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A General Uniqueness Theorem concerning the Stability of AQCQ Type Functional Equations

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Abstract. In this paper, we prove a general uniqueness theorem which is useful for proving the uniqueness of the relevant additive mapping, quadratic mapping, cubic mapping, quartic mapping, or the additive-quadratic-cubic-quartic mapping when we investigate the (generalized) Hyers-Ulam stability.

1. Introduction

From now on, we assume that V and W are vector spaces and a is a fixed real number larger than 1. If a mapping $f : V \to W$ is given, then we set

$$\begin{aligned}
f_1(x) &:= f_o(ax) - a^3 f_o(x),\\
f_2(x) &:= f_e(ax) - a^4 f_e(x),\\
f_3(x) &:= f_o(ax) - a f_o(x),\\
f_4(x) &:= f_e(ax) - a^2 f_e(x),\\
Af(x,y) &:= f(x+y) - f(x) - f(y),\\
Q_2 f(x,y) &:= f(x+y) + f(x-y) - 2f(x) - 2f(y),\\
Cf(x,y) &:= f(2x+y) - 3f(x+y) + 3f(y) - f(-x+y) - 6f(x),
\end{aligned}$$

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$$\begin{aligned} Q_4 f(x,y) \;:=\; & f(2x+y) - 4f(x+y) + 6f(y) - 4f(-x+y) \\ & + f(-2x+y) - 24f(x) \end{aligned}$$

for $x, y \in V$, where f_o and f_e denote the odd and even part of f, respectively.

If a mapping $f : V \to W$ satisfies the equation $Af(x,y) = 0$, $Q_2 f(x,y) = 0$, $Cf(x,y) = 0$, or $Q_4 f(x,y) = 0$ for all $x, y \in V$, then it is called an additive mapping, a quadratic mapping, a cubic mapping, or a quartic mapping, respectively. We notice that the real-valued mappings $f(x) = ax$, $g(x) = ax^2$, $h(x) = ax^3$, and $k(x) = ax^4$ are solutions to $Af(x,y) = 0$, $Q_2 f(x,y) = 0$, $Cg(x,y) = 0$, and $Q_4 h(x,y) = 0$, respectively.

We call a mapping $f : V \to W$ an additive-quadratic-cubic-quartic mapping if it is expressed as the sum of an additive mapping, a quadratic mapping, a cubic mapping, and a quartic mapping, and vice versa. An additive-quadratic-cubic-quartic type functional equation is just the functional equation, each of whose solutions is an additive-quadratic-cubic-quartic mapping. For example, the real-valued mapping $f(x) = ax^4 + bx^3 + cx^2 + dx$ defined on $\mathbb{R}$ is a solution to the additive-quadratic-cubic-quartic type functional equation.

Whenever we investigate the stability problems for additive-quadratic-cubic-quartic type functional equations or others, we encounter some uniqueness problems. To our best of knowledge, however, no author has succeeded in proving even a uniqueness theorem for these cases ([1, 2, 7, 12, 14, 15, 17]) except our papers ([8, 9, 10, 11]). The ideas of the present paper are strongly based on the previous papers [8, 9, 10, 11]. But this paper includes more general results than the previous three papers.

In this paper, a general uniqueness theorem will be proved which is useful for proving the uniqueness of the relevant additive mapping, quadratic mapping, cubic mapping, quartic mapping, or the additive-quadratic-cubic-quartic mapping when we investigate the (generalized) Hyers-Ulam stability. In Section 3, we make use of our uniqueness theorem to improve the stability theorems presented in [5, 13], where the uniqueness of exact solutions have not been proved.

2. Main Result

We assume that V is a real vector space and Y is a real normed space. The following somewhat surprising theorem states that if for any given mapping f, there exists a mapping F (near f) with some properties (which are possessed by quadratic, cubic, quartic, or possessed by additive-quadratic-cubic-quartic mappings), then F is uniquely determined.

Theorem 2.1. *Assume that $a > 1$ is a fixed real number and $\Phi : V \backslash \{0\} \to [0, \infty)$ is a function satisfying one of the following properties*

$$\lim_{n\to\infty} \frac{1}{a^n} \Phi(a^n x) = 0, \tag{2.1}$$

$$\lim_{n\to\infty} a^n \Phi\left(\frac{x}{a^n}\right) = \lim_{n\to\infty} \frac{1}{a^{2n}} \Phi(a^n x) = 0, \tag{2.2}$$

$$\lim_{n\to\infty} a^{2n}\Phi\left(\frac{x}{a^n}\right) = \lim_{n\to\infty}\frac{1}{a^{3n}}\Phi(a^n x) = 0, \tag{2.3}$$

$$\lim_{n\to\infty} a^{3n}\Phi\left(\frac{x}{a^n}\right) = \lim_{n\to\infty}\frac{1}{a^{4n}}\Phi(a^n x) = 0, \tag{2.4}$$

$$\lim_{n\to\infty} a^{4n}\Phi\left(\frac{x}{a^n}\right) = 0 \tag{2.5}$$

for all $x \in V\backslash\{0\}$. *Suppose* $f : V \to Y$ *is an arbitrary mapping. If a mapping* $F : V \to Y$ *satisfies the inequality*

$$\|f(x) - F(x)\| \le \Phi(x) \tag{2.6}$$

for all $x \in V\backslash\{0\}$ *and if* F *satisfies each of the following equalities*

$$\begin{aligned} F_1(ax) &= aF_1(x), \qquad F_2(ax) = a^2F_2(x), \\ F_3(ax) &= a^3F_3(x), \qquad F_4(ax) = a^4F_4(x) \end{aligned} \tag{2.7}$$

for all $x \in V$, *then* F *is given as*

$$F(x) = \begin{cases} \lim_{n\to\infty}\left[\frac{1}{a^4-a^2}\left(\frac{f_4(a^n x)}{a^{4n}} - \frac{f_2(a^n x)}{a^{2n}}\right) + \frac{1}{a^3-a}\left(\frac{f_3(a^n x)}{a^{3n}} - \frac{f_1(a^n x)}{a^n}\right)\right] & \text{in the case of (2.1)}, \\ \lim_{n\to\infty}\left[\frac{1}{a^4-a^2}\left(\frac{f_4(a^n x)}{a^{4n}} - \frac{f_2(a^n x)}{a^{2n}}\right) + \frac{1}{a^3-a}\left(\frac{f_3(a^n x)}{a^{3n}} - a^n f_1\left(\frac{x}{a^n}\right)\right)\right] & \text{in the case of (2.2)}, \\ \lim_{n\to\infty}\left[\frac{1}{a^4-a^2}\left(\frac{f_4(a^n x)}{a^{4n}} - a^{2n} f_2\left(\frac{x}{a^n}\right)\right) + \frac{1}{a^3-a}\left(\frac{f_3(a^n x)}{a^{3n}} - a^n f_1\left(\frac{x}{a^n}\right)\right)\right] & \text{in the case of (2.3)}, \\ \lim_{n\to\infty}\left[\frac{1}{a^4-a^2}\left(\frac{f_4(a^n x)}{a^{4n}} - a^{2n} f_2\left(\frac{x}{a^n}\right)\right) + \frac{1}{a^3-a}\left(a^{3n} f_3\left(\frac{x}{a^n}\right) - a^n f_1\left(\frac{x}{a^n}\right)\right)\right] & \text{in the case of (2.4)}, \\ \lim_{n\to\infty}\left[\frac{1}{a^4-a^2}\left(a^{4n} f_4\left(\frac{x}{a^n}\right) - a^{2n} f_2\left(\frac{x}{a^n}\right)\right) + \frac{1}{a^3-a}\left(a^{3n} f_3\left(\frac{x}{a^n}\right) - a^n f_1\left(\frac{x}{a^n}\right)\right)\right] & \text{in the case of (2.5)} \end{cases} \tag{2.8}$$

for all $x \in V\backslash\{0\}$. *In other words,* F *is the unique mapping satisfying* (2.6) *and* (2.7).

Proof. Suppose a mapping F satisfies (2.6) and (2.7) for a given $f : V \to Y$.
(i) We consider $F_1(x) = F_o(ax) - a^3 F_o(x)$. When $\Phi : V\backslash\{0\} \to [0, \infty)$ has the property (2.1), we make use of (2.7) to get

$$\begin{aligned}
&\left\| F_1(x) - \frac{1}{a^n} f_1(a^n x) \right\| \\
&= \frac{1}{a^n} \| F_1(a^n x) - f_1(a^n x) \| \\
&\le \frac{1}{a^n} \| F_o(a^{n+1} x) - f_o(a^{n+1} x) \| + \frac{a^3}{a^n} \| f_o(a^n x) - F_o(a^n x) \| \\
&\le \frac{1}{2a^n} \Big(\Phi(a^{n+1} x) + \Phi(-a^{n+1} x) + a^3 \Phi(a^n x) + a^3 \Phi(-a^n x) \Big) \\
&\to 0, \quad \text{as} \quad n \to \infty,
\end{aligned}$$

for all $x \in V\backslash\{0\}$. Thus, $F_1(x) = \lim_{n\to\infty} \frac{1}{a^n} f_o(a^n x)$ for all $x \in V\backslash\{0\}$ provided Φ has the property (2.1).

When $\Phi : V\backslash\{0\} \to [0, \infty)$ has the property (2.2), (2.3), (2.4), or (2.5), then we use (2.7) to show that

$$\begin{aligned}
&\left\| F_1(x) - a^n f_1\left(\frac{x}{a^n}\right) \right\| \\
&= a^n \left\| F_1\left(\frac{x}{a^n}\right) - f_1\left(\frac{x}{a^n}\right) \right\| \\
&\le a^n \left\| F_o\left(\frac{ax}{a^n}\right) - f_o\left(\frac{ax}{a^n}\right) \right\| + a^{n+3} \left\| f_o\left(\frac{x}{a^n}\right) - F_o\left(\frac{x}{a^n}\right) \right\| \\
&\le \frac{a^n}{2} \left(\Phi\left(\frac{ax}{a^n}\right) + \Phi\left(-\frac{ax}{a^n}\right) + a^3 \Phi\left(\frac{x}{a^n}\right) + a^3 \Phi\left(-\frac{x}{a^n}\right) \right) \\
&\to 0, \quad \text{as} \quad n \to \infty,
\end{aligned}$$

for all $x \in V\backslash\{0\}$. Hence, $F_o(x) = \lim_{n\to\infty} a^n f_1(\frac{x}{a^n})$ for all $x \in V\backslash\{0\}$ provided Φ has the property (2.2), (2.3), (2.4), or (2.5).
(ii) We consider the mapping $F_2(x) = F_e(ax) - a^4 F_e(x)$. When $\Phi : V\backslash\{0\} \to [0, \infty)$ has the property (2.1) or (2.2), then we apply (2.7) to verify

$$\begin{aligned}
&\left\| F_2(x) - \frac{1}{a^{2n}} f_2(a^n x) \right\| \\
&= \frac{1}{a^{2n}} \| F_2(a^n x) - f_2(a^n x) \| \\
&\le \frac{1}{a^{2n}} \| (F_e - f_e)(a^{n+1} x) \| + \frac{1}{a^{2n}} \| a^4 (F_e - f_e)(a^n x) \| \\
&\le \frac{1}{2a^{2n}} \Big(\Phi(a^{n+1} x) + \Phi(-a^{n+1} x) + a^4 \Phi(a^n x) + a^4 \Phi(-a^n x) \Big) \\
&\to 0, \quad \text{as} \quad n \to \infty,
\end{aligned}$$

for all $x \in V\backslash\{0\}$. Then, $F_2(x) = \lim_{n\to\infty} \frac{1}{a^{2n}} f_2(a^n x)$ is true for all $x \in V\backslash\{0\}$ provided Φ has the property (2.1) or (2.2).

When $\Phi : V\backslash\{0\} \to [0,\infty)$ satisfies (2.3), (2.4) or (2.5), we get

$$\begin{aligned} &\left\| F_2(x) - a^{2n} f_2\left(\frac{x}{a^n}\right)\right\| \\ &\quad = a^{2n}\left\| F_2\left(\frac{x}{a^n}\right) - f_2\left(\frac{x}{a^n}\right)\right\| \\ &\quad \le a^{2n}\left\|(F_e - f_e)\left(\frac{x}{a^{n-1}}\right)\right\| + a^{2n}\left\| a^4(F_e - f_e)\left(\frac{x}{a^n}\right)\right\| \\ &\quad \le \frac{a^{2n}}{2}\left(\Phi\left(\frac{x}{a^{n-1}}\right) + \Phi\left(\frac{-x}{a^{n-1}}\right) + a^4\Phi\left(\frac{x}{a^n}\right) + a^4\Phi\left(\frac{-x}{a^n}\right)\right) \\ &\quad \to 0, \quad \text{as} \quad n \to \infty, \end{aligned}$$

for all $x \in V\backslash\{0\}$. Thus, $F_2(x) = \lim_{n\to\infty} a^{2n} f_2(\frac{x}{a^n})$ for all $x \in V\backslash\{0\}$ provided Φ has the property (2.3), (2.4) or (2.5).

(iii) We now consider $F_3(x) = F_o(ax) - aF_o(x)$. When $\Phi : V\backslash\{0\} \to [0,\infty)$ has the property (2.1), (2.2) or (2.3), then we make use of (2.7) to see

$$\begin{aligned} &\left\| F_3(x) - \frac{1}{a^{3n}} f_3(a^n x)\right\| \\ &\quad = \frac{1}{a^{3n}}\| F_3(a^n x) - f_3(a^n x)\| \\ &\quad \le \frac{1}{a^{3n}}\|(F_o - f_o)(a^{n+1}x)\| + \frac{1}{a^{3n}}\| a(F_o - f_o)(a^n x)\| \\ &\quad \le \frac{1}{2a^{3n}}\Big(\Phi(a^{n+1}x) + \Phi(-a^{n+1}x) + a\Phi(a^n x) + a\Phi(-a^n x)\Big) \\ &\quad \to 0, \quad \text{as} \quad n \to \infty, \end{aligned}$$

for all $x \in V\backslash\{0\}$. Hence, $F_3(x) = \lim_{n\to\infty} \frac{1}{a^{3n}} f_3(a^n x)$ holds for all $x \in V\backslash\{0\}$ provided Φ has the property (2.1), (2.2), or (2.3).

When $\Phi : V\backslash\{0\} \to [0,\infty)$ has the property (2.4) or (2.5), we then obtain

$$\begin{aligned} &\left\| F_3(x) - a^{3n} f_3\left(\frac{x}{a^n}\right)\right\| \\ &\quad = a^{3n}\left\| F_3\left(\frac{x}{a^n}\right) - f_3\left(\frac{x}{a^n}\right)\right\| \\ &\quad \le a^{3n}\left\|(F_o - f_o)\left(\frac{x}{a^{n-1}}\right)\right\| + a^{2n}\left\| a(F_o - f_o)\left(\frac{x}{a^n}\right)\right\| \\ &\quad \le \frac{a^{3n}}{2}\left(\Phi\left(\frac{x}{a^{n-1}}\right) + \Phi\left(\frac{-x}{a^{n-1}}\right) + a\Phi\left(\frac{x}{a^n}\right) + a\Phi\left(\frac{-x}{a^n}\right)\right) \\ &\quad \to 0, \quad \text{as} \quad n \to \infty, \end{aligned}$$

for all $x \in V\backslash\{0\}$. Therefore, we have $F_3(x) = \lim_{n\to\infty} a^{3n} f_3(\frac{x}{a^n})$ for all $x \in V\backslash\{0\}$ provided Φ satisfies (2.4) or (2.5).

(iv) Finally, we consider $F_4(x) = F_e(ax) - a^2 F_e(x)$. If $\Phi : V\backslash\{0\} \to [0,\infty)$ has the property (2.1), (2.2), (2.3) or (2.4), then it follows from (2.7) that

$$\begin{aligned}
&\left\| F_4(x) - \frac{1}{a^{4n}} f_4(a^n x) \right\| \\
&= \frac{1}{a^{4n}} \| F_4(a^n x) - f_4(a^n x) \| \\
&\le \frac{1}{a^{4n}} \| (F_e - f_e)(a^{n+1} x) \| + \frac{1}{a^{4n}} \| a^2 (f_e - F_e)(a^n x) \| \\
&\le \frac{1}{2a^{4n}} \Big(\Phi(a^{n+1} x) + \Phi(-a^{n+1} x) + a^2 \Phi(a^n x) + a^2 \Phi(-a^n x) \Big) \\
&\to 0, \quad \text{as} \quad n \to \infty,
\end{aligned}$$

for all $x \in V\backslash\{0\}$ provided Φ has the property (2.1), (2.2), (2.3), or (2.4). That is, $F_4(x) = \lim_{n\to\infty} \frac{1}{a^{4n}} f_4(a^n x)$ holds for all $x \in V\backslash\{0\}$.

Now, we deal with the case when $\Phi : V\backslash\{0\} \to [0,\infty)$ satisfies (2.5). It holds that

$$\begin{aligned}
&\left\| F_4(x) - a^{4n} f_4\left(\frac{x}{a^n}\right) \right\| \\
&= a^{4n} \left\| F_4\left(\frac{x}{a^n}\right) - f_4\left(\frac{x}{a^n}\right) \right\| \\
&\le a^{4n} \left\| (F_e - f_e)\left(\frac{x}{a^{n-1}}\right) \right\| + a^{4n} \left\| a^2 (f_e - F_e)\left(\frac{x}{a^n}\right) \right\| \\
&\le \frac{a^{4n}}{2} \left(\Phi\left(\frac{x}{a^{n-1}}\right) + \Phi\left(\frac{-x}{a^{n-1}}\right) + a^2 \Phi\left(\frac{x}{a^n}\right) + a^2 \Phi\left(\frac{-x}{a^n}\right) \right) \\
&\to 0, \quad \text{as} \quad n \to \infty,
\end{aligned}$$

for all $x \in V\backslash\{0\}$. Thus, it holds that $F_4(x) = \lim_{n\to\infty} a^{4n} f_4(\frac{x}{a^n})$ for all $x \in V\backslash\{0\}$ provided Φ satisfies (2.5).

Consequently, since $F(x) = \frac{F_4(x) - F_2(x)}{a^4 - a^2} + \frac{F_3(x) - F_1(x)}{a^3 - a}$, F is expresses as one of equalities in (2.8) and F is uniquely determined in each case. □

3. Applications

Theorem 2.1 seems to be impractical for applications in general cases. Thus, it is necessary to introduce some corollaries which are easily applicable to the uniqueness problems for the generalized Hyers-Ulam stability. For the exact definition of the generalized Hyers-Ulam stability, we refer the reader to [3, 6].

Corollary 3.1. *Assume that $a > 1$ is a fixed real number and $\phi : V\backslash\{0\} \to [0,\infty)$*

satisfies either

$$\Phi(x) := \sum_{i=0}^{\infty} \frac{1}{a^i} \phi(a^i x) < \infty \quad \textit{for all } x \in V \backslash \{0\} \tag{3.1}$$

or

$$\Phi(x) := \sum_{i=0}^{\infty} a^{4i} \phi\left(\frac{x}{a^i}\right) < \infty \quad \textit{for all } x \in V \backslash \{0\}. \tag{3.2}$$

Suppose $f : V \to Y$ is an arbitrary mapping. If a mapping $F : V \to Y$ satisfies (2.6) *for all $x \in V\backslash\{0\}$ and* (2.7) *for all $x \in V$, then F is uniquely determined.*

Proof. When ϕ satisfies (3.1), it is obvious that

$$\lim_{n\to\infty} \frac{1}{a^n} \Phi(a^n x) = \lim_{n\to\infty} \sum_{i=0}^{\infty} \frac{1}{a^{n+i}} \phi(a^{n+i} x) = \lim_{n\to\infty} \sum_{i=n}^{\infty} \frac{1}{a^i} \phi(a^i x) = 0,$$

i.e., Φ has the property (2.1) for all $x \in V\backslash\{0\}$. For the case of (3.2), it is clear that

$$\lim_{n\to\infty} a^{4n} \Phi\left(\frac{x}{a^n}\right) = \lim_{n\to\infty} \sum_{i=0}^{\infty} a^{4n+4i} \phi\left(\frac{x}{a^{n+i}}\right) = \lim_{n\to\infty} \sum_{i=n}^{\infty} a^{4i} \phi\left(\frac{x}{a^i}\right) = 0,$$

i.e., Φ has the property (2.5) for all $x \in V\backslash\{0\}$. Hence, our assertion is true in view of Theorem 2.1. □

Corollary 3.2. *Assume that $a > 1$ is a fixed real number and the functions $\phi, \psi : V\backslash\{0\} \to [0,\infty)$ satisfy each of the following conditions*

$$\begin{aligned} &\sum_{i=0}^{\infty} a^i \psi\left(\frac{x}{a^i}\right) < \infty, && \sum_{i=0}^{\infty} \frac{1}{a^{2i}} \phi(a^i x) < \infty, \\ &\tilde{\Phi}(x) := \sum_{i=0}^{\infty} a^i \phi\left(\frac{x}{a^i}\right) < \infty, && \tilde{\Psi}(x) := \sum_{i=0}^{\infty} \frac{1}{a^{2i}} \psi(a^i x) < \infty \end{aligned} \tag{3.3}$$

for all $x \in V\backslash\{0\}$. Suppose $f : V \to Y$ is an arbitrary mapping. If a mapping $F : V \to Y$ satisfies the inequality

$$\|f(x) - F(x)\| \le \tilde{\Phi}(x) + \tilde{\Psi}(x) \tag{3.4}$$

for all $x \in V\backslash\{0\}$ and if F satisfies each condition of (2.7) *for all $x \in V$, then F is uniquely determined.*

Proof. We set $\Phi(x) := \tilde{\Phi}(x) + \tilde{\Psi}(x)$ and then use (3.3) to obtain

$$\frac{1}{a^{4n}} \Phi(a^{2n} x) = \sum_{i=0}^{\infty} \frac{1}{a^{4n-i}} \phi(a^{2n-i} x) + \sum_{i=0}^{\infty} \frac{1}{a^{4n+2i}} \psi(a^{2n+i} x)$$

for all $x \in V\backslash\{0\}$. We make change of the summation indices in the preceding equality with $j = i - 2n$ and $k = 2n + i$ to get

$$\begin{aligned}
&\frac{1}{a^{4n}}\Phi(a^{2n}x) \\
&\quad = \frac{1}{a^{2n}}\sum_{j=-2n}^{\infty} a^j \phi\left(\frac{x}{a^j}\right) + \sum_{k=2n}^{\infty}\frac{1}{a^{2k}}\psi(a^k x) \\
&\quad = \frac{1}{a^{2n}}\sum_{i=1}^{2n}\frac{1}{a^i}\phi(a^i x) + \frac{1}{a^{2n}}\sum_{i=0}^{\infty} a^i\phi\left(\frac{x}{a^i}\right) + \sum_{i=2n}^{\infty}\frac{1}{a^{2i}}\psi(a^i x) \\
&\quad = \frac{1}{a^n}\sum_{i=1}^{n-1}\frac{a^i}{a^n}\frac{1}{a^{2i}}\phi(a^i x) + \sum_{i=n}^{2n}\frac{a^i}{a^{2n}}\frac{1}{a^{2i}}\phi(a^i x) + \frac{1}{a^{2n}}\tilde{\Phi}(x) + \sum_{i=2n}^{\infty}\frac{1}{a^{2i}}\psi(a^i x) \\
&\quad \le \frac{1}{a^n}\sum_{i=1}^{\infty}\frac{1}{a^{2i}}\phi(a^i x) + \sum_{i=n}^{\infty}\frac{1}{a^{2i}}\phi(a^i x) + \frac{1}{a^{2n}}\tilde{\Phi}(x) + \sum_{i=2n}^{\infty}\frac{1}{a^{2i}}\psi(a^i x)
\end{aligned}$$

for any $x \in V\backslash\{0\}$. Hence, we obtain

$$\lim_{n\to\infty}\frac{1}{a^{4n}}\Phi(a^{2n}x) = 0$$

for all $x \in V\backslash\{0\}$.

On the other hand, we make use of the above equality to prove

$$\lim_{n\to\infty}\frac{1}{a^{4n+2}}\Phi(a^{2n+1}x) = \frac{1}{a^2}\lim_{n\to\infty}\frac{1}{a^{4n}}\Phi(a^{2n}ax) = 0$$

for all $x \in V\backslash\{0\}$. Using the two equalities above, we get

$$\lim_{n\to\infty}\frac{1}{a^{2n}}\Phi(a^n x) = 0$$

for all $x \in V\backslash\{0\}$.

Similarly, it holds that

$$a^{2n}\Phi\left(\frac{x}{a^{2n}}\right) = \sum_{i=0}^{\infty} a^{2n+i}\phi\left(\frac{x}{a^{2n+i}}\right) + \sum_{i=0}^{\infty}\frac{1}{a^{2i-2n}}\psi(a^{i-2n}x)$$

for all $x \in V\backslash\{0\}$. If we make change of the summation indices in the last equality with $j = i + 2n$ and $k = i - 2n$, then we get

$$\begin{aligned} & a^{2n}\Phi\left(\frac{x}{a^{2n}}\right) \\ = \quad & \sum_{j=2n}^{\infty} a^j \phi\left(\frac{x}{a^j}\right) + \frac{1}{a^{2n}} \sum_{k=-2n}^{\infty} \frac{1}{a^{2k}} \psi(a^k x) \\ = \quad & \sum_{i=2n}^{\infty} a^i \phi\left(\frac{x}{a^i}\right) + \frac{1}{a^{2n}} \sum_{i=1}^{2n} a^{2i} \psi\left(\frac{x}{a^i}\right) + \frac{1}{a^{2n}} \sum_{i=0}^{\infty} \frac{1}{a^{2i}} \psi(a^i x) \\ = \quad & \sum_{i=2n}^{\infty} a^i \phi\left(\frac{x}{a^i}\right) + \frac{1}{a^n} \sum_{i=1}^{n-1} \frac{a^i}{a^n} a^i \psi\left(\frac{x}{a^i}\right) + \sum_{i=n}^{2n} \frac{a^i}{a^{2n}} a^i \psi\left(\frac{x}{a^i}\right) + \frac{1}{a^{2n}} \tilde{\Psi}(x) \\ \leq \quad & \sum_{i=2n}^{\infty} a^i \phi\left(\frac{x}{a^i}\right) + \frac{1}{a^n} \sum_{i=1}^{\infty} a^i \psi\left(\frac{x}{a^i}\right) + \sum_{i=n}^{\infty} a^i \psi\left(\frac{x}{a^i}\right) + \frac{1}{a^{2n}} \tilde{\Psi}(x) \end{aligned}$$

for any $x \in V\backslash\{0\}$. Thus, we obtain

$$\lim_{n\to\infty} a^{2n}\Phi\left(\frac{x}{a^{2n}}\right) = 0,$$

$$\lim_{n\to\infty} a^{2n+1}\Phi\left(\frac{x}{a^{2n+1}}\right) = a \lim_{n\to\infty} a^{2n}\Phi\left(\frac{1}{a^{2n}}\frac{x}{a}\right) = 0$$

for any $x \in V\backslash\{0\}$. Hence, it holds that

$$\lim_{n\to\infty} a^n\Phi\left(\frac{x}{a^n}\right) = 0$$

for $x \in V\backslash\{0\}$.

Theorem 2.1 implies that our conclusion for this corollary is true. □

Corollary 3.3. *Assume that $a > 1$ is a fixed real number and the functions $\phi, \psi : V\backslash\{0\} \to [0, \infty)$ satisfy each of the following conditions*

$$\begin{aligned} & \sum_{i=0}^{\infty} a^{2i}\psi\left(\frac{x}{a^i}\right) < \infty, && \sum_{i=0}^{\infty} \frac{1}{a^{3i}}\phi(a^i x) < \infty, \\ & \tilde{\Phi}(x) := \sum_{i=0}^{\infty} a^{2i}\phi\left(\frac{x}{a^i}\right) < \infty, && \tilde{\Psi}(x) := \sum_{i=0}^{\infty} \frac{1}{a^{3i}}\psi(a^i x) < \infty \end{aligned} \tag{3.5}$$

for all $x \in V\backslash\{0\}$. Suppose $f : V \to Y$ is an arbitrary mapping. If a mapping $F : V \to Y$ satisfies the inequality

$$\|f(x) - F(x)\| \leq \tilde{\Phi}(x) + \tilde{\Psi}(x) \tag{3.6}$$

for all $x \in V\backslash\{0\}$ and each of the conditions in (2.7) for all $x \in V$, then F is uniquely determined.

Proof. We set $\Phi(x) := \tilde{\Phi}(x) + \tilde{\Psi}(x)$ and then use (3.5) to show

$$\frac{1}{a^{6n}}\Phi(a^{2n}x) = \sum_{i=0}^{\infty}\frac{1}{a^{6n-2i}}\phi(a^{2n-i}x) + \sum_{i=0}^{\infty}\frac{1}{a^{6n+3i}}\psi(a^{2n+i}x)$$

for all $x \in V\backslash\{0\}$. We change the summation indices in the preceding equality with $j = i - 2n$ and $k = 2n + i$ to get

$$\begin{aligned}
&\frac{1}{a^{6n}}\Phi(a^{2n}x)\\
&= \frac{1}{a^{2n}}\sum_{j=-2n}^{\infty} a^{2j}\phi\left(\frac{x}{a^j}\right) + \sum_{k=2n}^{\infty}\frac{1}{a^{3k}}\psi(a^k x)\\
&= \frac{1}{a^{2n}}\sum_{i=1}^{2n}\frac{1}{a^{2i}}\phi(a^i x) + \frac{1}{a^{2n}}\sum_{i=0}^{\infty} a^{2i}\phi\left(\frac{x}{a^i}\right) + \sum_{i=2n}^{\infty}\frac{1}{a^{3i}}\psi(a^i x)\\
&= \frac{1}{a^n}\sum_{i=1}^{n-1}\frac{1}{a^{2i+n}}\phi(a^i x) + \sum_{i=n}^{2n}\frac{1}{a^{2n+2i}}\phi(a^i x) + \frac{1}{a^{2n}}\tilde{\Phi}(x) + \sum_{i=2n}^{\infty}\frac{1}{a^{3i}}\psi(a^i x)\\
&\leq \frac{1}{a^n}\sum_{i=1}^{\infty}\frac{1}{a^{3i}}\phi(a^i x) + \sum_{i=n}^{\infty}\frac{1}{a^{3i}}\phi(a^i x) + \frac{1}{a^{2n}}\tilde{\Phi}(x) + \sum_{i=2n}^{\infty}\frac{1}{a^{3i}}\psi(a^i x)
\end{aligned}$$

for any $x \in V\backslash\{0\}$. Hence, we get

$$\lim_{n\to\infty}\frac{1}{a^{6n}}\Phi(a^{2n}x) = 0$$

for all $x \in V\backslash\{0\}$.

On the other hand, it follows from the above equality that

$$\lim_{n\to\infty}\frac{1}{a^{6n+3}}\Phi(a^{2n+1}x) = \frac{1}{a^3}\lim_{n\to\infty}\frac{1}{a^{6n}}\Phi(a^{2n}ax) = 0$$

for each $x \in V\backslash\{0\}$. By two equalities above, it holds that

$$\lim_{n\to\infty}\frac{1}{a^{3n}}\Phi(a^n x) = 0$$

for all $x \in V\backslash\{0\}$.

Similarly, we have

$$a^{4n}\Phi\left(\frac{x}{a^{2n}}\right) = \sum_{i=0}^{\infty} a^{4n+2i}\phi\left(\frac{x}{a^{2n+i}}\right) + \sum_{i=0}^{\infty}\frac{1}{a^{3i-4n}}\psi(a^{i-2n}x)$$

for all $x \in V\backslash\{0\}$. If we change the summation indices in the last equality with $j = i + 2n$ and $k = i - 2n$, then we get

$$\begin{aligned}
&a^{4n}\Phi\left(\frac{x}{a^{2n}}\right)\\
&= \sum_{j=2n}^{\infty} a^{2j}\phi\left(\frac{x}{a^j}\right) + \frac{1}{a^{2n}}\sum_{k=-2n}^{\infty}\frac{1}{a^{3k}}\psi(a^k x)\\
&= \sum_{i=2n}^{\infty} a^{2i}\phi\left(\frac{x}{a^i}\right) + \frac{1}{a^{2n}}\sum_{i=1}^{2n} a^{3i}\psi\left(\frac{x}{a^i}\right) + \frac{1}{a^{2n}}\sum_{i=0}^{\infty}\frac{1}{a^{3i}}\psi(a^i x)\\
&= \sum_{i=2n}^{\infty} a^{2i}\phi\left(\frac{x}{a^i}\right) + \frac{1}{a^n}\sum_{i=1}^{n-1}\frac{a^i}{a^n}a^{2i}\psi\left(\frac{x}{a^i}\right) + \sum_{i=n}^{2n}\frac{a^i}{a^{2n}}a^{2i}\psi\left(\frac{x}{a^i}\right) + \frac{1}{a^{2n}}\tilde{\Psi}(x)\\
&\leq \sum_{i=2n}^{\infty} a^{2i}\phi\left(\frac{x}{a^i}\right) + \frac{1}{a^n}\sum_{i=1}^{\infty} a^{2i}\psi\left(\frac{x}{a^i}\right) + \sum_{i=n}^{\infty} a^{2i}\psi\left(\frac{x}{a^i}\right) + \frac{1}{a^{2n}}\tilde{\Psi}(x)
\end{aligned}$$

for any $x \in V\backslash\{0\}$. Thus, we obtain

$$\lim_{n\to\infty} a^{4n}\Phi\left(\frac{x}{a^{2n}}\right) = 0,$$

$$\lim_{n\to\infty} a^{4n+2}\Phi\left(\frac{x}{a^{2n+1}}\right) = a^2\lim_{n\to\infty} a^{4n}\Phi\left(\frac{1}{a^{2n}}\frac{x}{a}\right) = 0$$

for every $x \in V\backslash\{0\}$. Hence, we have

$$\lim_{n\to\infty} a^{2n}\Phi\left(\frac{x}{a^n}\right) = 0$$

for each $x \in V\backslash\{0\}$. Theorem 2.1 implies that our assertion is true. $\square$

Corollary 3.4. *Assume that $a > 1$ is a fixed real number and the functions $\phi, \psi : V\backslash\{0\} \to [0,\infty)$ satisfy each of the following conditions*

$$\begin{aligned}
&\sum_{i=0}^{\infty} a^{3i}\psi\left(\frac{x}{a^i}\right) < \infty, \qquad \sum_{i=0}^{\infty}\frac{1}{a^{4i}}\phi(a^i x) < \infty,\\
&\tilde{\Phi}(x) := \sum_{i=0}^{\infty} a^{3i}\phi\left(\frac{x}{a^i}\right) < \infty, \qquad \tilde{\Psi}(x) := \sum_{i=0}^{\infty}\frac{1}{a^{4i}}\psi(a^i x) < \infty
\end{aligned} \tag{3.7}$$

for all $x \in V\backslash\{0\}$. Suppose $f : V \to Y$ is an arbitrary mapping. If a mapping $F : V \to Y$ satisfies the inequality

$$\|f(x) - F(x)\| \leq \tilde{\Phi}(x) + \tilde{\Psi}(x) \tag{3.8}$$

for all $x \in V\backslash\{0\}$ *as well as the conditions in* (2.7) *for all* $x \in V$*, then* F *is uniquely determined.*

Proof. Let us set $\Phi(x) := \tilde{\Phi}(x) + \tilde{\Psi}(x)$ and use (3.7) to get

$$\frac{1}{a^{4n}}\Phi(a^n x) = \sum_{i=0}^{\infty} \frac{1}{a^{4n-3i}}\phi(a^{n-i}x) + \sum_{i=0}^{\infty} \frac{1}{a^{4n+4i}}\psi(a^{n+i}x)$$

for all $x \in V\backslash\{0\}$. We make change of the summation indices in the preceding equality with $j = i - n$ and $k = n + i$ to get

$$\begin{aligned}
\frac{1}{a^{4n}}\Phi(a^n x) &= \frac{1}{a^{4n}} \sum_{j=-n}^{\infty} a^{3j}\phi\left(\frac{x}{a^j}\right) + \sum_{k=n}^{\infty} \frac{1}{a^{4k}}\psi(a^k x) \\
&= \frac{1}{a^{4n}} \sum_{i=1}^{n} \frac{1}{a^{3i}}\phi(a^i x) + \frac{1}{a^{4n}} \sum_{i=0}^{\infty} a^{3i}\phi\left(\frac{x}{a^i}\right) + \sum_{i=n}^{\infty} \frac{1}{a^{4i}}\psi(a^i x) \\
&= \frac{1}{a^{3n}} \sum_{i=1}^{n} \frac{a^i}{a^n}\frac{1}{a^{4i}}\phi(a^i x) + \frac{1}{a^{4n}}\tilde{\Phi}(x) + \sum_{i=n}^{\infty} \frac{1}{a^{4i}}\psi(a^i x) \\
&\le \frac{1}{a^{3n}} \sum_{i=1}^{\infty} \frac{1}{a^{4i}}\phi(a^i x) + \frac{1}{a^{4n}}\tilde{\Phi}(x) + \sum_{i=n}^{\infty} \frac{1}{a^{4i}}\psi(a^i x)
\end{aligned}$$

for any $x \in V\backslash\{0\}$. Hence, we get

$$\lim_{n\to\infty} \frac{1}{a^{4n}}\Phi(a^n x) = 0$$

for all $x \in V\backslash\{0\}$.

Similarly, we obtain

$$a^{3n}\Phi\left(\frac{x}{a^n}\right) = \sum_{i=0}^{\infty} a^{3n+3i}\phi\left(\frac{x}{a^{n+i}}\right) + \sum_{i=0}^{\infty} \frac{1}{a^{4i-3n}}\psi(a^{i-n}x)$$

for all $x \in V\backslash\{0\}$. If we change the summation indices in the last equality with $j = i + n$ and $k = i - n$, then we get

$$\begin{aligned}
a^{3n}\Phi\left(\frac{x}{a^n}\right) &= \sum_{j=n}^{\infty} a^{3j}\phi\left(\frac{x}{a^j}\right) + \frac{1}{a^{3n}} \sum_{k=-n}^{\infty} \frac{1}{a^{4k}}\psi(a^k x) \\
&= \sum_{i=n}^{\infty} a^{3i}\phi\left(\frac{x}{a^i}\right) + \frac{1}{a^{3n}} \sum_{i=1}^{n} a^{4i}\psi\left(\frac{x}{a^i}\right) + \frac{1}{a^{3n}} \sum_{i=0}^{\infty} \frac{1}{a^{4i}}\psi(a^i x) \\
&= \sum_{i=n}^{\infty} a^{3i}\phi\left(\frac{x}{a^i}\right) + \frac{1}{a^{2n}} \sum_{i=1}^{n} \frac{a^i}{a^n} a^{3i}\psi\left(\frac{x}{a^i}\right) + \frac{1}{a^{3n}}\tilde{\Psi}(x) \\
&\le \sum_{i=n}^{\infty} a^{3i}\phi\left(\frac{x}{a^i}\right) + \frac{1}{a^{2n}} \sum_{i=1}^{\infty} a^{3i}\psi\left(\frac{x}{a^i}\right) + \frac{1}{a^{3n}}\tilde{\Psi}(x)
\end{aligned}$$

for any $x \in V\backslash\{0\}$. Thus, we get

$$\lim_{n\to\infty} a^{3n}\Phi\left(\frac{x}{a^n}\right) = 0$$

for each $x \in V\backslash\{0\}$. Theorem 2.1 implies that our conclusion is true. □

The following corollary states that if, for any given mapping f, there exists an additive, a quadratic, a cubic, a quartic, or an additive-quadratic-quartic mapping F near f, then F is uniquely determined.

Corollary 3.5. *Assume that $a > 1$ is a fixed rational number and a function $\phi : V\backslash\{0\} \to [0,\infty)$ satisfies the condition (3.1) or (3.2). Suppose $f : V \to Y$ is an arbitrary mapping. If an additive, a quadratic, a cubic, a quartic, or an additive-quadratic-cubic-quartic mapping $F : V \to Y$ satisfies the inequality (2.6), then F is uniquely determined.*

The following corollaries are immediate consequences of Corollaries 3.2, 3.3, and 3.4, respectively, because each of additive, quadratic, cubic, quartic, and additive-quadratic-cubic-quartic mapping satisfies the conditions in (2.7) for any given rational number $a > 1$.

Corollary 3.6. *Assume that $a > 1$ is a fixed rational number and $\phi, \psi : V\backslash\{0\} \to [0,\infty)$ satisfy each of the conditions in (3.3). Suppose $f : V \to Y$ is an arbitrary mapping. If an additive, a quadratic, a cubic, a quartic, or an additive-quadratic-cubic-quartic mapping $F : V \to Y$ satisfies (3.4), then F is uniquely determined.*

Corollary 3.7. *Assume that $a > 1$ is a fixed rational number and $\phi, \psi : V\backslash\{0\} \to [0,\infty)$ satisfy each of the conditions in (3.5). Suppose $f : V \to Y$ is an arbitrary mapping. If an additive, a quadratic, a cubic, a quartic, or an additive-quadratic-cubic-quartic mapping $F : V \to Y$ satisfies (3.6), then F is uniquely determined.*

Corollary 3.8. *Assume that $a > 1$ is a fixed rational number and $\phi, \psi : V\backslash\{0\} \to [0,\infty)$ satisfy each of the conditions in (3.7). Suppose $f : V \to Y$ is an arbitrary mapping. If an additive, a quadratic, a cubic, a quartic, or an additive-quadratic-cubic-quartic mapping $F : V \to Y$ satisfies (3.8), then F is uniquely determined.*

If we set $\Phi(x) := \theta\|x\|^p$ for some constants $p \in \mathbf{R}\backslash\{1,2,3,4\}$ and $\theta > 0$, then

$$\Phi \text{ has the property } \begin{cases} (2.1) & \text{for } p < 1, \\ (2.2) & \text{for } 1 < p < 2, \\ (2.3) & \text{for } 2 < p < 3, \\ (2.4) & \text{for } 3 < p < 4, \\ (2.5) & \text{for } p > 4. \end{cases}$$

Hence, by Theorem 2.1, we have the following corollary concerning the Hyers-Ulam-Rassias stability. (For the exact definition of the Hyers-Ulam-Rassias stability, we refer to [4, 16, 18].)

Corollary 3.9. *Let $p \notin \{1, 2, 3, 4\}$ and $\theta > 0$ be real constants, let X and Y be real normed spaces, and let $f : X \to Y$ be an arbitrary mapping. If a mapping $F : X \to Y$ satisfies the inequality*

$$\|f(x) - F(x)\| \leq \theta \|x\|^p$$

for all $x \in X \backslash \{0\}$ as well as (2.7) for all $x \in X$, then F is uniquely determined.

Since each of additive, quadratic, cubic, quartic, or additive-quadratic-cubic-quartic mappings satisfies the conditions in (2.7), using Corollary 3.9, we can easily prove the following corollary.

Corollary 3.10. *Let $p \notin \{1, 2, 3, 4\}$ and $\theta > 0$ be real constants, let X and Y be real normed spaces, and let $f : X \to Y$ be an arbitrary mapping. If an additive, a quadratic, a cubic, a quartic, or an additive-quadratic-cubic-quartic mapping $F : X \to Y$ satisfies the inequality*

$$\|f(x) - F(x)\| \leq \theta \|x\|^p$$

for all $x \in X \backslash \{0\}$, then F is uniquely determined.

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