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A CHARACTERIZATION OF ADDITIVE DERIVATIONS ON C^*-ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a unital C^*-algebra. It is shown that additive map $\delta : \mathcal{A} \to \mathcal{A}$ which satisfies

$$\delta(|x|x) = \delta(|x|)x + |x|\delta(x), \ \forall x \in \mathcal{A}_N$$

is a Jordan derivation on $\mathcal{A}$. Here, $\mathcal{A}_N$ is the set of all normal elements in $\mathcal{A}$. Furthermore, if $\mathcal{A}$ is a semiprime C^*-algebra then δ is a derivation.

1. Introduction

Derivation has been the main subject of many researches done by mathematicians in recent years (see the articles [1,6,10] for example).

Recall that a ring $\mathcal{R}$ is prime ring if for $a, b \in \mathcal{R}$, $a\mathcal{R}b = (0)$ implies that $a = 0$ or $b = 0$ and is semiprime in case $a\mathcal{R}a = (0)$ implies that $a = 0$. Let $\mathcal{A}$ be a unital associative ring with unit e. Additive mapping $\delta : \mathcal{A} \longrightarrow \mathcal{A}$ is called a derivation (resp. Jordan derivation) if $\delta(xy) = \delta(x)y + x\delta(y)$ (resp. $\delta(x^2) = \delta(x)x + x\delta(x)$) holds for all $x, y \in \mathcal{A}$. Obviously, any derivation is a Jordan derivation, but in general the converse is not true. A classical result of Herstein [9] asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation. Cusack [5] generalized Herstein's theorem to 2-torsion free semiprime

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rings (see [2] for a alternative proof). It should be mentioned that Beidar, Bresar, Chebotar and Martidale [1] fairly generalized Herstein's theorem. Bresar [3] proved the following theorem.

Theorem 1.1. *Let $\mathcal{R}$ be a 2-tortion free semiprime ring and let $\delta : \mathcal{R} \longrightarrow \mathcal{R}$ be an additive mapping satisfying the relation*

$$\delta(xyx) = \delta(x)yx + x\delta(y)x + xy\delta(x).$$

for all pairs $x, y \in \mathcal{R}$. Then δ is a derivation.

In 1996, Johnson [7] proved that if A is a C^*-algebra and M is a Banach $A-$module, then each Jordan derivation $\delta : A \longrightarrow M$ is a derivation (see [8], Theorem 2.4).

In this paper we consider these results in situation of $\mathcal{A}$ be a C^*-algebras. We consider a more general problem concerning certain biadditive maps and then to the proof of the main result. Afterwards we use this result whenever $\mathcal{A}$ be a C^*-algebras and $\delta : \mathcal{A} \longrightarrow \mathcal{A}$ is an additive map satisfying (0.1). Then δ is a derivation on $\mathcal{A}$.

Furthermore, we prove that if $[x, y] = [x, \delta(y)] = 0$ or $[x, y] = [\delta(x), y] = 0$ for any pair of normal elements x, y of $\mathcal{A}$, then $\delta(y) = \delta(\lambda e)$ for some $\lambda \in \mathcal{C}$. In fact, it is an extention on the work of Shoichiro Sakai ([10], Theorem 2.2.7), in which he showed that :

Let $\mathcal{A}$ be a C^*-algebra, δ be a linear derivation on $\mathcal{A}$. If $[\delta(x), x] = 0$ for a normal element x of $\mathcal{A}$, then $\delta(x) = 0$. Throughout this paper let $\mathcal{A}_N$ be the set of all normal elements in $\mathcal{A}$.

2. Main Results

We begin with the following lemma which will be used to prove our main results.

Lemma 2.1. *Let $\mathcal{A}$ be a C^*-algebra, X be a vector space and $f : \mathcal{A} \times \mathcal{A} \longrightarrow X$ be a biadditive map which satisfies*

$$f(|x|, x) = 0 \ for\ all\ x \in \mathcal{A}_N. \tag{2.1}$$

Then $f(x, y) = 0$ for all pairs of binormal elements $x, y \in \mathcal{A}$.

Proof. Let a and b be two commuting self-adjoint operators in $\mathcal{A}$. We have

$$|a \pm ib| = \sqrt{a^2 + b^2}.$$

By using (2.1) it follows that:

$$f(\sqrt{a^2+b^2}, a \pm ib) = f(|a \pm ib|, a \pm ib) = 0,$$

which implies that

$$f(\sqrt{a^2+b^2}, a) = 0, \; f(\sqrt{a^2+b^2}, ib) = 0. \tag{2.2}$$

In particular, let a and b be two positive elements such that $ab = ba$. Then there exists a unique positive element c such that $c^2 = a^2 + 2ab$. By (2.2) we obtain following equations

$$\begin{aligned} f(a+b, b) &= f(\sqrt{(a+b)^2}, b) = f(\sqrt{a^2+2ab+b^2}, b) \\ &= f(\sqrt{c^2+b^2}, b) = 0, \end{aligned}$$

which implies that $f(a, b) = 0$ and also, $f(a, ib) = 0$.

Now, assume x and y are two commuting self-adjoint operators in $\mathcal{A}$. We can write each of two self-adjoint elements of x and y as the combination of two positive ones. Easily, can be shown that the positive and negative parts of x and y commute with the other one. Consequently:

$$f(x, y) = 0, \; f(x, iy) = 0. \tag{2.3}$$

Finally, we assume that x and y are two binormal operators in $\mathcal{A}$. Since real and imaginary parts x and y commute with each other's we conclude that $f(x, y) = 0$. The proof of the lemma is now completed. □

We use Lemma 2.1 to study additive maps which the image of the binormal pairs elements commutes (see [4]).

Corollary 2.2. *Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{B}$ be an algebra and $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive map which satisfies*

$$\phi(|x|)\phi(x) = \phi(x)\phi(|x|), \; for \; all \; x \in \mathcal{A}_N. \tag{2.4}$$

Then $\phi(x)\phi(y) = \phi(y)\phi(x)$ for all binormal elements $x, y \in \mathcal{A}$.

Proof. By defining $f(x, y) = \phi(x)\phi(y) - \phi(y)\phi(x)$ for all $x, y \in \mathcal{A}$ we can obtain the statement from Lemma 2.1. □

We now proceed to show that we can not conclude from Lemma 2.1 which $f(x, y) = 0$ for every $x, y \in \mathcal{A}$ which commute with each other.

Example 2.3. Let $\mathcal{A}$ be a C^*-algebra and map $f : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ with $f(x, y) = xy^* - y^*x$ be a biadditive map satisfying (2.1). Let x not to be a normal operator. ix and x commute with each other, but $f(x, ix) \neq 0$, because $f(x, ix) = 0$ implies that x is a normal operator. This contradiction shows the correctness of the assertion.

As an application of Lemma 1.1, we give the following theorem for characterization of derivation on C^*-algebras.

THEOREM 2.4. *Let $\mathcal{A}$ be a unital C^*-algebra with unit e. If $\delta : \mathcal{A} \longrightarrow \mathcal{A}$ is an additive map satisfying*

$$\delta(|x|x) = \delta(|x|)x + |x|\delta(x),\ \forall x \in \mathcal{A}_N,$$

then δ is a Jordan derivation on $\mathcal{A}$. Furthermore, if $\mathcal{A}$ is a semiprime C^*-algebra then δ is a derivation.

Proof. The proof is divided into several steps.

Step 1. $\delta(xy) = \delta(x)y + x\delta(y)$ for all binormal elements x, y.

Since $\delta : \mathcal{A} \longrightarrow \mathcal{A}$ is an additive map $f : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ with $f(x, y) = \delta(xy) - \delta(x)y - x\delta(y)$ for all $x, y \in \mathcal{A}$ is a biadditive map. Since δ satisfies in (0.1), $f(|x|, x) = \delta(|x|x) - \delta(|x|)x - |x|\delta(x) = 0$ for all $x \in \mathcal{A}_N$. Now, if x, y are binormal elements then Lemma 2.1 follows $f(x, y) = 0$, which means, $\delta(xy) = \delta(x)y + x\delta(y)$.

Step 2. $\delta(ix) = i\delta(x)$ for all $x \in \mathcal{A}$.

Let x be an arbitrary element in $\mathcal{A}$. In view of hypothesis we easily can show that $\delta(e) = 0$ and also

$$0 = -\delta(e) = ie\delta(ie) + ie\delta(ie)$$

which implies $\delta(ie) = 0$. So

$$\delta(ix) = ie\delta(x) + \delta(ie)x = i\delta(x).$$

Step 3. $f(x, y) + f(y, x) = 0$ for all self-adjoint operators $x, y \in \mathcal{A}$.

Clearly, we can show $f(x, x) = 0$ for all $x \in \mathcal{A}_s$. Let x and y be self-adjoint operators in $\mathcal{A}$. We can conclude

$$\begin{aligned} f(x, y) + f(y, x) &= f(x, y) + f(y, x) + f(x, x) + f(y, y) \\ &= f(x + y, x + y) \\ &= 0. \end{aligned}$$

Step 4. δ is a Jordan derivation.

Let f be as in Step 1, by Step 2 $f(ix, y) = f(x, iy) = if(x, y) = -f(ix, iy)$ for all $x, y \in \mathcal{A}_s$. Thus, if x is an arbitrary element of $\mathcal{A}$ by Step 3 we have

$$\begin{aligned} f(x, x) &= f(x_1 + ix_2, x_1 + ix_2) \\ &= f(x_1, x_1) - f(x_2, x_2) + if(x_1, x_2) + if(x_2, x_1) \\ &= 0, \end{aligned}$$

and this shows δ is a Jordan derivation.

Step 5. δ is a derivation.

By Step 4 we have δ is a derivation. One can easily prove that any Jordan derivation on an arbitrary 2-tortion free ring is a Jordan triple derivation. That is,

$$\delta(xyx) = \delta(x)yx + x\delta(y)x + xy\delta(x),$$

for all pairs of $x, y \in \mathcal{A}$ and so δ is a derivation by Theorem 1.1. $\square$

THEOREM 2.5. *Let $\mathcal{A}$ be a unital C^*-algebra with unit e. If $\delta : \mathcal{A} \longrightarrow \mathcal{A}$ is an additive map satisfying*

$$\delta(|x|x) = \delta(|x|)x^* + |x|\delta(x), \ \forall x \in \mathcal{A}_N,$$

then $\delta(xy) = \delta(x)y^* + x\delta(y)$, for all pairs of binormal elements $x, y \in \mathcal{A}$.

Proof. Since $\delta : \mathcal{A} \longrightarrow \mathcal{A}$ is an additive map $f : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ with $f(x, y) = \delta(xy) - \delta(x)y^* - x\delta(y)$ for all $x, y \in \mathcal{A}$ is a biadditive map. Similar to proof of Theorem 1.1 we can show $\delta(xy) = \delta(x)y^* + x\delta(y)$, for all pairs of binormal elements $x, y \in \mathcal{A}$. $\square$

THEOREM 2.6. *Let $\mathcal{A}$ be a unital C^*-algebra and $\delta : \mathcal{A} \longrightarrow \mathcal{A}$ be an additive map satisfying (0.1). If $[x, y] = [x, \delta(y)] = 0$ for any pair of normal elements x, y of $\mathcal{A}$, then $\delta(y) = \delta(\lambda e)$ for some $\lambda \in \mathcal{C}$.*

Proof. By (Fuglede) theorem we have $\delta(y)x^* = x^*\delta(y)$ and $x^*y = yx^*$ therefore by 2.4 we have $\delta(x^*y) = \delta(x^*)y + x^*\delta(y)$ also $\delta(yx^*) = \delta(y)x^* + y\delta(x^*)$. This implies that

$$\delta(x^*)y = y\delta(x^*).$$

Again by the 2.4 $\delta(xy) = \delta(x)y + x\delta(y)$ and $\delta(yx) = \delta(y)x + y\delta(x)$. Then

$$\delta(x)y = y\delta(x).$$

Let $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{A}$ generated by $\{e, y, \delta(x), \delta(x^*)\}$, then y belongs to the center of $\mathcal{B}$. Similar to proof of ([10],Theorem 2.2.7), let $y = y_1 + iy_2$, $(y_1 = y_1^*, y_2 = y_2^*)$ and let $\mathcal{P}$ be any closed primitive ideal of $\mathcal{B}$ then there is a real number λ_1 such that $y_1 - \lambda_1 e = a_1^2 - a_2^2$, where $a_1^2, a_2^2 \in \mathcal{P} \cap \mathcal{B}^+$ then

$$\delta(y_1 - \lambda_1 e) = \delta(a_1)a_1 + a_1\delta(a_1) - \delta(a_2)a_2 - a_2\delta(a_2).$$

Clearly $\delta(y_1 - \lambda_1 e) \in \mathcal{B}$. Let φ be any state on $\mathcal{A}$ such that $\varphi(\mathcal{P}) = 0$ then

$$|\varphi(\delta(y_1 - \lambda_1 e)| \leq |\varphi(\delta(a_1)a_1)| + |\varphi(a_1\delta(a_1))| + |\varphi(\delta(a_2)a_2)| + |\varphi(a_2\delta(a_2))|$$

$$\leq \varphi(\delta(a_1)^*\delta(a_1))^{\frac{1}{2}}\varphi(a_1^2)^{\frac{1}{2}} + \varphi(\delta(a_2)^*\delta(a_2))^{\frac{1}{2}}\varphi(a_2^2)^{\frac{1}{2}}$$

$$+\varphi(\delta(a_1)\delta(a_1)^*)^{\frac{1}{2}}\varphi(a_1^2)^{\frac{1}{2}} + \varphi(\delta(a_2)\delta(a_2)^*)^{\frac{1}{2}}\varphi(a_2^2)^{\frac{1}{2}} = 0$$

hence $\delta(y_1) \in \mathcal{P}$ and so $\delta(y_1 - \lambda_1 e) \in \bigcap_{\forall\mathcal{P}} \mathcal{P} = (0)$ sine every C^*-algebra is semi-simple. Similarly, $\delta(y_2) = \delta(\lambda_2 e)$. Hence $\delta(y) = \delta(\lambda e)$. □

THEOREM 2.7. *Let $\mathcal{A}$ be a C^*-algebra and $\delta : \mathcal{A} \longrightarrow \mathcal{A}$ be an additive map satisfies (0.1). If $[x, y] = [\delta(x), y] = 0$ for any pair of normal elements x, y of $\mathcal{A}$, then $\delta(y) = \delta(\lambda e)$ for some $\lambda \in \mathcal{C}$.*

Proof. By 2.4 $\delta(xy) = \delta(x)y + x\delta(y)$ and $\delta(yx) = \delta(y)x + y\delta(x)$. Then

$$\delta(y)x = x\delta(y).$$

Thus $[x, y] = [x, \delta(y)] = 0$. By Theorem 2.6 we have $\delta(y) = \delta(\lambda e)$ for some $\lambda \in \mathcal{C}$. □

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