(y_1)\ y_2]_\alpha + [y_1\ ad_x^\alpha(y_2)]_\alpha,
\end{aligned}$$

we have $ad_x^\alpha \in Der_{(\alpha)}(V)$. □

Note that $Z_{(\alpha)}(V) = \{z \in V : \ [x\ z]_\alpha = 0 \text{ for all } x \in V\}$ is an α-ideal of V and $Z_{(\alpha)}(V) = \bigcap_{x\in V} ad_x^\alpha$.

Let V be an associative Γ-algebra over a field F and $\alpha \in \Gamma$. We say that $T^\alpha : V \longrightarrow gl_{(\alpha)}(V)$ is an α-representation of V if T^α be a linear transformation and $T^\alpha([x\ y]_\alpha) = [T^\alpha(x)\ T^\alpha(y)]$ for all $x, y \in V$.

Proposition 3.5. *Let V be an associative Γ-algebra over a field F and $\alpha \in \Gamma$. Then, $T^\alpha : V \longrightarrow gl_{(\alpha)}(V)$ which sends x to ad_x^α is an α-representation of V.*

Proof. One can see that T^α is a linear transformation. Now, it is enough to show that for all $x_1, x_2 \in V$, $T^\alpha([x_1\ x_2]_\alpha) = [T^\alpha(x_1)\ T^\alpha(x_2)]$. Indeed, we have

$$\begin{aligned} T^\alpha([x_1\ x_2]_\alpha)(y) &= (ad^\alpha_{[x_1\ x_2]_\alpha})(y) \\ &= [[x_1\ x_2]_\alpha\ y]_\alpha \end{aligned}$$

and

$$\begin{aligned} ([T^\alpha(x_1)\ T^\alpha(x_2)])(y) &= [ad^\alpha_{x_1}\ ad^\alpha_{x_2}](y) \\ &= (ad^\alpha_{x_1} ad^\alpha_{x_2})(y) - (ad^\alpha_{x_2} ad^\alpha_{x_1})(y) \\ &= ad^\alpha_{x_1}([x_2\ y]_\alpha) - ad^\alpha_{x_2}([x_1\ y]_\alpha) \\ &= [x_1\ [x_2\ y]_\alpha]_\alpha - [x_2\ [x_1\ y]_\alpha]_\alpha \\ &= [[x_1\ x_2]_\alpha\ y]_\alpha. \end{aligned}$$

□

Definition 3.6. Let V be an associative Γ-algebra over a field F, $\alpha \in \Gamma$, $L_\alpha(V)$ be the constructed α-Lie algebra on V and $x \in V$. Then, we say that x is α-nilpotent if there exists $n \in \mathbb{N}$ such that $x^n_{(\alpha)} = x\alpha x\alpha ... x = 0$. ($x$ appears in this multiplication n times.)

It is clear that if $x \in V$ be α-nilpotent, then $ad_x^\alpha \in gl_{(\alpha)}(V) \subseteq gl(V)$ is nilpotent.

Definition 3.7. Let V be an associative Γ-algebra over a field F, $\alpha \in \Gamma$, $L_\alpha(V)$ be the constructed α-Lie algebra on V. Define a sequence of α-ideals of V (the *descending central series*, also called the *lower central series*) by $V^0_{(\alpha)} = V$, $V^1_{(\alpha)} = [V\ V]_\alpha$, $V^2_{(\alpha)} = [V\ V^1_{(\alpha)}]_\alpha$, ... , $V^i_{(\alpha)} = [V\ V^{i-1}_{(\alpha)}]_\alpha$. V is called *α-nilpotent* if $V^n_{(\alpha)} = 0$ for some $n \in \mathbb{N}$.

For example, any abelian α-Lie algebra is α-nilpotent.

Proposition 3.8. *Let V be an associative Γ-algebra over a field F, $\alpha \in \Gamma$, $L_\alpha(V)$ be the constructed α- lie algebra on V.*

(1) *If V is α-nilpotent then, so are all α-Lie subalgebras and α-homomorphic images of V.*
(2) *If $V/Z_{(\alpha)}(V)$ is α-nilpotent then, so is V.*
(3) *If V is α-nilpotent and non-zero then, $Z_{(\alpha)}(V) \neq 0$*

Proof. The proof is straightforward. □

The condition for V to be α-nilpotent can be rephrased as follows: For some $n \in \mathbb{N}$ (depending only on V), $ad^{\alpha}_{x_1} ad^{\alpha}_{x_2} ... ad^{\alpha}_{x_n}(y) = 0$ for all $x_i, y \in V$. In particular, $(ad^{\alpha}_x)^n = 0$ for all $x \in V$. Now if V is any α-Lie algebra and $x \in V$, we call x *ad^{α}-nilpotent* if ad^{α}_x is a nilpotent endomorphism. Using this language, our conclusion can be stated: If V be α-nilpotent then, all elements of V are ad^{α}-nilpotent. It is a pleasant surprise to find that the converse is also true.

THEOREM 3.9. *(Engle) If all elements of V are ad^{α}-nilpotent, then V is α-nilpotent.*

Proof. To proof, at first we recall a theorem from ordinary Lie algebra: (see [4]). Let L be a subalgebra of $gl(V)$, V finite dimensional. If L consists of nilpotent endomorphisms and $V \neq 0$, then there exists non-zero $v \in V$ for which $L.v = 0$. In this theorem let $L = ad^{\alpha}(V) = \{ad^{\alpha}_x \mid x \in V\} \subseteq gl_{(\alpha)}(V) \subseteq gl(V)$ (we can assume that $V \neq 0$). Thus, there exists $v \neq 0$ in V such that $ad^{\alpha}(V)(v) = 0$. This means that $ad^{\alpha}_x(v) = 0$ for all $x \in V$ or $v \in Z_{(\alpha)}(V)$, i.e. $Z_{(\alpha)}(V) \neq 0$. Now, $V/Z_{(\alpha)}(V)$ evidently consists of ad^{α}-nilpotent elements and has smaller dimension than V. Using induction on $dimV$, we find that $V/Z_{(\alpha)}(V)$ is nilpotent. Part (2) of Proposition 3.8 implies that V itself is α-nilpotent. □

Now one can see that many results in ordinary Lie algebra are true here.

References

[1] S. Chakraborty and A.C. Paul, *On Jordan isomorphisms of 2-torsion free prime gamma rings*, Novi Sad J. Math. **40** (2010), 1–5.
[2] W. E. Barnes, *On the* Γ*-rings of Nobusawa*, Pasific J. Math. **18** (1966), 411–422.
[3] B. Davvaz, R. M. Santilli and T. Vougiouklis, *Algebra, Hyperalgebra and Lie-Santilli Theory*, J. Gen. Lie Theory Appl. **9** (2) (2015), 231.
[4] J.E. Humphreys, *Introduction to Lie algebras and representation theory*, Second printing, revised. Graduate Texts in Mathematics, 9. Springer-Verlag, New York-Berlin, 1978.
[5] S. Kyuno, *On prime* Γ*-rings*, Pacific J. Math. **75** (1978), 185–190.

[6] J. Luh, *On the theory of simple* Γ*-rings*, Michigan Math. J. **16** (1969), 65–75.
[7] N. Nobusawa, *On generalization of the ring theory*, Osaka J. Math. **1** (1978), 185–190.
[8] D. Özden, M.A. Öztürk and Y.B. Jun, *Permuting tri-derivations in prime and semi-prime gamma rings*, Kyungpook Math. J. **46** (2006), 153–167.
[9] A.C. Paul and S. Uddin, *On Artinian gamma rings*, Aligarh Bull. Math. **28** (2009), 15–19.
[10] R.M. Santilli, *An introduction to Lie-admissible algebras*, Nuovo Cimento Suppl. (1) **6** (1968), 1225–1249.

A. H. Rezaei
Department of Mathematics
Yazd University
Yazd, Iran
E-mail: ahrezaei89@gmail.com

Bijan Davvaz
Department of Mathematics
Yazd University
Yazd, Iran
E-mail: davvaz@yazd.ac.ir