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On Some Spaces Isomorphic to the Space of Absolutely q-summable Double Sequences

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Abstract. Let $0 < q < \infty$. In this study, we introduce the spaces $\mathcal{BV}_q$ and $\mathcal{LS}_q$ of q-bounded variation double sequences and q-summable double series as the domain of four-dimensional backward difference matrix Δ and summation matrix S in the space $\mathcal{L}_q$ of absolutely q-summable double sequences, respectively. Also, we determine their α- and β-duals and give the characterizations of some classes of four-dimensional matrix transformations in the case $0 < q \leq 1$.

1. Introduction

We denote the set of all complex valued double sequences by Ω which forms a vector space with coordinatewise addition and scalar multiplication. Any vector subspace of Ω is called as *a double sequence space.*

By $\mathcal{M}_u$, we denote the space of all bounded double sequences, that is

$$\mathcal{M}_u := \left\{ x = (x_{kl}) \in \Omega : \|x\|_\infty = \sup_{k,l\in\mathbb{N}} |x_{kl}| < \infty \right\},$$

which is a Banach space with the norm $\|\cdot\|_\infty$; where $\mathbb{N} = \{0, 1, 2, \ldots\}$.

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If for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ and $L \in \mathbb{C}$ such that $|x_{kl} - L| < \varepsilon$ for all $k, l > N$, then we call that the double sequence $x = (x_{kl}) \in \Omega$ is *convergent to L in the Pringsheim's sense* (shortly, p-convergent to L) and write $p - \lim_{k,l\to\infty} x_{kl} = L$; where $\mathbb{C}$ denotes the complex field (see Pringsheim [16]). We denote the space of all p-convergent double sequences by $\mathcal{C}_p$.

It is well-known that in single sequence spaces a convergent single sequence is bounded. But, in double sequence spaces a p-convergent double sequence may be unbounded. A double sequence $x \in \mathcal{C}_p \cap \mathcal{M}_u$ is called *boundedly convergent to L in the Pringsheim's sense* (shortly, bp-convergent to L), where L is the p-limit of x. We denote the space of such sequences by $\mathcal{C}_{bp}$.

Throughout the text the summation without limits runs from 0 to ∞, for example $\sum_{k,l} x_{kl}$ means that $\sum_{k,l=0}^{\infty} x_{kl}$, and unless stated otherwise, we assume that ϑ denotes any of the symbols p or bp.

We denote the space of all absolutely q-summable double sequences by $\mathcal{L}_q$, that is,

$$\mathcal{L}_q := \left\{ x = (x_{kl}) \in \Omega : \sum_{k,l} |x_{kl}|^q < \infty \right\}, \quad (0 < q < \infty).$$

If we take $q = 1$, we obtain the space $\mathcal{L}_u$ of all absolutely summable double sequences.

Let $\mathbf{e^{kl}} = (\mathbf{e}_{ij}^{\mathbf{kl}})$ be a double sequence defined by

$$\mathbf{e}_{ij}^{\mathbf{kl}} := \begin{cases} 1 & , \quad (i,j) = (k,l), \\ 0 & , \quad (i,j) \neq (k,l) \end{cases}$$

for all $i, j, k, l \in \mathbb{N}$ and $\mathbf{e} = \sum_{k,l} \mathbf{e^{kl}}$ (coordinatewise sums), is a double sequence that all elements are one. All considered spaces are supposed to contain Φ, the set of all finitely non-zero double sequences; i.e.,

$$\begin{aligned} \Phi &:= \left\{ x = (x_{kl}) \in \Omega : \exists\, N \in \mathbb{N}\ \forall\ (k,l) \in \mathbb{N}^2 \backslash [0,N]^2, \quad x_{kl} = 0 \right\} \\ &:= \operatorname{span}\left\{ \mathbf{e^{kl}} : k, l \in \mathbb{N} \right\}. \end{aligned}$$

Let λ be a space of double sequences, converging with respect to some linear convergence rule $\vartheta - \lim : \lambda \to \mathbb{C}$. The sum of a double series $\sum_{i,j} x_{ij}$ with respect to this rule is defined by $\vartheta - \sum_{i,j} x_{ij} = \vartheta - \lim_{m,n\to\infty} \sum_{i,j=0}^{m,n} x_{ij}$. Then, the *$\alpha$-dual* λ^α and the *$\beta(\vartheta)$-dual* $\lambda^{\beta(\vartheta)}$ of a double sequence space λ are respectively defined by

$$\begin{aligned} \lambda^\alpha &:= \left\{ a = (a_{kl}) \in \Omega : \sum_{k,l} |a_{kl} x_{kl}| < \infty \text{ for all } x = (x_{kl}) \in \lambda \right\}, \\ \lambda^{\beta(\vartheta)} &:= \left\{ a = (a_{kl}) \in \Omega : \vartheta - \sum_{k,l} a_{kl} x_{kl} \ \text{ exists for all } x = (x_{kl}) \in \lambda \right\}. \end{aligned}$$

It is easy to see for any two spaces λ and μ of double sequences that $\mu^\alpha \subset \lambda^\alpha$ whenever $\lambda \subset \mu$.

Let λ and μ be two double sequence spaces, and $A = (a_{mnkl})$ be any four-dimensional complex infinite matrix. Then, we say that A defines a *matrix mapping* from λ into μ and we write $A : \lambda \to \mu$, if for every sequence $x = (x_{kl}) \in \lambda$ the A-transform $Ax = \{(Ax)_{mn}\}_{m,n\in\mathbb{N}}$ of x exists and belongs to μ; where

$$(Ax)_{mn} = \vartheta - \sum_{k,l} a_{mnkl}x_{kl} \quad \text{for each} \quad m,n \in \mathbb{N}. \tag{1.1}$$

We define the *ϑ-summability domain* $\lambda_A^{(\vartheta)}$ of A in a space λ of double sequences by

$$\lambda_A^{(\vartheta)} := \left\{ x = (x_{kl}) \in \Omega : Ax = \left(\vartheta - \sum_{k,l} a_{mnkl}x_{kl} \right)_{m,n\in\mathbb{N}} \text{exists and is in } \lambda \right\}.$$

We say with the notation (1.1) that A maps the space λ into the space μ if $\lambda \subset \mu_A^{(\vartheta)}$ and we denote the set of all four dimensional matrices transforming the space λ into the space μ by $(\lambda : \mu)$. Thus, $A = (a_{mnkl}) \in (\lambda : \mu)$ if and only if the double series on the right side of (1.1) converges in the sense of ϑ for each $m,n \in \mathbb{N}$, i.e, $A_{mn} \in \lambda^{\beta(\vartheta)}$ for all $m,n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax \in \mu$ for all $x \in \lambda$; where $A_{mn} = (a_{mnkl})_{k,l\in\mathbb{N}}$ for all $m,n \in \mathbb{N}$. In this paper, we only consider bp-summability domain.

For all $k,l,m,n \in \mathbb{N}$, we say that $A = (a_{mnkl})$ is *a triangular matrix* if $a_{mnkl} = 0$ for $k > m$ or $l > n$ or both, [1]. By following Adams [1], we also say that a triangular matrix $A = (a_{mnkl})$ is called a *triangle* if $a_{mnmn} \neq 0$ for all $m,n \in \mathbb{N}$. Referring to Cooke [13, Remark (a), p. 22], one can conclude that every triangle matrix has an unique inverse which is also a triangle.

We shall write throughout for simplicity in notation for all $k,l,m,n \in \mathbb{N}$ that

$$\begin{array}{rclrcl}
\Delta_{10}a_{kl} &=& a_{kl} - a_{k+1,l}, & \Delta_{10}^{kl}a_{mnkl} &=& a_{mnkl} - a_{mn,k+1,l},\\
\Delta_{01}a_{kl} &=& a_{kl} - a_{k,l+1}, & \Delta_{01}^{kl}a_{mnkl} &=& a_{mnkl} - a_{mnk,l+1},\\
\Delta_{11}a_{kl} &=& \Delta_{10}(\Delta_{01}a_{kl}), & \Delta_{11}^{kl}a_{mnkl} &=& \Delta_{10}^{kl}(\Delta_{01}^{kl}a_{mnkl}),\\
&=& \Delta_{01}(\Delta_{10}a_{kl}), & &=& \Delta_{01}^{kl}(\Delta_{10}^{kl}a_{mnkl}).
\end{array}$$

The four dimensional backward difference matrix $\Delta = (d_{mnkl})$ is defined by

$$d_{mnkl} := \begin{cases} (-1)^{m+n-(k+l)} & , \quad m-1 \leq k \leq m \quad \text{and} \quad n-1 \leq l \leq n,\\ 0 & , \quad \text{otherwise} \end{cases}$$

for all $k,l,m,n \in \mathbb{N}$. We suppose that the terms of the double sequences $x = (x_{kl})$ and $y = (y_{kl})$ are connected with the relation

$$y_{kl} = (\Delta x)_{kl} = \begin{cases} x_{00} & , \quad k,l = 0,\\ x_{0l} - x_{0,l-1} & , \quad k = 0 \text{ and } l \geq 1,\\ x_{k0} - x_{k-1,0} & , \quad l = 0 \text{ and } k \geq 1,\\ x_{k-1,l-1} - x_{k-1,l} \\ \quad -x_{k,l-1} + x_{kl} & , \quad k,l \geq 1 \end{cases} \tag{1.2}$$

for all $k, l \in \mathbb{N}$. Additionally, a direct calculation gives the inverse $\Delta^{-1} = S = (s_{mnkl})$ of the matrix Δ as follows:

$$s_{mnkl} := \begin{cases} 1 & , \quad 0 \leq k \leq m \quad \text{and} \quad 0 \leq l \leq n, \\ 0 & , \quad \text{otherwise} \end{cases}$$

for all $k, l, m, n \in \mathbb{N}$. Here, we can redefine the relation between the double sequences $x = (x_{kl})$ and $y = (y_{kl})$ by summation matrix S as follows:

$$x_{kl} = (Sy)_{kl} = \sum_{i,j=0}^{k,l} y_{ij} \tag{1.3}$$

for all $k, l \in \mathbb{N}$.

It is worth mentioning here that Altay and Başar [2] have defined the spaces $\mathcal{BS}$ and $\mathcal{CS}_\vartheta$ by using summation matrix S and also Demiriz and Duyar [14] recently defined the spaces $\mathcal{M}_u(\Delta)$ and $\mathcal{C}_\vartheta(\Delta)$ by using backward difference matrix Δ, as folllows:

$$\begin{aligned} \mathcal{BS} &:= \left\{ x = (x_{kl}) \in \Omega : \sup_{k,l \in \mathbb{N}} |(Sx)_{kl}| < \infty \right\}, \\ \mathcal{CS}_\vartheta &:= \left\{ x = (x_{kl}) \in \Omega : Sx = \{(Sx)_{kl}\}_{k,l \in \mathbb{N}} \in \mathcal{C}_\vartheta \right\}, \\ \mathcal{M}_u(\Delta) &:= \left\{ x = (x_{kl}) \in \Omega : \sup_{k,l \in \mathbb{N}} |(\Delta x)_{kl}| < \infty \right\}, \\ \mathcal{C}_\vartheta(\Delta) &:= \left\{ x = (x_{kl}) \in \Omega : \Delta x = \{(\Delta x)_{kl}\}_{k,l \in \mathbb{N}} \in \mathcal{C}_\vartheta \right\}. \end{aligned}$$

In this study, we introduce the spaces $\mathcal{BV}_q$ and $\mathcal{LS}_q$ of all double sequences whose Δ-transforms and S-transforms are absolutely q-summable, that is,

$$\begin{aligned} \mathcal{BV}_q &:= \left\{ x = (x_{kl}) \in \Omega : \sum_{k,l} |(\Delta x)_{kl}|^q < \infty \right\}, \\ \mathcal{LS}_q &:= \left\{ x = (x_{ij}) \in \Omega : \sum_{k,l} |(Sx)_{kl}|^q < \infty \right\}. \end{aligned}$$

One can easily observe that the sets $\mathcal{BV}_q$ and $\mathcal{LS}_q$ are the domain of the backward difference matrix Δ and summation matrix S in the space $\mathcal{L}_q$ which are q-normed spaces with

$$\widehat{\|x\|}_{\mathcal{BV}_q} = \sum_{k,l} |(\Delta x)_{kl}|^q \quad \text{and} \quad \widehat{\|x\|}_{\mathcal{LS}_q} = \sum_{k,l} |(Sx)_{kl}|^q$$

for $0 < q \leq 1$, and normed spaces with

$$\|x\|_{\mathcal{BV}_q} = \left[\sum_{k,l} |(\Delta x)_{kl}|^q\right]^{1/q} \quad \text{and} \quad \|x\|_{\mathcal{LS}_q} = \left[\sum_{k,l} |(Sx)_{kl}|^q\right]^{1/q}$$

for $1 < q < \infty$, respectively . In the special case $q = 1$, we obtain the space $\mathcal{BV} = (\mathcal{L}_u)_\Delta$, defined by Altay and Başar in [2], and the space $\mathcal{LS} = (\mathcal{L}_u)_S$.

2. New Sequence Spaces

In the present section, we examine some topological properties of the spaces $\mathcal{BV}_q$ and $\mathcal{LS}_q$, and also give important inclusion theorems related to them.

Theorem 2.1. *The spaces $\mathcal{BV}_q$ and $\mathcal{LS}_q$ are linearly isomorphic to the space $\mathcal{L}_q$, where $0 < q < \infty$.*

Proof. We will only show $\mathcal{BV}_q \cong \mathcal{L}_q$ with $0 < q < \infty$.

Let $0 < q < \infty$. With the notation of (1.2), consider the transformation T from $\mathcal{BV}_q$ to $\mathcal{L}_q$ defined by $x \mapsto Tx = \Delta x$. Then, clearly T is linear and injective. Let $y \in \mathcal{L}_q$ and define the sequence $x = Sy$ as in (1.3). Then, we have $\Delta x = \Delta(Sy) = y$ which gives $\widehat{\|x\|}_{\mathcal{BV}_q} = \widehat{\|y\|}_{\mathcal{L}_q}$ with $0 < q \leq 1$ and $\|x\|_{\mathcal{BV}_q} = \|y\|_{\mathcal{L}_q}$ with $1 < q < \infty$, i.e., $x \in \mathcal{BV}_q$. Hence, T is surjective and is norm preserving.

This completes the proof. □

Since $\mathcal{BV}_q \cong \mathcal{L}_q$ and $\mathcal{LS}_q \cong \mathcal{L}_q$, we can give following theorem without proof.

Theorem 2.2. *The sets $\mathcal{BV}_q$ and $\mathcal{LS}_q$ are linear spaces with the coordinatewise addition and scalar multiplication, and the following statements hold:*

(i) *Let $0 < q < 1$. Then, $\mathcal{BV}_q$ and $\mathcal{LS}_q$ are complete q-normed spaces with $\widehat{\|\cdot\|}_{\mathcal{BV}_q}$ and $\widehat{\|\cdot\|}_{\mathcal{LS}_q}$, respectively.*

(ii) *Let $1 \leq q < \infty$. Then, $\mathcal{BV}_q$ and $\mathcal{LS}_q$ are Banach spaces with $\|\cdot\|_{\mathcal{BV}_q}$ and $\|\cdot\|_{\mathcal{LS}_q}$, respectively.*

Now, we define the double sequences $\mathbf{b^{kl}} = \left(\mathbf{b}_{ij}^{\mathbf{kl}}\right)$ and $\mathbf{d^{kl}} = \left(\mathbf{d}_{ij}^{\mathbf{kl}}\right)$ by

$$\mathbf{b}_{ij}^{\mathbf{kl}} := \begin{cases} 1 & , \quad i \geq k \text{ and } j \geq l, \\ 0 & , \quad \text{otherwise,} \end{cases}$$
$$\mathbf{d}_{ij}^{\mathbf{kl}} := \begin{cases} 1 & , \quad (i,j) = (k,l), (k+1,l+1), \\ -1 & , \quad (i,j) = (k+1,l), (k,l+1), \\ 0 & , \quad \text{otherwise} \end{cases}$$

for all $i, j, k, l \in \mathbb{N}$. Then it is obvious that the sets $\left\{\mathbf{e}, \mathbf{e^{kl}}, \mathbf{b^{kl}}, \mathbf{d^{kl}}; k, l \in \mathbb{N}\right\} \subset \mathcal{BV}_q$ and $\left\{\mathbf{d^{kl}}; k, l \in \mathbb{N}\right\} \subset \mathcal{LS}_q$. These double sequences will be used in the rest of the study.

Definition 2.3.([18, p. 225]) A double sequence space λ is said to be *monotone* if $xu = (x_{kl}u_{kl}) \in \lambda$ (coordinatwise product) for every $x = (x_{kl}) \in \lambda$ and $u = (u_{kl}) \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}$, where $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$ denotes the set of all double sequences consisting of 0's and 1's.

If λ is monotone, then $\lambda^{\alpha} = \lambda^{\beta(\vartheta)}$, but the converse is not true in general.

Theorem 2.4. *The spaces $\mathcal{BV}_q$ and $\mathcal{LS}_q$ are not monotone, where $0 < q < \infty$.*

Proof. Let λ be a double sequence space. To show λ is not monotone, we must find a sequence $u = (u_{kl}) \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}$ such that $xu = (x_{kl}u_{kl}) \notin \lambda$ for a sequence $x = (x_{kl}) \in \lambda$.

Let us define the double sequence $u = (u_{kl})$ by

$$u_{kl} := \begin{cases} 0 & , \quad k \text{ or } l \text{ odd,} \\ 1 & , \quad \text{otherwise} \end{cases}$$

for all $k, l \in \mathbb{N}$. Then $\mathbf{e} \in \mathcal{BV}_q$, but $\mathbf{e}u = u \notin \mathcal{BV}_q$. Hence, the space $\mathcal{BV}_q$ is not monotone.

To show $\mathcal{LS}_q$ is not monotone take $u = \mathbf{e^{kl}}$. Then, $\mathbf{d^{kl}} \in \mathcal{LS}_q$, but $\mathbf{d^{kl}}\mathbf{e^{kl}} = \mathbf{e^{kl}} \notin \mathcal{LS}_q$. □

Theorem 2.5. *Let $0 < q < \infty$. Then, the inclusion $\mathcal{L}_q \subset \mathcal{BV}_q$ is strict.*

Proof. Let $x = (x_{kl}) \in \mathcal{L}_q$. Then, by neglecting negative indexed terms of x, we obtain

$$\begin{aligned} \widehat{\|x\|}_{\mathcal{BV}_q} &= \sum_{k,l} |x_{k-1,l-1} - x_{k-1,l} - x_{k,l-1} + x_{kl}|^q \\ &\leq 4 \sum_{k,l} |x_{kl}|^q = 4\widehat{\|x\|}_{\mathcal{L}_q} \end{aligned}$$

for $0 < q \leq 1$ and by using Minkowski's inequality

$$\begin{aligned} \|x\|_{\mathcal{BV}_q} &= \left(\sum_{k,l} |x_{k-1,l-1} - x_{k-1,l} - x_{k,l-1} + x_{kl}|^q \right)^{1/q} \\ &\leq 4 \left(\sum_{k,l} |x_{kl}|^q \right)^{1/q} = 4\|x\|_{\mathcal{L}_q} \end{aligned}$$

for $1 < q < \infty$, that is, $x \in \mathcal{BV}_q$ for $0 < q < \infty$. Also, by $\mathbf{e} \in \mathcal{BV}_q \backslash \mathcal{L}_q$, the inclusion $\mathcal{L}_q \subset \mathcal{BV}_q$ is strict. □

Since backward difference matrix Δ and summation matrix S are opposite working matrices we can give the following inclusion theorem without proof.

Theorem 2.6. *Let $0 < q < \infty$. Then, the inclusion $\mathcal{LS}_q \subset \mathcal{L}_q$ is strict.*

Theorem 2.7. *Let $1 < q < \infty$. Then, the sets $\mathcal{L}_q$ and $\mathcal{BV}$ do not contain each other.*

Proof. It is immediate that $\mathbf{e} \in \mathcal{BV} \setminus \mathcal{L}_q$ and $\mathbf{e^{kl}} \in \mathcal{BV} \cap \mathcal{L}_q$. Consider the sequence $x = (x_{kl})$ defined by

$$x_{kl} := \frac{(-1)^{k+l}}{(k+1)(l+1)}$$

for all $k, l \in \mathbb{N}$. Since $q > 1$, the series

$$\sum_{k,l} |x_{kl}|^q = \sum_{k,l} \frac{1}{[(k+1)(l+1)]^q}$$

is convergent, that is, $x \in \mathcal{L}_q$. Nevertheless, we get from

$$(\Delta x)_{kl} = \begin{cases} 1 & , \quad k,l = 0, \\ (-1)^l \dfrac{2l+1}{l(l+1)} & , \quad k = 0 \text{ and } l \geq 1, \\ (-1)^k \dfrac{2k+1}{k(k+1)} & , \quad l = 0 \text{ and } k \geq 1, \\ (-1)^{k+l} \dfrac{(k+1)(l+1) + (k+1)l + k(l+1) + kl}{kl(k+1)(l+1)} & , \quad k,l \geq 1 \end{cases} \tag{2.1}$$

that the series

$$\begin{aligned} \sum_{k,l} |(\Delta x)_{kl}| &= 1 + \sum_{l=1}^{\infty} \frac{2l+1}{l(l+1)} + \sum_{k=1}^{\infty} \frac{2k+1}{k(k+1)} \\ &\quad + \sum_{k,l=1}^{\infty} \frac{(k+1)(l+1) + (k+1)l + k(l+1) + kl}{kl(k+1)(l+1)} \\ &\geq 1 + \sum_{l=1}^{\infty} \frac{l+1}{l(l+1)} + \sum_{k=1}^{\infty} \frac{k+1}{k(k+1)} + \sum_{k,l=1}^{\infty} \frac{(k+1(l+1)}{kl(k+1)(l+1)} \\ &= 1 + \sum_{l=1}^{\infty} \frac{1}{l} + \sum_{k=1}^{\infty} \frac{1}{k} + \sum_{k,l=1}^{\infty} \frac{1}{kl} \end{aligned}$$

diverges which gives the fact that $x \notin \mathcal{BV}$. Therefore, $x \in \mathcal{L}_q \setminus \mathcal{BV}$. □

Theorem 2.8. *Let $1 < q < \infty$. Then, the sets $\mathcal{L}_u$ and $\mathcal{LS}_q$ do not contain each other.*

Proof. One can easily see that $\mathbf{e^{kl}} \in \mathcal{L}_u \setminus \mathcal{LS}_q$ and $\mathbf{d^{kl}} \in \mathcal{L}_u \cap \mathcal{LS}_q$. Consider the sequence $\Delta x = \{(\Delta x)_{kl}\}$ as in (2.1) for all $k, l \in \mathbb{N}$. Then, we obtain

$$\sum_{k,l} |\{S(\Delta x)\}_{kl}|^q = \sum_{k,l} |x_{kl}|^q = \sum_{k,l} \frac{1}{[(k+1)(l+1)]^q} < \infty,$$

i.e., $\Delta x \in \mathcal{LS}_q$, but $\Delta x \notin \mathcal{L}_u$ by Theorem 2.7. □

Theorem 2.9. *Let $0 < q < 1$. Then, the sets $\mathcal{L}_u$ and $\mathcal{BV}_q$ do not contain each other.*

Proof. It is clear that $\mathbf{e} \in \mathcal{BV}_q \setminus \mathcal{L}_u$ and $\mathbf{e^{kl}} \in \mathcal{BV}_q \cap \mathcal{L}_u$. Define $x = (x_{kl})$ by

$$x_{kl} := \frac{(-1)^{k+l}}{[(k+1)(l+1)]^{1/q}}$$

for all $k, l \in \mathbb{N}$. Since $1/q > 1$, the series

$$\sum_{k,l} |x_{kl}| = \sum_{k,l} \frac{1}{[(k+1)(l+1)]^{1/q}}$$

is convergent. On the other hand, we see from

(2.2)
$$(\Delta x)_{kl} = \begin{cases} 1 & , \quad k, l = 0, \\ (-1)^l \dfrac{l^{1/q} + (l+1)^{1/q}}{[l(l+1)]^{1/q}} & , \quad k = 0 \text{ and } l \geq 1, \\ (-1)^k \dfrac{k^{1/q} + (k+1)^{1/q}}{[k(k+1)]^{1/q}} & , \quad l = 0 \text{ and } k \geq 1, \\ (-1)^{k+l} \dfrac{[(k+1)(l+1)]^{1/q} + [(k+1)l]^{1/q}}{[kl(k+1)(l+1)]^{1/q}} \\ \quad + (-1)^{k+l} \dfrac{[k(l+1)]^{1/q} + (kl)^{1/q}}{[kl(k+1)(l+1)]^{1/q}} & , \quad k, l \geq 1 \end{cases}$$

that the series

$$\begin{aligned}
&\sum_{k,l} |(\Delta x)_{kl}|^q \\
&= 1 + \sum_{l=1}^{\infty} \left| \frac{l^{1/q} + (l+1)^{1/q}}{[l(l+1)]^{1/q}} \right|^q + \sum_{k=1}^{\infty} \left| \frac{k^{1/q} + (k+1)^{1/q}}{[k(k+1)]^{1/q}} \right|^q \\
&\quad + \sum_{k,l=1}^{\infty} \left| \frac{[(k+1)(l+1)]^{1/q} + [(k+1)l]^{1/q} + [k(l+1)]^{1/q} + (kl)^{1/q}}{[kl(k+1)(l+1)]^{1/q}} \right|^q \\
&\geq 1 + \sum_{l=1}^{\infty} \left| \frac{(l+1)^{1/q}}{[l(l+1)]^{1/q}} \right|^q + \sum_{k=1}^{\infty} \left| \frac{(k+1)^{1/q}}{[k(k+1)]^{1/q}} \right|^q + \sum_{k,l=1}^{\infty} \left| \frac{[(k+1)(l+1)]^{1/q}}{[kl(k+1)(l+1)]^{1/q}} \right|^q \\
&= 1 + \sum_{l=1}^{\infty} \frac{1}{l} + \sum_{k=1}^{\infty} \frac{1}{k} + \sum_{k,l=1}^{\infty} \frac{1}{kl}
\end{aligned}$$

is divergent. Hence, $x \in \mathcal{L}_u \setminus \mathcal{BV}_q$. □

Theorem 2.10. *Let $0 < q < 1$. Then, the sets $\mathcal{L}_q$ and $\mathcal{LS}$ do not contain each other.*

Proof. It is easy to see that $\mathbf{e^{kl}} \in \mathcal{L}_q \setminus \mathcal{LS}$ and $\mathbf{d^{kl}} \in \mathcal{L}_q \cap \mathcal{LS}$. If we consider the sequence x in (2.2), then it is immediate that $x \in \mathcal{LS} \setminus \mathcal{L}_q$. □

Let $0 < q < s < \infty$. It is known that the inclusions $\mathcal{L}_q \subset \mathcal{L}_s \subset \mathcal{M}_u$ strictly hold. By combining this fact with Theorem 2.1, we can give the following theorem without proof.

Theorem 2.11. *Let $0 < q < s < \infty$. Then, the inclusions $\mathcal{BV}_q \subset \mathcal{BV}_s \subset \mathcal{M}_u(\Delta)$ and $\mathcal{LS}_q \subset \mathcal{LS}_s \subset \mathcal{BS}$ strictly hold.*

Theorem 2.12. *Let λ denotes any of the spaces $\mathcal{M}_u$ or $\mathcal{C}_\vartheta$ and $1 < q < \infty$. Then, neither of the spaces $\mathcal{BV}_q$ and λ includes the other one.*

Proof. It is clear that $\mathbf{e} \in \mathcal{BV}_q \cap \lambda$. Define $x = (x_{kl})$ and $y = (y_{kl})$ by

$$x_{kl} := \sum_{i,j=0}^{k,l} \frac{1}{(i+1)(j+1)} \quad \text{and} \quad y_{kl} := \begin{cases} 1 & , \quad k = 0 \text{ and } l \text{ even}, \\ 0 & , \quad \text{otherwise} \end{cases}$$

for all $k, l \in \mathbb{N}$. Then, since

$$(\Delta x)_{kl} := \frac{1}{(k+1)(l+1)} \quad \text{and} \quad (\Delta y)_{kl} := \begin{cases} (-1)^{k+l} & , \quad k = 0, 1 \text{ and } l \in \mathbb{N}, \\ 0 & , \quad \text{otherwise} \end{cases}$$

one can conclude that $x \in \mathcal{BV}_q \setminus \lambda$ and $y \in \lambda \setminus \mathcal{BV}_q$. Hence, the spaces $\mathcal{BV}_q$ and λ are overlap but neither contains the other. □

3. Dual Spaces

In this section, we give the α- and $\beta(bp)$-duals of the spaces $\mathcal{BV}_q$ and $\mathcal{LS}_q$ in the case $0 < q \leq 1$. It is worth mentioning that although the alpha dual of a double sequence space is unique its beta dual may be more than one with respect to ϑ-convergence rule. By $\lambda^{n\zeta}$, we mean that $\{\lambda^{(n-1)\zeta}\}^{\zeta}$ for a double sequence space λ and $n \in \mathbb{N}_1$, the set of positive integers. It is well-known that $\mathcal{L}_u^{\alpha} = \mathcal{M}_u$ and $\mathcal{M}_u^{\alpha} = \mathcal{L}_u$.

Theorem 3.1. *Let $0 < q \leq 1$. Then, the followings hold for all $k \in \mathbb{N}_1$:*

$$\mathcal{BV}_q^{n\alpha} := \begin{cases} \mathcal{L}_u & , \quad n = 2k-1, \\ \mathcal{M}_u & , \quad n = 2k, \end{cases} \tag{3.1}$$

$$\mathcal{LS}_q^{n\alpha} := \begin{cases} \mathcal{M}_u & , \quad n = 2k-1, \\ \mathcal{L}_u & , \quad n = 2k. \end{cases}$$

Proof. Let $0 < q \leq 1$.
(i) $\mathcal{BV}_q^{\alpha} = \mathcal{L}_u$.
$\mathcal{L}_u \subset \mathcal{BV}_q^{\alpha}$: Let us consider $a = (a_{kl}) \in \mathcal{L}_u$ and $x = (x_{kl}) \in \mathcal{BV}_q$. Then, we have by relation (1.3) that $y \in \mathcal{L}_q \subset \mathcal{L}_u$ which gives

$$\sum_{k,l} |a_{kl}x_{kl}| \leq \sum_{k,l} |a_{kl}| \sum_{i,j=0}^{k,l} |y_{ij}| \leq \|a\|_{\mathcal{L}_u} \|y\|_{\mathcal{L}_u},$$

that is, $ax \in \mathcal{L}_u$. Hence, $a \in \mathcal{BV}_q^{\alpha}$.

$\mathcal{BV}_q^{\alpha} \subset \mathcal{L}_u$: Suppose that $a \in \mathcal{BV}_q^{\alpha} \setminus \mathcal{L}_u$. Then, we have $ax \in \mathcal{L}_u$ for $x \in \mathcal{BV}_q$ but $a \notin \mathcal{L}_u$. If we consider $\mathbf{e} \in \mathcal{BV}_q$, then we obtain $a\mathbf{e} = a \notin \mathcal{L}_u$, that is, $a \notin \mathcal{BV}_q^{\alpha}$, a contradiction. Hence, a must be in $\mathcal{L}_u$.
(ii) $\mathcal{LS}_q^{\alpha} = \mathcal{M}_u$.
$\mathcal{M}_u \subset \mathcal{LS}_q^{\alpha}$: Take $a \in \mathcal{M}_u$ and $x \in \mathcal{LS}_q \subset \mathcal{L}_u$. Then, we get by

$$\|ax\|_{\mathcal{L}_u} \leq \|a\|_{\infty} \|x\|_{\mathcal{L}_u}$$

that $a \in \mathcal{LS}_q^{\alpha}$.

$\mathcal{LS}_q^{\alpha} \subset \mathcal{M}_u$: Consider $a \in \mathcal{LS}_q^{\alpha} \setminus \mathcal{M}_u$. Since $a \notin \mathcal{M}_u$, there exist the subsequences of natural numbers $\{k(i)\}$ and $\{l(i)\}$, at least one of them is strictly increasing, such that

$$a_{k(i),l(i)} > (i+1)^{2/q}$$

for all $i \in \mathbb{N}$. If we define the sequence x by using the double sequence $\mathbf{d^{kl}}$ as

$$x = \sum_i (i+1)^{-2/q} \mathbf{d^{k(i),l(i)}},$$

then we obtain that

$$\sum_{i,j=0}^{k,l} x_{ij} := \begin{cases} (i+1)^{-2/q} & , \quad k = k(i) \text{ and } l = l(i), \\ 0 & , \quad k \neq k(i) \text{ and } l \neq l(i) \end{cases}$$

which leads us to the fact that

$$\sum_{k,l} \left| \sum_{i,j=0}^{k,l} x_{ij} \right|^q = \sum_i \frac{1}{(i+1)^2} < \infty,$$

i.e., $x \in \mathcal{LS}_q$. Nevertheless, by choosing $k(i)+1 < k(i+1)$ or $l(i)+1 < l(i+1)$ we get

$$\begin{aligned} \sum_{k,l} |a_{kl} x_{kl}| &> \sum_i |a_{k(i),l(i)} x_{k(i),l(i)}| \\ &> \sum_i (i+1)^{2/q} |x_{k(i),l(i)}| \\ &= \sum_i 1 = \infty, \end{aligned}$$

i.e., $a \notin \mathcal{LS}_q^\alpha$, a contradiction. Hence, $a \in \mathcal{M}_u$.

Now, by using the facts $\mathcal{L}_u^\alpha = \mathcal{M}_u$ and $\mathcal{M}_u^\alpha = \mathcal{L}_u$, one can easily show that (3.1) holds with mathematical induction. □

We give the following lemma which is needed in proving the $\beta(\vartheta)$-dual of the spaces $\mathcal{BV}_q$ and $\mathcal{LS}_q$.

Lemma 3.2.([17]) *Let $0 < q \leq 1$. Then, a four-dimensional matrix $A = (a_{mnkl}) \in (\mathcal{L}_q : \mathcal{C}_\vartheta)$ if and only if the following conditions hold:*

(3.2) $$\sup_{m,n,k,l \in \mathbb{N}} |a_{mnkl}| < \infty,$$

(3.3) $$\exists \alpha_{kl} \in \mathbb{C} \ni \vartheta - \lim_{m,n \to \infty} a_{mnkl} = \alpha_{kl} \text{ for each } k,l \in \mathbb{N}.$$

Now, we may give the beta-duals of the new spaces with respect to the ϑ-convergence rule using the technique in [4] and [5] for the spaces of single sequences.

Let us define the sets $\mathcal{BS}(u)$ and $\mathcal{CS}_\vartheta(u)$ via the double sequence u, as follows:

$$\begin{aligned} \mathcal{BS}(u) &:= \{a = (a_{ij}) \in \Omega : au = (a_{ij} u_{ij})_{i,j \in \mathbb{N}} \in \mathcal{BS}\}, \\ \mathcal{CS}_\vartheta(u) &:= \{a = (a_{ij}) \in \Omega : au = (a_{ij} u_{ij})_{i,j \in \mathbb{N}} \in \mathcal{CS}_\vartheta\}. \end{aligned}$$

Theorem 3.3. *Let $0 < q \leq 1$. Then, the $\beta(\vartheta)$-dual of the space $\mathcal{BV}_q$ is $\mathcal{CS}_{bp}(\mathbf{b^{kl}})$.*

Proof. We will determine the necessary and sufficient conditions in order to the sequence $t = (t_{mn})$ defined by

$$t_{mn} := \sum_{i,j=0}^{m,n} a_{ij}x_{ij}; \quad x = (x_{ij}) \in \mathcal{BV}_q$$

for all $m, n \in \mathbb{N}$ to be ϑ-convergent for a sequence $a = (a_{ij}) \in \Omega$.

Let us define the sequence $x = (x_{ij}) \in \mathcal{BV}_q$ by the relation (1.3) which gives $y = (y_{kl}) \in \mathcal{L}_q$. Then, we can write $t = (t_{mn})$ in the matrix form, as follows:

$$\begin{aligned} t_{mn} &= \sum_{i,j=0}^{m,n} x_{ij}a_{ij} = \sum_{i,j=0}^{m,n} \left(\sum_{k,l=0}^{i,j} y_{kl} \right) a_{ij} \\ &= \sum_{k,l=0}^{m,n} \left(\sum_{i,j=k,l}^{m,n} a_{ij} \right) y_{kl} \\ &= \sum_{k,l=0}^{m,n} b_{mnkl}y_{kl} = (By)_{mn}, \end{aligned}$$

where the four-dimensional matrix $B = (b_{mnkl})$ is defined by

$$(3.4) \qquad b_{mnkl} := \begin{cases} \sum_{i,j=k,l}^{m,n} a_{ij} & , \quad 0 \le k \le m \text{ and } 0 \le l \le n, \\ 0 & , \quad \text{otherwise} \end{cases}$$

for all $k, l, m, n \in \mathbb{N}$. Now, it is easy to see that $ax = (a_{ij}x_{ij}) \in \mathcal{CS}_\vartheta$ whenever $x = (x_{ij}) \in \mathcal{BV}_q$ if and only if $t = (t_{mn}) \in \mathcal{C}_\vartheta$ whenever $y = (y_{kl}) \in \mathcal{L}_q$ which leads us to the fact that $B \in (\mathcal{L}_q : \mathcal{C}_\vartheta)$. Therefore, by using the conditions (3.2) and (3.3) of Lemma 3.2, we obtain the conditions

$$(3.5) \qquad \sup_{m,n,k,l \in \mathbb{N}} |b_{mnkl}| = \sup_{m,n,k,l \in \mathbb{N}} \left| \sum_{i,j=k,l}^{m,n} a_{ij} \right| = \sup_{m,n,k,l \in \mathbb{N}} \left| \sum_{i,j=0}^{m,n} a_{ij}\mathbf{b}_{ij}^{\mathbf{kl}} \right| < \infty,$$

$$(3.6) \qquad \vartheta - \lim_{m,n\to\infty} b_{mnkl} = \vartheta - \lim_{m,n\to\infty} \sum_{i,j=0}^{m,n} a_{ij}\mathbf{b}_{ij}^{\mathbf{kl}} \quad \text{exists}$$

for all $k, l \in \mathbb{N}$. By means of (3.5) and (3.6), we can say that $a\mathbf{b}^{\mathbf{kl}} = \left(a_{ij}\mathbf{b}_{ij}^{\mathbf{kl}}\right) \in \{\mathcal{BS}, \mathcal{CS}_\vartheta\}$, in other words, $a \in \mathcal{BS}(\mathbf{b}^{\mathbf{kl}}) \cap \mathcal{CS}_\vartheta(\mathbf{b}^{\mathbf{kl}}) = \mathcal{CS}_{bp}(\mathbf{b}^{\mathbf{kl}})$.

This completes the proof. □

Theorem 3.4. *Let $0 < q \le 1$. Then, the $\beta(\vartheta)$-dual of the space $\mathcal{LS}_q$ is the set $\mathcal{M}_u$.*

Proof. We prove the theorem it by the similar way used in the proof of Theorem 3.3.

Consider $y = (y_{kl}) \in \mathcal{LS}_q$ by (1.2). Then, $x = (x_{kl}) \in \mathcal{L}_q$. Therefore, we obtain by applying Abel's generalized transformation for double sequences that

$$\begin{aligned} t_{mn} &= \sum_{k,l=0}^{m,n} a_{kl} y_{kl} \\ &= \sum_{k,l=0}^{m-1,n-1} (\Delta_{11} a_{kl}) x_{kl} + \sum_{k=0}^{m-1} (\Delta_{10} a_{kn}) x_{kn} + \sum_{l=0}^{n-1} (\Delta_{01} a_{ml}) x_{ml} + a_{mn} x_{mn} \\ &= \sum_{k,l=0}^{m,n} c_{mnkl} x_{kl} = (Cx)_{mn}, \end{aligned}$$

where the four-dimensional matrix $C = (c_{mnkl})$ is defined by

$$c_{mnkl} := \begin{cases} \Delta_{11} a_{kl} & , \quad 0 \leq k \leq m-1 \text{ and } 0 \leq l \leq n-1, \\ \Delta_{10} a_{kn} & , \quad 0 \leq k \leq m-1 \text{ and } l = n, \\ \Delta_{01} a_{ml} & , \quad 0 \leq l \leq n-1 \text{ and } k = m, \\ a_{mn} & , \quad k = m \text{ and } l = n, \\ 0 & , \quad \text{otherwise} \end{cases}$$

for all $k, l, m, n \in \mathbb{N}$. By using similar approach in Theorem 3.3, $C \in (\mathcal{L}_q : \mathcal{C}_\vartheta)$. Therefore, we get by Lemma 3.2 that

$$\vartheta - \lim_{m,n\to\infty} c_{mnkl} = \Delta_{11} a_{kl},$$

i.e., $\vartheta - \lim_{m,n\to\infty} c_{mnkl}$ always exists for each $k, l \in \mathbb{N}$. Also, from the condition

$$\sup_{m,n,k,l\in\mathbb{N}} |c_{mnkl}| < \infty$$

we have $(a_{mn}) \in \mathcal{M}_u$, $(\Delta_{01} a_{ml}) \in \mathcal{M}_u$, $(\Delta_{10} a_{kn}) \in \mathcal{M}_u$ and $(\Delta_{11} a_{kl}) \in \mathcal{M}_u$ for all $k, l, m, n \in \mathbb{N}$. It is easy to show that the condition $a = (a_{mn}) \in \mathcal{M}_u$ is sufficient for the matrix $C = (c_{mnkl})$ to be bounded for all $k, l, m, n \in \mathbb{N}$.

This completes the proof. □

4. Matrix Transformations

In the present section, we characterize the classes $(\mathcal{L}_q : \mathcal{L}_{q_1})$, $(\mathcal{BV}_q : \mathcal{L}_{q_1})$, $(\mathcal{LS}_q : \mathcal{L}_{q_1})$, $(\mathcal{BV}_q : \mathcal{C}_{bp})$ and $(\mathcal{LS}_q : \mathcal{C}_{bp})$ together with a corollary characterizing some classes of four-dimensional matrices without proof; where $0 < q \leq 1$ and $0 < q_1 < \infty$.

Theorem 4.1. *Let $0 < q \leq 1$ and $0 < q_1 < \infty$. Then, $A = (a_{mnkl}) \in (\mathcal{L}_q : \mathcal{L}_{q_1})$ if and only if the following condition holds:*

$$\sup_{k,l\in\mathbb{N}} \sum_{m,n} |a_{mnkl}|^{q_1} < \infty. \tag{4.1}$$

Proof. Let us consider $A = (a_{mnkl}) \in (\mathcal{L}_q : \mathcal{L}_{q_1})$ with $0 < q \leq 1$ and $0 < q_1 < \infty$. Then, Ax exists and belongs to $\mathcal{L}_{q_1}$ for all $x \in \mathcal{L}_q$. Since $\mathbf{e^{kl}} \in \mathcal{L}_q$, we obtain

$$\sum_{m,n} \left| \sum_{i,j} a_{mnij} e_{ij}^{kl} \right|^{q_1} = \sum_{m,n} |a_{mnkl}|^{q_1} < \infty$$

for all $k, l \in \mathbb{N}$. Hence, (4.1) is necessary.

Conversely, suppose that the condition (4.1) holds and $x = (x_{kl}) \in \mathcal{L}_q$. Since $\mathcal{L}_q \subset \mathcal{L}_u$ for $0 < q \leq 1$, x also belongs to $\mathcal{L}_u$. Thus, we have

$$\begin{aligned}
\sum_{m,n} \left| \sum_{k,l} a_{mnkl} x_{kl} \right|^{q_1} &\leq \sum_{m,n} \left(\sum_{k,l} |a_{mnkl}||x_{kl}| \right)^{q_1} \\
&\leq \sum_{m,n} \left(|a_{mnk_0l_0}| \sum_{k,l} |x_{kl}| \right)^{q_1} \\
&\leq \|x\|_{\mathcal{L}_u}^{q_1} \sum_{m,n} |a_{mnk_0l_0}|^{q_1} < \infty
\end{aligned}$$

for any fixed $k_0, l_0 \in \mathbb{N}$. Therefore, $A \in (\mathcal{L}_q : \mathcal{L}_{q_1})$.

This completes the proof. □

Theorem 4.2. *Let $0 < q \leq 1$ and $0 < q_1 < \infty$. Then, $A = (a_{mnkl}) \in (\mathcal{BV}_q : \mathcal{L}_{q_1})$ if and only if the following condition holds:*

$$\sup_{k,l \in \mathbb{N}} \sum_{m,n} \left| \sum_{i,j=k,l}^{\infty} a_{mnij} \right|^{q_1} < \infty. \tag{4.2}$$

Proof. We obtain the necessity of the condition (4.2) by choosing the double sequence $\mathbf{b^{kl}} \in \mathcal{BV}_q$.

Let us define $x = (x_{ij}) \in \mathcal{BV}_q$ by (1.3) which gives $y = (y_{kl}) \in \mathcal{L}_q$. Then, we derive by the s,t-th rectangular partial sum of the series $\sum_{i,j} a_{mnij} x_{ij}$ that

$$\begin{aligned}
(Ax)_{mn}^{[s,t]} &= \sum_{i,j=0}^{s,t} \left(\sum_{k,l=0}^{i,j} y_{kl} \right) a_{mnij} \\
&= \sum_{k,l=0}^{s,t} \left(\sum_{i,j=k,l}^{s,t} a_{mnij} \right) y_{kl}
\end{aligned}$$

for all $m, n, s, t \in \mathbb{N}$. Therefore, we see by letting $s, t \to \infty$ that

$$(Ax)_{mn} = \sum_{k,l} \left(\sum_{i,j=k,l}^{\infty} a_{mnij} \right) y_{kl}$$

for all $m, n \in \mathbb{N}$. Thus, we get the desired result by the same way used in proving Theorem 4.1. □

Theorem 4.3. *Let* $0 < q \leq 1$ *and* $0 < q_1 < \infty$. *Then,* $A = (a_{mnkl}) \in (\mathcal{LS}_q : \mathcal{L}_{q_1})$ *if and only if the following conditions hold:*

$$\sup_{k,l \in \mathbb{N}} \sum_{m,n} \left| \Delta_{11}^{kl} a_{mnkl} \right|^{q_1} < \infty, \tag{4.3}$$

$$\sup_{k,l \in \mathbb{N}} \sum_{m,n} \left| \Delta_{10}^{kl} a_{mnkl} \right|^{q_1} < \infty, \tag{4.4}$$

$$\sup_{k,l \in \mathbb{N}} \sum_{m,n} \left| \Delta_{01}^{kl} a_{mnkl} \right|^{q_1} < \infty, \tag{4.5}$$

$$\sup_{k,l \in \mathbb{N}} \sum_{m,n} \left| a_{mnkl} \right|^{q_1} < \infty. \tag{4.6}$$

Proof. Take $x = (x_{kl}) \in \mathcal{LS}_q$ by the relation $x = \Delta y$ which gives $y = (y_{kl}) \in \mathcal{L}_q$. Let us define the four-dimensional matrix $A^{st} = (a^{st}_{mnkl})$ by

$$a^{st}_{mnkl} := \begin{cases} a_{mnkl} & , \quad 0 \leq k \leq s \text{ and } 0 \leq l \leq t, \\ 0 & , \quad k > s \text{ or } l > t \end{cases}$$

for each $s, t \in \mathbb{N}$ and all $m, n, k, l \in \mathbb{N}$. By using generalized Abel transformation for double series, we obtain the equalities

$$\begin{aligned} \left(A^{st}x\right)_{mn} &= \sum_{k,l=0}^{s,t} a_{mnkl} x_{kl} \\ &= \sum_{k,l=0}^{s-1,t-1} \left(\Delta_{11}^{kl} a_{mnkl}\right) y_{kl} + \sum_{k=0}^{s-1} \left(\Delta_{10}^{kl} a_{mnkt}\right) y_{kt} \\ &\quad + \sum_{l=0}^{t-1} \left(\Delta_{01}^{kl} a_{mnsl}\right) y_{sl} + a_{mnst} y_{st} \\ &= \left(D^{st} y\right)_{mn} \end{aligned}$$

which gives that $A^{st} \in (\mathcal{LS}_q : \mathcal{L}_{q_1})$ if and only if $D^{st} \in (\mathcal{L}_q : \mathcal{L}_{q_1})$, where the four-dimensional matrix $D^{st} = (d^{st}_{mnkl})$ defined by

$$d^{st}_{mnkl} := \begin{cases} \Delta^{kl}_{11} a_{mnkl} & , \quad 0 \leq k \leq s-1 \text{ and } 0 \leq l \leq t-1, \\ \Delta^{kl}_{10} a_{mnkl} & , \quad 0 \leq k \leq s-1 \text{ and } l = t, \\ \Delta^{kl}_{01} a_{mnkl} & , \quad 0 \leq l \leq t-1 \text{ and } k = s, \\ a_{mnkl} & , \quad k = s \text{ and } l = t, \\ 0 & , \quad k > s \text{ or } l > t \end{cases}$$

for each $s, t \in \mathbb{N}$ and all $m, n, k, l \in \mathbb{N}$. Now, one can easily derive the conditions (4.3)-(4.6) for all $s, t \in \mathbb{N}$. □

By using Theorem 4.2 and Theorem 4.3, we can give the following two theorems without proof.

Theorem 4.4. *Let $0 < q \leq 1$. Then, $A = (a_{mnkl}) \in (\mathcal{BV}_q : \mathcal{C}_\vartheta)$ if and only if the following conditions hold:*

$$\sup_{m,n,k,l \in \mathbb{N}} \left| \sum_{i,j=k,l}^{\infty} a_{mnij} \right| < \infty, \tag{4.7}$$

$$\vartheta - \lim_{m,n \to \infty} \sum_{i,j=k,l}^{\infty} a_{mnij} \text{ exists for all } k, l \in \mathbb{N}. \tag{4.8}$$

Theorem 4.5. *Let $0 < q \leq 1$. Then, $A = (a_{mnkl}) \in (\mathcal{LS}_q : \mathcal{C}_\vartheta)$ if and only if the conditions (3.2) and (3.3) hold.*

Theorem 4.6. *Suppose that the elements of the four-dimensional infinite matrices $E = (e_{mnkl})$ and $F = (f_{mnkl})$ are connected with the relation*

$$e_{klij} = \sum_{m,n=0}^{k,l} f_{mnij} \tag{4.9}$$

for all $i, j, k, l, m, n \in \mathbb{N}$ and λ, μ be any given two double sequence spaces. Then, $E \in (\lambda : \mu_\Delta)$ if and only if $F \in (\lambda : \mu)$ and also $F \in (\lambda : \mu_S)$ if and only if $E \in (\lambda : \mu)$.

Proof. Let $x = (x_{ij}) \in \lambda$. By using (4.9), we derive that

$$\sum_{i,j=0}^{s,t} e_{klij} x_{ij} = \sum_{m,n=0}^{k,l} \sum_{i,j=0}^{s,t} f_{mnij} x_{ij}$$

for all $k, l, m, n, s, t \in \mathbb{N}$ and by letting $s, t \to \infty$ that

$$\sum_{i,j} e_{klij} x_{ij} = \sum_{m,n=0}^{k,l} \sum_{i,j} f_{mnij} x_{ij}$$

which lead us to the fact

$$(4.10) \qquad (Ex)_{kl} = \sum_{m,n=0}^{k,l} (Fx)_{mn}$$

for all $k, l \in \mathbb{N}$. Then, it is easy to see by (4.10) that $Ex \in \mu_\Delta$ whenever $x \in \lambda$ if and only if $Fx \in \mu$ whenever $x \in \lambda$ and similarly, $Fx \in \mu_S$ whenever $x \in \lambda$ if and only if $Ex \in \mu$ whenever $x \in \lambda$.

This completes the proof. □

As a consequence of Theorem 4.6, we can give the following corollary.

Corollary 4.7. *Let $0 < q \leq 1$, $0 < q_1 < \infty$ and the elements of the four-dimensional matrices $E = (e_{mnkl})$ and $F = (f_{mnkl})$ are connected with the relation (4.9). Then, the following statements hold:*

(i) $E = (e_{mnkl}) \in (\mathcal{L}_q : \mathcal{C}_\vartheta(\Delta))$ *if and only if the conditions (3.2) and (3.3) hold with f_{mnkl} instead of a_{mnkl}.*

(ii) $F = (e_{mnkl}) \in (\mathcal{L}_q : \mathcal{CS}_\vartheta)$ *if and only if the conditions (3.2) and (3.3) hold with e_{mnkl} instead of a_{mnkl}.*

(iii) $E = (e_{mnkl}) \in (\mathcal{L}_q : \mathcal{BV}_{q_1})$ *if and only if the condition (4.1) holds with f_{mnkl} instead of a_{mnkl}.*

(iv) $F = (e_{mnkl}) \in (\mathcal{L}_q : \mathcal{LS}_{q_1})$ *if and only if the condition (4.1) holds with e_{mnkl} instead of a_{mnkl}.*

(v) $E = (e_{mnkl}) \in (\mathcal{BV}_q : \mathcal{BV}_{q_1})$ *if and only if the condition (4.2) holds with f_{mnkl} instead of a_{mnkl}.*

(vi) $F = (e_{mnkl}) \in (\mathcal{BV}_q : \mathcal{LS}_{q_1})$ *if and only if the condition (4.2) holds with e_{mnkl} instead of a_{mnkl}.*

(vii) $E = (e_{mnkl}) \in (\mathcal{LS}_q : \mathcal{BV}_{q_1})$ *if and only if the conditions (4.3)-(4.6) hold with f_{mnkl} instead of a_{mnkl}.*

(viii) $F = (e_{mnkl}) \in (\mathcal{LS}_q : \mathcal{LS}_{q_1})$ *if and only if the conditions (4.3)-(4.6) hold with e_{mnkl} instead of a_{mnkl}.*

(ix) $E = (e_{mnkl}) \in (\mathcal{BV}_q : \mathcal{C}_\vartheta(\Delta))$ *if and only if the conditions (4.7) and (4.8) hold with f_{mnkl} instead of a_{mnkl}.*

(x) $F = (e_{mnkl}) \in (\mathcal{BV}_q : \mathcal{CS}_\vartheta)$ *if and only if the conditions (4.7) and (4.8) hold with e_{mnkl} instead of a_{mnkl}.*

(xi) $E = (e_{mnkl}) \in (\mathcal{LS}_q : \mathcal{C}_\vartheta(\Delta))$ *if and only if the conditions (3.2) and (3.3) hold with f_{mnkl} instead of a_{mnkl}.*

(xii) $F = (e_{mnkl}) \in (\mathcal{LS}_q : \mathcal{CS}_\vartheta)$ *if and only if the conditions (3.2) and (3.3) hold with e_{mnkl} instead of a_{mnkl}.*

5. Conclusion

As the domain of backward difference matrix in the space ℓ_p of absolutely p-summable sequences, the space bv_p of p-bounded variation single sequences were studied in the case $1 < p \leq \infty$ by Başar and Altay [5], and in the case $0 < p \leq 1$ by Altay and Başar [3]. Later, by introducing the space $\widehat{\ell_p}$ as the domain of double band matrix $B(r,s)$ in the space ℓ_p Kirişçi and Başar [15] generalized the space bv_p. Besides this, the space bv_p was extended to the paranormed space $bv(u,p)$ of single sequences by Başar et al. [6].

Recently, the space $\mathcal{L}_q$ of absolutely q-summable double sequences with $q > 1$ was introduced by Başar and Sever [9], and some complementary results related to the space $\mathcal{L}_q$ have been recently given by Yeşilkayagil and Başar [17]. We introduce the space $\mathcal{LS}_q$ as the domain of four dimensional summation matrix S in the space $\mathcal{L}_q$ with $0 < q < \infty$. It is natural to expect the extension of the space $\mathcal{LS}_q$ to the paranormed space $\mathcal{LS}_q(t)$ as a generalization of the space $\overline{\ell(p)}$ derived by Choudhary and Misra [12] as the domain of the two dimensional summation matrix in the paranormed space $\ell(p)$ of single sequences. Our main goal is to investigate the space $\mathcal{BV}_q$ of q-bounded variation double sequences and is to extend to the results obtained for the space bv_q. Of course, it is worth mentioning here that the domain of the backward difference matrix Δ in the paranormed space $\ell(p)$ and also the investigation of the results for double sequences corresponding to Başar et al. [6] remains open.

Additionally, one can generalize the main results of the present paper related to the space $\mathcal{BV}_q$ by using the four dimensional triangle matrix $B(r,s,t,u) = \{b_{mnkl}(r,s,t,u)\}$ instead of the four dimensional backward difference matrix Δ, where $r,s,t,u \in \mathbb{R}$ with $r,t \neq 0$ and

$$b_{mnkl}(r,s,t,u) := \begin{cases} rt & , \quad (k,l) = (m,n), \\ st & , \quad (k,l) = (m-1,n), \\ ru & , \quad (k,l) = (m,n-1), \\ su & , \quad (k,l) = (m-1,n-1), \\ 0 & , \quad \text{otherwise} \end{cases}$$

for all $m,n,k,l \in \mathbb{N}$. Furthermore, following Başar and Çapan [7, 8] and Çapan and Başar [10, 11], one can also extend the main results of this paper to the paranormed case.

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