On the Fekete–Szegö Problem for a Certain Class of Meromorphic Functions Using $q$–Derivative Operator

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Abstract. In this paper, we obtain Fekete-Szegö inequalities for certain class of meromorphic functions $f(z)$ for which

\[-\frac{(1-\alpha)(qzD_qf(z) + \alpha zD_q[zD_qf(z)])}{(1-\alpha)f(z) + \alpha zD_qf(z)} \prec \varphi(z) (\alpha \in \mathbb{C} \setminus (0,1], \ 0 < q < 1).\]

Sharp bounds for the Fekete-Szegö functional $|a_1 - \mu a_0^2|$ are obtained.

1. Introduction

The theory of $q$–analysis has important role in many areas of mathematics and physics, for example, in the areas of ordinary fractional calculus, optimal control problems, $q$–difference, $q$–integral equations and in $q$–transform analysis (see for instance [1, 6, 8, 9]). The study of $q$–calculus has gained momentum years mainly due to the pioneer work of M. E. H. Ismail et al. [7] in recent years; it was followed by such works as those by S. Kanas and D. Raducanu [10] and S. Sivasubramanian and M. Govindaraj [19]. Let $\Sigma$ denote the class of meromorphic functions of the form:

\[(1.1) \quad f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k,\]

* Corresponding Author.
Received October 6, 2017; revised March 27, 2018; accepted March 29, 2018.
2010 Mathematics Subject Classification: 30C45, 30C50.
Key words and phrases: Analytic, meromorphic, $q$–starlike and convex functions, Fekete-Szegö problem, convolution.

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which are analytic in the open punctured unit disc
\[ U^* = \{ z : z \in \mathbb{C} \text{ and } 0 < |z| < 1 \} = \mathbb{U}\setminus\{0\}. \]

A function \( f \in \Sigma \) is meromorphic starlike of order \( \beta \), denoted by \( \Sigma^*(\beta) \), if
\[
-\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \beta \quad (0 \leq \beta < 1; \: z \in U).
\]

The class \( \Sigma^*(\beta) \) was introduced and studied by Pommerenke [16] (see also Miller [14]). Let \( \varphi(z) \) be an analytic function with positive real part on \( U \) satisfies \( \varphi(0) = 1 \) and \( \varphi'(0) > 0 \) which maps \( U \) onto a region starlike with respect to 1 and symmetric with respect to the real axis. Let \( \Sigma^*(\varphi) \) be the class of functions \( f(z) \in \Sigma \) for which
\[
-\frac{zf'(z)}{f(z)} < \varphi(z) \quad (z \in U).
\]

The class \( \Sigma^*(\varphi) \) was introduced and studied by Silverman et al. [18]. The class \( \Sigma^*(\beta) \) is the special case of \( \Sigma^*(\varphi) \) when \( \varphi(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \) \( (0 \leq \beta < 1) \). Let \( \mathcal{A} \) denote the class of functions \( f(z) \) of the form
\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k,
\]
which are analytic in the open unit disc \( \mathbb{U} \) and let \( \mathcal{S} \) be the subclass of \( \mathcal{A} \) consisting of functions which are analytic and univalent in \( U \). Ma and Minda [13] introduced and studied the class \( \mathcal{S}^*(\varphi) \) which consists of functions \( f(z) \in \mathcal{S} \) for which
\[
\frac{zf'(z)}{f(z)} < \varphi(z) \quad (z \in \mathbb{U}),
\]
and the class \( \mathcal{C}(\varphi) \) consists of functions \( f(z) \in \mathcal{S} \) for which
\[
1 + \frac{zf''(z)}{f'(z)} < \varphi(z) \quad (z \in \mathbb{U}).
\]

Following Ma and Minda [13], Shanmugam and Sivasubramanian [17] defined a more general class \( \mathcal{M}_\alpha(\varphi) \) consists of functions \( f(z) \in \mathcal{S} \) for which
\[
\frac{zf'(z) + \alpha z^2 f''(z)}{(1 - \alpha)f(z) + \alpha z f'(z)} < \varphi(z) \quad (\alpha \geq 0).
\]

Analogous to the class \( \mathcal{M}_\alpha(\varphi) \), Aouf et al. [4] defined the class \( \mathcal{T}_\alpha^*(\varphi) \) as follows: For \( \alpha \in \mathbb{C}\setminus\{0, 1\} \), let \( \mathcal{T}_\alpha^*(\varphi) \) be the subclass of \( \Sigma \) consisting of functions \( f(z) \) of the form (1.1) and satisfying the analytic criterion:
\[
\frac{zf'(z) + \alpha z^2 f''(z)}{(1 - \alpha)f(z) + \alpha z f'(z)} < \varphi(z).
\]
For a function \( f(z) \in \Sigma \) given by (1.1) and \( 0 < q < 1 \), the \(-q\)-derivative of a function \( f(z) \) is defined by (see Gasper and Rahman [6])

\[
D_q f(z) = \frac{f(qz) - f(z)}{(q - 1)z} \quad \text{if } z \in U^*.
\]

From (1.2), we deduce that \( D_q f(z) \) for a function \( f(z) \) of the form (1.1) is given by

\[
D_q f(z) = -\frac{1}{qz^2} + \sum_{k=0}^{\infty} [k]_q a_k z^{k-1} (z \neq 0),
\]

where

\[
[i]_q = \frac{1 - q^i}{1 - q}.
\]

As \( q \to 1^- \), \([k]_q \to k\), we have

\[
\lim_{q \to 1^-} D_q f(z) = f'(z).
\]

Making use of the \(-q\)-derivative \( D_q \), we introduce the subclass \( F^*_{q,\alpha}(\varphi) \) as follows:

For \( \alpha \in \mathbb{C} \setminus (0,1] \), \( 0 < q < 1 \), a function \( f(z) \in \Sigma \) is said to be in the class \( F^*_{q,\alpha}(\varphi) \), if and only if

\[
-(1 - \frac{2}{q})q z D_q f(z) + \alpha q z D_q [z D_q f(z)] (1 - \frac{2}{q}) f(z) + \alpha z D_q f(z) \prec \varphi(z) (z \in U).
\]

We note that:

(i) \( \lim_{q \to 1^-} F^*_{q,\alpha}(\varphi) = F^*_\alpha(\varphi) \) (see Aouf et al. [4]);

(ii) \( \lim_{q \to 1^-} F^*_{q,0}(\varphi) = \Sigma^*(\varphi) \) (see Silverman et al. [18] and Ali and Ravichandran [2]);

(iii) \( \lim_{q \to 1^-} F^*_{q,0} \left( \frac{1 + z}{1 - z} \right) = F^*(1) = F^* \) (see Aouf [3, with \( b = 1 \)]);

(iv) \( \lim_{q \to 1^-} F^*_{q,0} \left( \frac{1 + (1 - 2\beta)z}{1 - z} \right) = \Sigma^*(\beta) (0 \leq \beta < 1) \) (see Pommerenke [16]);

(v) \( \lim_{q \to 1^-} F^*_{q,0} \left( \frac{1 + \beta(1 - 2\gamma)z}{1 + \beta(1 - 2\gamma)z} \right) = \Sigma(\eta, \beta, \gamma) (0 \leq \eta < 1, 0 < \beta \leq 1, \frac{1}{2} \leq \gamma \leq 1) \) (see Kulkarni and Joshi [12]);

(vi) \( \lim_{q \to 1^-} F^*_{q,0} \left( \frac{1 + Az}{1 + Bz} \right) = K_1(A, B) (0 \leq B < 1, -B < A < B) \) (see Karunakaran [11]).
2. Fekete-Szegő Problem

To prove our results, we need the following lemmas.

**Lemma 1.** ([13]) If \( p(z) = 1 + c_1 z + c_2 z^2 + \cdots \) is a function with positive real part in \( \mathbb{U} \) and \( \mu \) is a complex number, then

\[
|c_2 - \mu c_1^2| \leq 2 \max\{1; |2\mu - 1|\}.
\]

The result is sharp for the functions given by

\[
p(z) = \frac{1 + z^2}{1 - z^2} \quad \text{and} \quad p(z) = \frac{1 + z}{1 - z}.
\]

**Lemma 2.** ([13]) If \( p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots \) is a function with positive real part in \( \mathbb{U} \), then

\[
|c_2 - \nu c_1^2| \leq \begin{cases} 
-4\nu + 2 & \text{if } \nu \leq 0, \\
2 & \text{if } 0 \leq \nu \leq 1, \\
4\nu - 2 & \text{if } \nu \geq 1.
\end{cases}
\]

When \( \nu < 0 \) or \( \nu > 1 \), the equality holds if and only if \( p_1(z) = \frac{1 + z}{1 - z} \) or one of its rotations. If \( 0 < \nu < 1 \), then the equality holds if and only if \( p_1(z) = \frac{1 + z^2}{1 - z^2} \) or one of its rotations. If \( \nu = 0 \), the equality holds if and only if

\[
p_1(z) = \left( \frac{1}{2} + \frac{\lambda}{2} \right) \frac{1 + z}{1 - z} + \left( \frac{1}{2} - \frac{\lambda}{2} \right) \frac{1 - z}{1 + z} \quad (0 \leq \lambda \leq 1),
\]

or one of its rotations. If \( \nu = 1 \), the equality holds if and only if

\[
\frac{1}{p_1(z)} = \left( \frac{1}{2} + \frac{\lambda}{2} \right) \frac{1 + z}{1 - z} + \left( \frac{1}{2} - \frac{\lambda}{2} \right) \frac{1 - z}{1 + z} \quad (0 \leq \lambda \leq 1),
\]

or one of its rotations. Also the above upper bound is sharp and it can be improved as follows when \( 0 < \nu < 1 \):

\[
|c_2 - \nu c_1^2| + \nu |c_1|^2 \leq 2 \left( 0 < \nu \leq \frac{1}{2} \right),
\]

and

\[
|c_2 - \nu c_1^2| + (1 - \nu) |c_1|^2 \leq 2 \left( \frac{1}{2} < \nu < 1 \right).
\]

Unless otherwise mentioned, we assume throughout this paper that \( \alpha \in \mathbb{C} \setminus (0, 1) \) and \( 0 < q < 1 \).
Theorem 1. Let \( \varphi(z) = 1 + B_1 z + B_2 z^2 + \ldots \). If \( f(z) \) given by (1.1) belongs to the class \( \mathcal{F}_{q, \alpha}^* (\varphi) \) and \( \mu \) is a complex number, then

(2.1) (i) \[ |a_1 - \mu a_0^2| \leq \frac{1}{1 + q} \left| \frac{(q - 2\alpha) B_1}{(q - \alpha + \alpha q)} \right| \times \max \left\{ 1, \left| \frac{B_2}{B_1} - \left[ 1 - \mu \left( \frac{q - 2\alpha}{q - \alpha + \alpha q} \right) \frac{(q - \alpha + \alpha q) (q + 1)}{(q - \alpha)^2} \right] B_1 \right| \right\} \quad (B_1 \neq 0), \]

and

(2.2) (ii) \[ |a_1| \leq \frac{1}{1 + q} \left| \frac{(q - 2\alpha) B_2}{(q - \alpha + \alpha q)} \right| \quad (B_1 = 0). \]

The result is sharp.

Proof. If \( f(z) \in \mathcal{F}_{q, \alpha}^* (\varphi) \), then there is a Schwarz function \( w(z) \) in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) in \( U \) and such that

(2.3) \[ - \frac{(1 - \frac{q}{2}) q z D_q f(z) + \alpha q z D_q [z D_q f(z)]}{(1 - \frac{q}{2}) f(z) + \alpha z D_q f(z)} = \varphi(w(z)). \]

Define the function \( p_1(z) \) by

(2.4) \[ p_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \ldots. \]

Since \( w(z) \) is a Schwarz function, we see that \( \Re \{ p_1(z) \} > 0 \) and \( p_1(0) = 1 \). Define

(2.5) \[ p(z) = - \frac{(1 - \frac{q}{2}) q z D_q f(z) + \alpha q z D_q [z D_q f(z)]}{(1 - \frac{q}{2}) f(z) + \alpha z D_q f(z)} = 1 + b_1 z + b_2 z^2 + \ldots. \]

In view of (2.3), (2.4) and (2.5), we have

(2.6) \[ p(z) = \varphi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right). \]

Since

\[ \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2} \left[ c_1 z + \left( c_2 - \frac{c_1^2}{2} \right) z^2 + \left( c_3 + \frac{c_1^3}{4} - c_1 c_2 \right) z^3 + \ldots \right]. \]

Therefore, we have

\[ \varphi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right) = 1 + \frac{1}{2} B_1 c_1 z + \left[ \frac{1}{2} B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2 \right] z^2 + \ldots, \]

and from this equation and (2.6), we obtain

\[ b_1 = \frac{1}{2} B_1 c_1. \]
and
\[ b_2 = \frac{1}{2} B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2. \]

Then, from (2.5) and (1.1), we see that
\[ b_1 = -\left( \frac{q - \alpha}{q - 2\alpha} \right) a_0, \]
and
\[ b_2 = \left( \frac{q - \alpha}{q - 2\alpha} \right)^2 a_0^2 - \frac{(q + 1)(q - \alpha + \alpha q)}{q - 2\alpha} a_1, \]
or, equivalently, we have
\[ (2.7) \quad a_0 = -\left( \frac{q - 2\alpha}{q - \alpha} \right), \]
and
\[ (2.8) \quad a_1 = -\frac{(q - 2\alpha) B_1}{2(1 + q)(q - \alpha + \alpha q)} \left[ c_2 - \frac{c_1^2}{2} \left( 1 - \frac{B_2}{B_1} + B_1 \right) \right]. \]
Therefore
\[ a_1 - \mu a_0^2 = -\frac{(q - 2\alpha) B_1}{2(1 + q)(q - \alpha + \alpha q)} \left\{ c_2 - \nu c_1^2 \right\}, \]
where
\[ (2.9) \quad \nu = \frac{1}{2} \left[ 1 - \frac{B_2}{B_1} + B_1 - \mu \frac{(q - 2\alpha)(q - \alpha + \alpha q)(q + 1) B_1}{(q - \alpha)^2} \right]. \]

Now, the result (2.1) follows by an application of Lemma 1. Also, if \( B_1 = 0 \), then
\[ a_0 = 0 \quad \text{and} \quad a_1 = -\frac{(q - 2\alpha) B_2 c_1^2}{4(1 + q)(q - \alpha + \alpha q)}. \]
Since \( p(z) \) has positive real part, \( |c_1| \leq 2 \) (see Nehari [15]), so that
\[ |a_1| \leq \frac{1}{1 + q} \left| \frac{(q - 2\alpha) B_2}{(q - \alpha + \alpha q)} \right|, \]
this proving (2.2). The result is sharp for the functions
\[ -\frac{(1 - \frac{q}{q}) q z D_0 f(z) + \alpha q z D_0 [D_0 f(z)]}{(1 - \frac{q}{q}) f(z) + \alpha D_0 f(z)} = \varphi(z^2), \]
and
\[ -\frac{(1 - \frac{q}{q}) q z D_0 f(z) + \alpha q z D_0 [D_0 f(z)]}{(1 - \frac{q}{q}) f(z) + \alpha D_0 f(z)} = \varphi(z). \]

This completes the proof of Theorem 1.
Remark 1.

(i) For \( q \to 1^- \) in Theorem 1, we obtain the result obtained by Aouf et al. [4, Theorem 2.1];

(ii) For \( q \to 1^- \) and \( \alpha = 0 \) in Theorem 1, we obtain the result obtained by Silverman et al. [18, Theorem 2.1].

By using Lemma 2, we can obtain the following theorem.

**Theorem 2.** Let \( \varphi(z) = 1 + B_1z + B_2z^2 + \ldots \) \( (B_i > 0, \ i \in \{1, 2\}, \ 0 < \alpha < \frac{q}{1 + q}) \).

If \( f(z) \) given by (1.1) belongs to the class \( S_{q, \alpha}^*(\varphi) \), then

\[
|a_1 - \mu a_0^2| \leq \begin{cases} 
\frac{(q-2\alpha)B_2^2}{(1+q)(q-\alpha+aq)} \left\{ -B_2 + \left[ 1 - \mu \frac{[q - \alpha(1+q)][(1+q)(q-2\alpha)]}{q(1-\alpha)^2} \right] B_1^2 \right\} & \text{if } \mu \leq \sigma_1, \\
\frac{(q-2\alpha)B_1}{(1+q)(q-\alpha+aq)} \left\{ B_2 - \left[ 1 - \mu \frac{[q - \alpha(1+q)][(1+q)(q-2\alpha)]}{q(1-\alpha)^2} \right] B_1^2 \right\} & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\
\frac{(q-2\alpha)B_1^2}{(1+q)(q-\alpha+aq)} \left\{ B_2 + \left[ 1 - \mu \frac{[q - \alpha(1+q)][(1+q)(q-2\alpha)]}{q(1-\alpha)^2} \right] B_1^2 \right\} & \text{if } \mu \geq \sigma_2,
\end{cases}
\]

where

\[
\sigma_1 = \frac{(q-\alpha)^2 \left[ -B_1 - B_2 + B_1^2 \right]}{(q-2\alpha)(q-\alpha+aq)(1+q)B_1^2} \quad \text{and} \quad \sigma_2 = \frac{(q-\alpha)^2 \left( B_1 - B_2 + B_1^2 \right)}{(q-2\alpha)(q-\alpha+aq)(1+q)B_1^2}.
\]

The result is sharp. Further, let

\[
\sigma_3 = \frac{(q-\alpha)^2 \left[ -B_2 + B_1^2 \right]}{(q-2\alpha)(q-\alpha+aq)(1+q)B_1^2}.
\]

(i) If \( \sigma_1 \leq \mu \leq \sigma_3 \), then

\[
|a_1 - \mu a_0^2| + \frac{(q-\alpha)^2}{(q-2\alpha)(q-\alpha+aq)(1+q)B_1^2} \times \left\{ (B_1 + B_2) + \left[ \mu \frac{(1+q)(q-2\alpha)(q-\alpha+aq)}{q(1-\alpha)^2} - 1 \right] B_1^2 \right\} |a_0|^2 \leq \frac{(q-2\alpha)B_1}{(1+q)(q-\alpha+aq)}.\]

(ii) If \( \sigma_3 \leq \mu \leq \sigma_2 \), then

\[
|a_1 - \mu a_0^2| + \frac{(q-\alpha)^2}{(q-2\alpha)(q-\alpha+aq)(1+q)B_1^2} \times \left\{ (B_1 - B_2) + \left[ 1 - \mu \frac{(1+q)(q-2\alpha)(q-\alpha+aq)}{q(1-\alpha)^2} \right] B_1^2 \right\} |a_0|^2 \leq \frac{(q-2\alpha)B_1}{(1+q)(q-\alpha+aq)}.\]
Proof. First, let $\mu \leq \sigma_1$. Then

$$|a_1 - \mu a_0^2| \leq \frac{(q - 2\alpha) B_1}{(1 + q)(q - \alpha + \alpha q)} \left\{ \frac{B_2}{B_1} + \left[ 1 - \mu \frac{[q - \alpha(1 + q)]}{q(1 - \alpha)^2} \right] B_1 \right\} \leq \frac{(q - 2\alpha) B_1}{(1 + q)(q - \alpha + \alpha q)} \left\{ -B_2 + \left[ 1 - \mu \frac{[q - \alpha(1 + q)]}{q(1 - \alpha)^2} \right] B_1 \right\}.$$

Let, now $\sigma_1 \leq \mu \leq \sigma_2$. Then, using the above calculations, we obtain

$$|a_1 - \mu a_0^2| \leq \frac{(q - 2\alpha) B_1}{(1 + q)(q - \alpha + \alpha q)}.$$

Finally, if $\mu \geq \sigma_2$, then

$$|a_1 - \mu a_0^2| \leq \frac{(q - 2\alpha) B_1}{(1 + q)(q - \alpha + \alpha q)} \left\{ \frac{B_2}{B_1} - \left[ 1 - \mu \frac{[q - \alpha(1 + q)]}{q(1 - \alpha)^2} \right] B_1 \right\} \leq \frac{(q - 2\alpha) B_1}{(1 + q)(q - \alpha + \alpha q)} \left\{ B_2 - \left[ 1 - \mu \frac{[q - \alpha(1 + q)]}{q(1 - \alpha)^2} \right] B_1 \right\}.$$

To show that the bounds are sharp, we define the functions $K_{\varphi n} (n \geq 2)$ by

$$- \frac{(1 - \frac{2}{q}) qzD_q K_{\varphi n}(z) + \alpha qzD_q[zD_q K_{\varphi n}(z)]}{(1 - \frac{2}{q}) K_{\varphi n}(z) + \alpha qzD_q K_{\varphi n}(z)} = \varphi(z^{n-1}),$$

$$z^2 K_{\varphi n}(z) \mid_{z=0} = 0 = -z^2 K'_{\varphi n}(z) \mid_{z=0} - 1,$$

and the functions $F_\gamma$ and $G_\gamma (0 \leq \gamma \leq 1)$ by

$$- \frac{(1 - \frac{2}{q}) qzD_q F_\gamma(z) + \alpha qzD_q[zD_q F_\gamma(z)]}{(1 - \frac{2}{q}) F_\gamma(z) + \alpha qzD_q F_\gamma(z)} = \varphi \left( \frac{z(\gamma + 1)}{1 + \gamma z} \right),$$

$$z^2 F_\gamma(z) \mid_{z=0} = 0 = -z^2 F'_\gamma(z) \mid_{z=0} - 1,$$

and

$$- \frac{(1 - \frac{2}{q}) qzD_q G_\gamma(z) + \alpha qzD_q[zD_q G_\gamma(z)]}{(1 - \frac{2}{q}) G_\gamma(z) + \alpha qzD_q G_\gamma(z)} = \varphi \left( \frac{z(\gamma + 1)}{1 + \gamma z} \right),$$

$$z^2 G_\gamma(z) \mid_{z=0} = 0 = -z^2 G'_\gamma(z) \mid_{z=0} - 1.$$

Clari the functions $K_{\varphi n}$, $F_\gamma$ and $G_\gamma \in F^\varphi q,\alpha (\varphi)$. Also we write $K_\varphi = K_{\varphi 2}$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if $f(z)$ is $K_\varphi$ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if $f(z)$ is $K_{\varphi 3}$ or one of its rotations. If $\mu = \sigma_1$, then the equality holds if and only if $f(z)$ is $F_\gamma$ or one of its rotations. If $\mu = \sigma_2$, then the equality holds if and only if $f(z)$ is $G_\gamma$ or one of its rotations. This completes the proof of Theorem 2. \hfill \Box

Remark 2.

(i) For $q \to 1^-$ in Theorem 2, we obtain the result obtained by Aouf et al. [4, Theorem 2];
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(ii) Putting $q \to 1^-$ and $\alpha = 0$ in Theorem 2, we obtain the result obtained by Ali and Ravichandran [2, Theorem 5.1].

3. Applications to Functions Defined by $q$–Bessel Function

We recall some definitions of $q$–calculus which we will be used in our paper. For any complex number $\alpha$, the $q$–shifted factorials are defined by

\[ (\alpha; q)_0 = 1; \quad (\alpha; q)_n = \prod_{k=0}^{n-1} (1 - \alpha q^k) \quad (n \in \mathbb{N} = \{1, 2, \ldots\}). \]

If $|q| < 1$, the definition (3.1) remains meaningful for $n = \infty$ as a convergent infinite product

\[ (\alpha; q)_\infty = \prod_{j=0}^{\infty} (1 - \alpha q^j). \]

In terms of the analogue of the gamma function

\[ (q^\alpha; q)_n = \frac{\Gamma_q(\alpha + n)(1 - q)^n}{\Gamma_q(\alpha)} \quad (n > 0), \]

where the $q$–gamma function is defined by

\[ \Gamma_q(x) = \frac{(q; q)_\infty(1 - q)^{1-x}}{(q^x; q)_\infty} \quad (0 < q < 1). \]

We note that

\[ \lim_{q \to 1^-} (\alpha; q)_n = (\alpha)_n, \]

where

\[ (\alpha)_n = \begin{cases} 1 & \text{if } n = 0, \\ \alpha(\alpha + 1)(\alpha + 2)\ldots(\alpha + n - 1) & \text{if } n \in \mathbb{N}. \end{cases} \]

Now, consider the $q$–analogue of Bessel function defined by (Jackson [8])

\[ J_v^{(1)}(z; q) = \frac{(q^{v+1}; q)_\infty}{(q; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{(q; q)_k(q^{v+1}; q)_k} (\frac{z^2}{2})^{2k+v} (0 < q < 1). \]

Also, let us define

\[ \mathcal{L}_v(z; q) = \frac{2^v(q; q)_\infty}{(q^{v+1}; q)_\infty(1 - q)^{v/2 + 1/2}} J_v^{(1)} \left( z^{1/2}(1 - q); q \right) \]

\[ = \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(1 - q)^{2(k+1)}}{4^{k+1} (q; q)_{k+1}(q^{v+1}; q)_{k+1}} z^k \quad (z \in \mathbb{U}). \]
By using the Hadamard product (or convolution), we define the linear operator \( \mathcal{L}_{q,\nu} : \Sigma \to \Sigma \), as follows:

\[
(\mathcal{L}_{q,\nu}f)(z) = \mathcal{L}_{\nu}(z; q) \ast f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(1 - q)^{2(k+1)}}{4^{(k+1)}_{q} q_{k+1}(q^{v+1}; q)_{k+1}} a_{k+1} z^{k}.
\]

As \( q \to 1^{-} \), the linear operator \( \mathcal{L}_{q,\nu} \) reduces to the operator \( \mathcal{L}_{\nu} \), introduced and studied by Aouf et al. [5]. For \( 0 < q < 1 \) and \( \alpha \in \mathbb{C} \setminus (0, 1] \), let \( \mathcal{F}_{q,\alpha,\nu}(\varphi) \) be the subclass of \( \Sigma \) consisting of functions \( f(z) \) of the form (1.1) and satisfies the analytic criterion:

\[
\frac{(1 - \frac{\alpha}{q}) q z D_{q}(\mathcal{L}_{q,\nu}f) + \alpha q z D_{q}[D_{q}(\mathcal{L}_{q,\nu}f)]}{(1 - \frac{\alpha}{q}) (\mathcal{L}_{q,\nu}f) + \alpha z D_{q}(\mathcal{L}_{q,\nu}f)} \prec \varphi(z) \quad (z \in \mathbb{U}).
\]

Using similar arguments to those in the proof of the above theorems, we obtain the following theorems.

**Theorem 3.** Let \( \varphi(z) = 1 + B_{1} z + B_{2} z^{2} + \cdots \). If \( f(z) \) given by (1.1) belongs to the class \( \mathcal{F}_{q,\alpha,\nu}(\varphi) \) and \( \mu \) is a complex number, then

(i) \( |a_{1} - \mu a_{0}^{2}| \leq \frac{4^{2} (1 - q^{v+1})(1 - q^{v+2})}{(1 - q)^{2}} \left| B_{1} \frac{(q - 2\alpha)}{q + \alpha q - \alpha} \right| \times \max \left\{ 1, \left| \frac{B_{2}}{B_{1}} - \left[ 1 - \mu \frac{(q - 2\alpha)(1 - q^{v+1})(q - \alpha + \alpha q)}{(1 - q^{v+2})(q - \alpha)^{2}} \right] B_{1} \right| \right\} \quad (B_{1} \neq 0), \)

(ii) \( |a_{1}| \leq \frac{4^{2} (1 - q^{v+1})(1 - q^{v+2})}{(1 - q)^{2}} \left| \frac{B_{2} (q - 2\alpha)}{q + \alpha q - \alpha} \right| \quad (B_{1} = 0). \)

The result is sharp.

**Theorem 4.** Let \( \varphi(z) = 1 + B_{1} z + B_{2} z^{2} + \ldots, (B_{i} > 0, \ i \in \{1, 2\}, \ \alpha > 0) \). If \( f(z) \) given by (1.1) belongs to the class \( \mathcal{F}_{q,\alpha,\nu}(\varphi) \), then

\[
|a_{1} - \mu a_{0}^{2}| \leq \begin{cases} 
\frac{4^{2} (1 - q^{v+1})(1 - q^{v+2})}{(1 - q)^{2}} (q - 2\alpha) B_{1}^{2} \times \left\{ -B_{2} + \left[ 1 - \mu \frac{(q - 2\alpha)(1 - q^{v+1})(q - \alpha + \alpha q)}{(1 - q^{v+2})(q - \alpha)^{2}} \right] B_{1} \right\} & \text{if } \mu \leq \sigma_{1}^{*}, \ \frac{4^{2} (1 - q^{v+1})(1 - q^{v+2})}{(1 - q)^{2}} (q - \alpha(q + 1)) B_{1} \times \left\{ B_{2} - \left[ 1 - \mu \frac{(q - 2\alpha)(1 - q^{v+1})(q - \alpha + \alpha q)}{(1 - q^{v+2})(q - \alpha)^{2}} \right] B_{1} \right\} & \text{if } \sigma_{1}^{*} \leq \mu \leq \sigma_{2}^{*}, \ \frac{4^{2} (1 - q^{v+1})(1 - q^{v+2})}{(1 - q)^{2}} (q - \alpha(q + 1)) \times \left\{ B_{2} - \left[ 1 - \mu \frac{(q - 2\alpha)(1 - q^{v+1})(q - \alpha + \alpha q)}{(1 - q^{v+2})(q - \alpha)^{2}} \right] B_{1} \right\} & \text{if } \mu \geq \sigma_{2}^{*}, \end{cases}
\]
where
\[ \sigma_1^* = \frac{(q - \alpha)^2(1 - q^{v+2})}{(q - 2\alpha)(q - \alpha + \alpha q)(1 - q^{v+1})B_1^2} \left[ -B_1 - B_2 + B_1^2 \right], \]
and
\[ \sigma_2^* = \frac{(q - \alpha)^2(1 - q^{v+2})}{(q - 2\alpha)(q - \alpha + \alpha q)(1 - q^{v+1})B_1^2} \left[ B_1 - B_2 + B_1^2 \right]. \]
The result is sharp. Further, let
\[ \sigma_3^* = \frac{(q - \alpha)^2(1 - q^{v+2})}{(q - 2\alpha)(q - \alpha + \alpha q)(1 - q^{v+1})B_1^2} \left[ B_2 - B_1 + B_1^2 \right]. \]

(i) If \( \sigma_1^* \leq \mu \leq \sigma_3^* \), then
\[
\left| a_1 - \mu a_0 \right|^2 + \frac{(1 - q^{v+2})(q - \alpha)^2}{(q - 2\alpha)(q - \alpha + \alpha q)(1 - q^{v+1})B_1^2} \times \left\{ (B_1 + B_2) + \left[ \frac{\mu (q - 2\alpha)(1 - q^{v+1})(1 - q^{v+2})(q - \alpha + \alpha q)}{(1 - q^{v+2})(q - \alpha)^2} - 1 \right] B_1^2 \right\} \left| a_0 \right|^2 \\
\leq \frac{4^2 (q - 2\alpha)(1 - q^{v+1})(1 - q^{v+2})B_1}{(1 - q)^2(q - \alpha + \alpha q)}.
\]

(ii) If \( \sigma_3^* \leq \mu \leq \sigma_2^* \), then
\[
\left| a_1 - \mu a_0 \right|^2 + \frac{(1 - q^{v+2})(q - \alpha)^2}{(q - 2\alpha)(q - \alpha + \alpha q)(1 - q^{v+1})B_1^2} \times \left\{ (B_1 - B_2) + \left[ 1 - \mu \frac{\left( q - 2\alpha \right)(1 - q^{v+1})(1 - q^{v+2})(q - \alpha + \alpha q)}{(1 - q^{v+2})(q - \alpha)^2} \right] B_1^2 \right\} \left| a_0 \right|^2 \\
\leq \frac{4^2 (q - 2\alpha)(1 - q^{v+1})(1 - q^{v+2})B_1}{(1 - q)^2(q - \alpha + \alpha q)}.
\]

References


