KYUNGPOOK Math. J. 58(2018), 333-346
https://doi.org/10.5666/KMJ.2018.58.2.333
pISSN 1225-6951 eISSN 0454-8124

A New Technique for Solving Optimal Control Problems of the Time-delayed Systems

Fateme Ghomanjani
Department of Mathematics, Kashmar Higher Education Institute, Kashmar, Iran
e-mail : fatemeghomanjani@gmail.com

Abstract. An approximation scheme utilizing Bezier curves is considered for solving time-delayed optimal control problems with terminal inequality constraints. First, the problem is transformed, using a Páde approximation, to one without a time-delayed argument. Terminal inequality constraints, if they exist, are converted to equality constraints. A computational method based on Bezier curves in the time domain is then proposed for solving the obtained non-delay optimal control problem. Numerical examples are introduced to verify the efficiency and accuracy of the proposed technique. The findings demonstrate that the proposed method is accurate and easy to implement.

1. Introduction

The control of systems with time delay has been of considerable concern. Delays occur frequently in biological, chemical, electronic and transportation systems. Wu, et al. [14] built up a computational method for solving an optimal control problem which is represented by a switched dynamical system with time delay. Kharatishidi [7] has approached this problem by extending Pontryagin's maximum principle to time delay systems. The actual solution involves a two-point boundary-value problem in which advances and delays are exhibited. In addition, this solution does not yield a feedback controller. Time-optimal control of delay systems has been considered by Oguztoreli [11] who obtained several results concerning bang-bang controls which parallel those of LaSalle [9] for non delay systems. For a time-invariant system with an infinite upper limit in the performance measure, Krasovskii [8] has developed the forms of the controller and the performance measure. Ross [12] has acquired a set of differential equations for the unknowns in the forms of Krasovskii. However, Ross's results are not applicable to time-varying systems with a finite

Received July 12, 2017; revised February 23, 2018; accepted March 9, 2018.
2010 Mathematics Subject Classification: 65K10, 26A33, 49K15.
Key words and phrases: Bezier curve, Páde approximation, optimal control problems, time-delay.

limit in the performance measure. In [1], the authors presented an optimal regular for a linear system with multiple state and input delays and a quadratic criterion. The optimal regulator equations were obtained reducing the original problem to the linear-quadratic regulator design for a system without delays (see [3] and [4]).
B-splines (where Bezier form is a special case of B-splines), due to numerical stability and arbitrary order of accuracy, have become popular tools for solving differential equations. The use of Bezier curves for solving time-delayed optimal control systems with Páde approximation is a novel idea. The stated technique reduces the CPU time and the computer memory comparing with existing methods such as methods in [2, 10] and at the same time keeps the solution accuracy. Although the stated technique is very easy to utilize and straightforward, the obtained results are satisfactory (see numerical results). In this paper, one may utilize the Bezier polynomials. There are many papers and books deal with the Bezier curves or surface techniques [6]. The organization of this study is arranged as follows: In Section 2, problem transformation is presented. Section 3 is referred to the Bezier curve technique. Some Numerical examples are provided in Section 4. Section 5 is devoted the conclusion.

2. Problem Transformation

Consider the time-delayed optimal control problem

$$\min_{u,\pi} J = \eta(x_{t_f}, \pi) + \int_0^{t_f} L(x(t), x(t-\sigma), u(t), \pi, t)\ dt \tag{2.1}$$

$$\dot{x}(t) = f(x(t), x(t-\sigma), u(t), \pi, t), \tag{2.2}$$

$$x(t) = \xi(t), \quad -\sigma \leq t \leq 0, \tag{2.3}$$

$$\eta(x_f, \pi) \geq 0 \tag{2.4}$$

where $x \in \mathcal{R}^n$, $u \in \mathcal{R}^m$ and $\pi \in \mathcal{R}^p$ are the state, control and unknown parameter vectors respectively, $\eta \in R^q$ represents the terminal inequality constraints, t_f is end time, and σ is the delay time associated with the state vector x. For the sake of simplicity, our discussion will be confined to the case of a single time delay σ. However, all the results can be extended in a straightforward manner to the case of multiple time delays. The time-delayed optimal control problem (2.1) can be transformed to one without a time-delayed argument utilizing the following approximation scheme. Let $X(s)$ be a two-sided Laplace transform of $x(t)$ (see [10]):

$$x(t) \Leftrightarrow X(s) \triangleq \int_{-\infty}^{+\infty} e^{-st} x(t) dt \tag{2.5}$$

Presently, $X(s)$ is defined over the strip of convergence $\alpha < Re(s) < \beta$, where $Re(s)$ denotes the real part of s, and $' \Leftrightarrow '$ denotes a two-sided Laplace transformation. The two-sided Laplace transform is utilized because $x(t) = \xi(t) \neq 0$ for $t \leq 0$.
Given that $x(t) \Leftrightarrow X(s)$, the two-sided Laplace transforms of $x(t-\sigma)$ and $\dot{x}(t)$ are

presented by

$$x(t-\sigma) \Leftrightarrow e^{-\sigma s}X(s), \quad \alpha < Re(s) < \beta, \tag{2.6}$$

$$\dot{x}(t) \Leftrightarrow sX(s), \quad \alpha_1 < Re(s) < \beta_1, \tag{2.7}$$

where the strip of convergence after the differentiation is assumed to exist and might or might not be the same as that for $X(s)$. To omit variables with a time-delayed argument in (2.1), let us define $y(t) \triangleq x(t-\sigma)$ and $y(t) \Leftrightarrow Y(s)$. The two-sided Laplace transforms $Y(s)$ and $X(s)$ are referred by (see (2.6))

$$Y(s) = e^{-\sigma s}X(s). \tag{2.8}$$

Utilizing a first-order Páde approximation, one may have

$$Y(s) \doteq \frac{\frac{2}{\sigma} - s}{\frac{2}{\sigma} + 2}X(s), \tag{2.9}$$

$$(\frac{2}{\sigma} + s)Y(s) \doteq (\frac{2}{\sigma} - s)X(s). \tag{2.10}$$

If an inverse Laplace transformation was performed on the last equation, one may have (see (2.2) and (2.7))

$$\dot{y}(t) \doteq \frac{2}{\sigma}[x(t) - y(t)] - \dot{x}(t) \tag{2.11}$$

$$\doteq \frac{2}{\sigma}[x(t) - y(t)] - f(x(t), y(t), u(t), \pi, t) \tag{2.12}$$

The time-delayed optimal control problem is approximately changed to one of minimizing $J(x(t), y(t), u(t), \pi, t)$ subject to

$$\dot{x}(t) = f(x(t), y(t), u(t), \pi, t) \tag{2.13}$$

$$\dot{y}(t) \doteq \frac{2}{\sigma}[x(t) - y(t)] - f(x(t), y(t), u(t), \pi, t) \tag{2.14}$$

with the following conditions

$$x(0) = \xi(0), \quad y(0) = \xi(-\sigma) \tag{2.15}$$

and the terminal condition $\eta(x_f, \pi) \geq 0$. An alternative approximation scheme (but equivalent) is to state $Y(s)$ in (2.9) as follows:

$$Y(s) \doteq (-1 + \frac{\frac{4}{\sigma}}{\frac{2}{\sigma} + s})X(s) \doteq -X(s) + Q(s) \tag{2.16}$$

where $Q(s)$ is defined as

$$Q(s) = \frac{\frac{4}{\sigma}}{\frac{2}{\sigma} + s}X(s) \tag{2.17}$$

The inverse Laplace transforms of (2.16) and (2.17) are

$$x(t-\sigma) = y(t) = -x(t) + q(t) \tag{2.18}$$

$$\dot{q}(t) = -\frac{2}{\sigma}q(t) + \frac{4}{\sigma}x(t) \tag{2.19}$$

Utilizing this technique, the original optimization problem is transformed to one of minimizing $J(x(t), q(t), u(t), \pi, t)$ subject to

$$\dot{x}(t) = f(x(t), q(t), u(t), \pi, t) \tag{2.20}$$

$$\dot{q}(t) = -\frac{2}{\sigma}q(t) + \frac{4}{\sigma}x(t) \tag{2.21}$$

with the following conditions

$$x(0) \;=\; \xi(0), \quad q(0) = \xi(0) + \xi(-\sigma) \tag{2.22}$$

and the terminal condition $\eta(x_f, \pi) \geq 0$.
To enhance the accuracy of the above-described approximation schemes, the time delay σ can be subdivided into smaller sections. For example, one may define

$$y(t) \;\triangleq\; x(t-\frac{\sigma}{2}) \tag{2.23}$$

$$z(t) \;\triangleq\; y(t-\frac{\sigma}{2}) = x(t-\sigma) \tag{2.24}$$

Again, utilizing a first-order Páde approximation, one may obtain

$$\dot{y}(t) \;\doteq\; \frac{4}{\sigma}[x(t) - y(t)] - f(x(t), z(t), u(t), \pi, t) \tag{2.25}$$

$$\dot{z}(t) \;\doteq\; \frac{4}{\sigma}[2y(t) - z(t) - x(t)] + f(x(t), z(t), u(t), \pi, t) \tag{2.26}$$

The time-delayed problem is changed to one of minimizing $J(x(t), z(t), u(t), \pi, t)$ subject to

$$\dot{x}(t) \;\doteq\; f(x(t), q(t), u(t), \pi, t) \tag{2.27}$$

$$\dot{y}(t) \;\doteq\; \frac{4}{\sigma}[x(t) - y(t)] - f(x(t), z(t), u(t), \pi, t) \tag{2.28}$$

$$\dot{z}(t) \;\doteq\; \frac{4}{\sigma}[2y(t) - z(t) - x(t)] + f(x(t), z(t), u(t), \pi, t) \tag{2.29}$$

with the initial conditions

$$x(0) \;=\; \xi(0), \quad y(0) = \xi(-\frac{\sigma}{2}), \quad z(0) = \xi(-\sigma) \tag{2.30}$$

and the final condition (2.4). Repeated applications of this method will result progressively improved achievement. However, this is at the expense of an increase in the system order, resulting in an increase in the computation time.

3. Bezier Curve Method

Our strategy is utilizing Bezier curves to approximate the solutions $x_i(t)$ and $u(t)$ where $x_i(t)$ are given below. Define the Bezier polynomials of degree n for $t \in [t_0, t_f]$ as follows:

$$x_i(t) \simeq \sum_{r=0}^{n} a_r^i B_{r,n}(\frac{t-t_0}{h}), \tag{3.1}$$

$$u(t) \simeq \sum_{r=0}^{n} b_r B_{r,n}(\frac{t-t_0}{h}) \tag{3.2}$$

where $h = t_f - t_0$, and

$$B_{r,n}(\frac{t-t_0}{h}) = \binom{n}{r} \frac{1}{h^n} (t_f - t)^{n-r} (t - t_0)^r,$$

is the Bernstein polynomial of degree n for $t \in [t_0, t_f]$, a_r^i and b_r are the control points. By substituting $x_i(t)$ and $u(t)$ in (2.27)-(2.30), one may define the problem which can be solved by Maple 16.

Ghomanjani et al. [6] proved the convergence of this method where $n \to \infty$ when the optimal control system solved by Bezier curve method (for more explanation, see [5]). For the convergence of the time-delayed optimal control problem, one may use Páde approximation, then the problem is converted to an optimal control problem (OCP), where the convergence of OCP is in [6].
Termination criterions are $\|x_i(t) - x_{i,exact}(t)\|_\infty < \epsilon$, and $\|u(t) - u_{exact}(t)\| < \epsilon$

4. Numerical Application

In this section, some numerical examples are presented for illustrating the proposed technique.

Example 4.1. Consider the following time-delay system:

$$\begin{aligned} &\min \quad J = 5(x_1(2))^2 + \frac{1}{2}\int_0^2 u^2(t)\, dt \\ &\dot{x}_1(t) = x_2(t), \quad 0 \le t \le 2 \\ &\dot{x}_2(t) = -x_1(t) - x_2(t-1) + u(t), \quad 0 \le t \le 2 \\ &x_1(t) = 10, \quad x_0(t) = 0, \quad -1 \le t \le 0, \end{aligned} \tag{4.1}$$

For this example the exact solution is given by [10] as follows:

$$u(t) = \begin{cases} \delta \sin(2-t) + (\frac{\delta}{2})(1-t)\sin(t-1) & 0 \le t \le 1 \\ \delta \sin(2-t) & 1 \le t \le 2 \end{cases} \tag{4.2}$$

where $\delta = 2.5599$. Utilizing the proposed method, one may have

$$x_3(t) \triangleq x_2(t - \frac{1}{2}) \tag{4.3}$$

$$x_4(t) \triangleq x_3(t - \frac{1}{2}) = x_2(t-1) \tag{4.4}$$

The delayed differential equations (4.1) then become

$$\begin{aligned}
&\dot{x}_1(t) = x_2(t)\\
&\dot{x}_2(t) = -x_1(t) - x_4(t) + u(t)\\
&\dot{x}_3(t) = x_1(t) + 4x_2(t) - 4x_3(t) + x_4(t) - u(t)\\
&\dot{x}_4(t) = -x_1(t) - 4x_2(t) + 8x_3(t) - 5x_4(t) + u(t)\\
&x_1(0) = 10,\ x_2(0) = 0,\ x_3(0) = 0,\ x_4(0) = 0,
\end{aligned}$$

by using this method, we have

$$\begin{aligned}
x_1(t) &= (1 - (\frac{1}{2})t)^2 a1_0 + t(1 - (\frac{1}{2})t)a1_1,\\
x_2(t) &= (1 - (\frac{1}{2})t)^2 a2_0 + t(1 - (\frac{1}{2})t)a2_1,\\
x_3(t) &= (1 - (\frac{1}{2})t)^2 a3_0 + t(1 - (\frac{1}{2})t)a3_1,\\
x_4(t) &= (1 - (\frac{1}{2})t)^2 a4_0 + t(1 - (\frac{1}{2})t)a4_1,\\
u(t) &= (1 - (\frac{1}{2})t)^2 b_0 + t(1 - (\frac{1}{2})t)b_1.
\end{aligned}$$

The optimal value of J in proposed method is 3.04246518971317. This value compares well with that given in [10] ($J = 3.256613$). In proposed method, one may obtain

$$\begin{aligned}
x_1(t) &= 10(1 - (\frac{1}{2})t)^2 + 4.84426425991426t(1 - (\frac{1}{2})t),\\
x_2(t) &= -7.57786787004287t(1 - (\frac{1}{2})t)^2 - 1.21106606497857t^2,\\
x_3(t) &= 8.74933030407381t(1 - (\frac{1}{2})t) - 1.39164977425000t^2,\\
x_4(t) &= 0.674964953083765t(1 - (\frac{1}{2})t) - 0.683400902532152t^2,\\
u(t) &= (1 - (\frac{1}{2})t)^2(2.5599\sin(2) - 1.279950000\sin(1))\\
&\quad + 3.68282830005636t(1 - (\frac{1}{2})t),
\end{aligned}$$

the graphs of approximated and exact solution $u(t)$ and $x_i(t)$ for $i = 1, 2, 3, 4$ are respectively plotted in Figs. 1, 2, 3, 4 and 5.

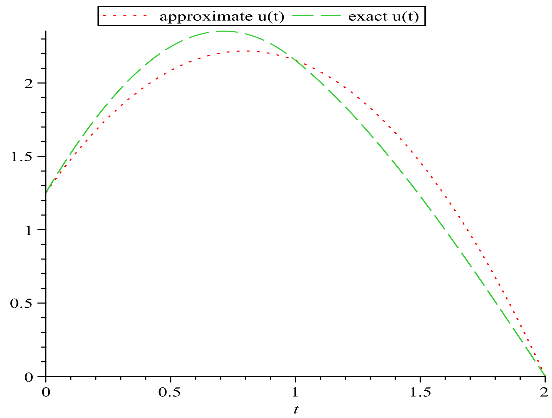

Figure 1: The graphs of approximated and exact solution $u(t)$ for Example 4.1

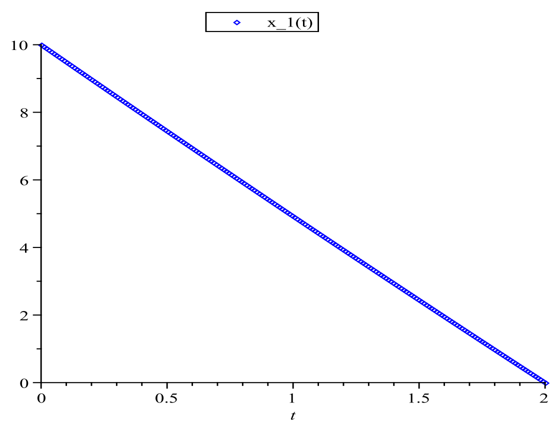

Figure 2: The graphs of approximated solution $x_1(t)$ for Example 4.1

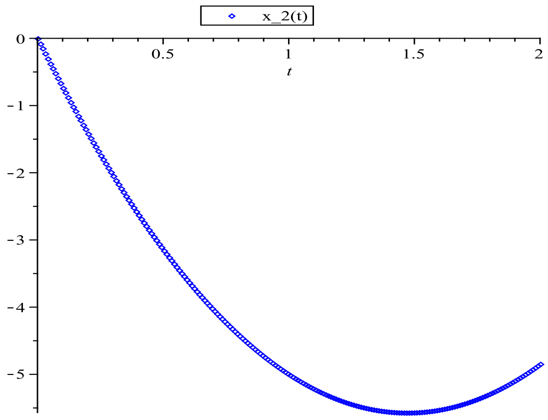

Figure 3: The graphs of approximated solution $x_2(t)$ for Example 4.1

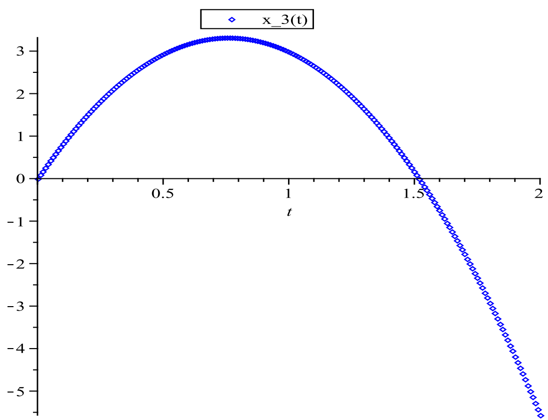

Figure 4: The graphs of approximated solution $x_3(t)$ for Example 4.1

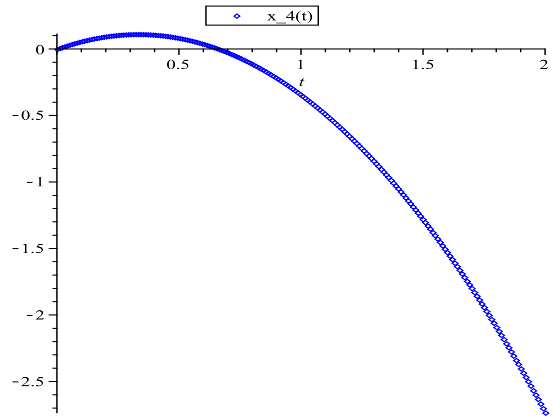

Figure 5: The graphs of approximated solution $x_4(t)$ for Example 4.1

Example 4.2. Consider the following time-delay system (see [10]):

$$\begin{aligned}
&\min \quad J = \frac{1}{2}((x_1(2))^2 + (x_1(2))^2) + \frac{1}{2}\int_0^2 u^2(t)\ dt \\
&\dot{x}_1(t) = x_2(t), \quad 0 \le t \le 2 \\
&\dot{x}_2(t) = -x_2(t) - x_1(t-1) + u(t), \quad 0 \le t \le 2 \\
&x_1(t) = 1, \quad x_2(t) = 0, \quad -1 \le t \le 0,
\end{aligned}$$

where the exact solution is

$$u(t) = \begin{cases} (\mu+\delta)e^{t-2} + (2\mu - 3\delta - (\mu-\delta)t)e^{t-1} + \delta(t+2) - \mu, & 0 \le t \le 1 \\ (\mu-\delta)e^{t-2} + \delta, & 1 \le t \le 2 \end{cases}$$
$$\mu \approx 0.5226194, \quad \delta \approx -0.5259256.$$

Utilizing this method, one may achieve

$$\begin{aligned}
x_3(t) &\triangleq x_1(t - \frac{1}{2}), \\
x_4(t) &\triangleq x_3(t - \frac{1}{2}) = x_1(t-1),
\end{aligned}$$

presently, one may have

$$\begin{aligned}
&\dot{x}_1(t) = x_2(t), \\
&\dot{x}_2(t) = -x_2(t) - x_4(t) + u(t), \\
&\dot{x}_3(t) = 4x_1(t) + x_2(t) - 3x_3(t) - u(t), \\
&\dot{x}_4(t) = 8x_3(t) - 5x_4(t) - x_2(t) - 4x_1(t) + u(t), \\
&x_1(0) = 1,\ x_2(0) = 0,\ x_3(0) = 1,\ x_4(0) = 1,
\end{aligned}$$

The optimal value of J in proposed method is 0.0671730978076868. This value compares well with that given in [10] ($J = 0.1967$). The graphs of approximated and exact solution $u(t)$ are plotted in Fig. 6.

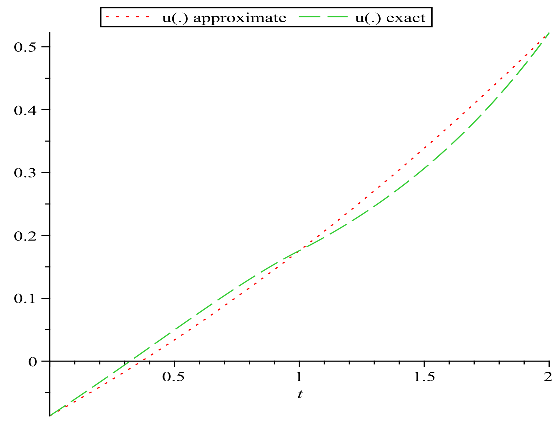

Figure 6: The graphs of approximated and exact solution $u(t)$ for Example 4.2

Example 4.3. Consider the following time-delay system (see [10]):

$$\begin{aligned}
&\min \quad J = \frac{1}{2}x_1(2))^2 + \frac{1}{2}\int_0^2 x_1^2(t) + u^2(t)\ dt \\
&\dot{x}_1(t) = x_1(t)sin(x_1) + x_1(t-1) + u(t), \quad 0 \le t \le 2 \\
&x_1(t) = 10, \qquad -1 \le t \le 0,
\end{aligned}$$

now, one may have

$$\begin{aligned}
x_2(t) &\triangleq x_1(t - \frac{1}{2}), \\
x_3(t) &\triangleq x_2(t - \frac{1}{2}) = x_1(t-1),
\end{aligned}$$

utilizing this method, one may achieve

$$\begin{aligned}
&\dot{x}_1(t) = x_1(t)sin(x_1) + x_3(t) + u(t), \\
&\dot{x}_2(t) = 4x_1(t) - 4x_2(t) - x_3(t) - x_1 sin(x_1) - u(t), \\
&\dot{x}_3(t) = 8x_2(t) - 3x_3(t) - 4x_1(t) + x_1 sin(x_1) + u(t), \\
&x_1(0) = 10,\ x_2(0) = 10,\ x_3(0) = 10, \qquad (4.5)
\end{aligned}$$

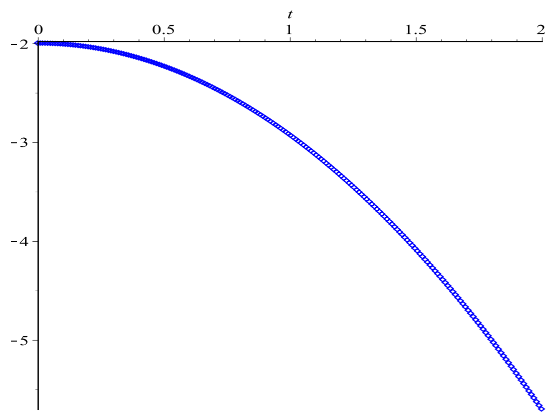

Figure 7: The graph of approximated solution $u(t)$ for Example 4.3

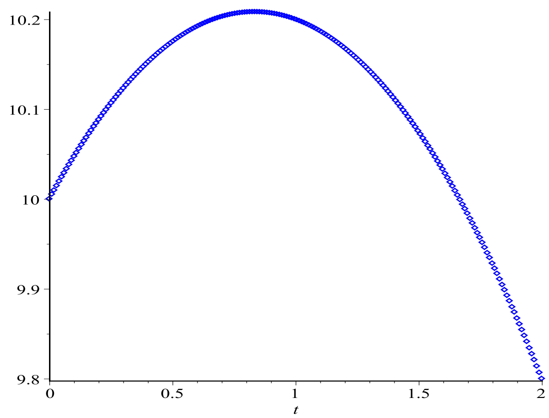

Figure 8: The graph of approximated solution $x_1(t)$ for Example 4.3

using Bezier curve, one may have

$$
\begin{aligned}
x_{1,bezier} &= (1-\frac{1}{2}t)^2 p1[0] + t(1-\frac{1}{2}t)p1[1] + \frac{1}{4}t^2 p1[2],\\
x_{2,bezier} &= (1-\frac{1}{2}t)^2 p2[0] + t(1-\frac{1}{2}t)p2[1] + \frac{1}{4}t^2 p2[2],\\
x_{3,bezier} &= (1-\frac{1}{2}t)^2 p3[0] + t(1-\frac{1}{2}t)p3[1] + \frac{1}{4}t^2 p3[2],\\
u_{bezier} &= (1-\frac{1}{2}t)^2 q[0] + t(1-\frac{1}{2}t)q[1] + \frac{1}{4}t^2 q[2], \qquad (4.6)
\end{aligned}
$$

then one may substitute (4.6) in (4.5), and solve this system by using Maple 16 software. The optimal value of J in proposed method is 161.712666656000, when the values of J in [10], Banks [2], and Wong et al. [13] are respectively $J = 161.88$, $J = 162.019$, and $J = 162.104$. The graphs of approximated solution $u(t)$ and $x_1(t)$ are plotted in Figs. 7 and 8.

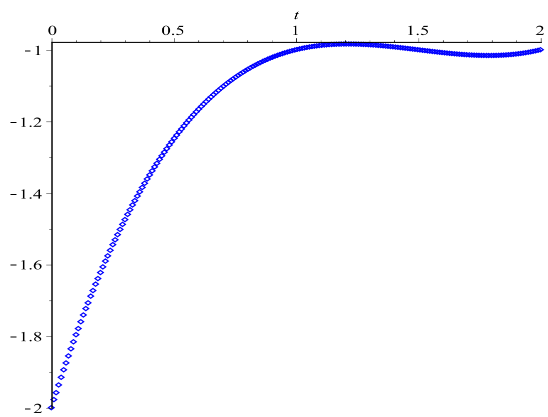

Figure 9: The graph of approximated solution $u(t)$ for Example 4.4

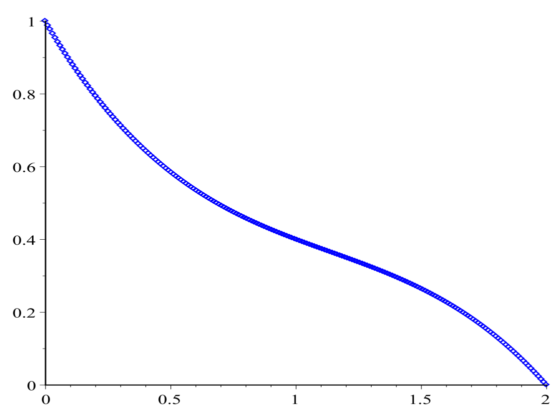

Figure 10: The graph of approximated solution $x_1(t)$ for Example 4.4

Example 4.4. Consider the following time-delay system (see [10]):

$$\min \quad J = \frac{1}{2}10^5 \times x_1(2))^2 + \frac{1}{2}\int_0^2 u^2(t)\ dt$$
$$\dot{x}_1(t) = x_1(t-1) + u(t), \quad 0 \le t \le 2$$
$$x_1(t) = 1, \quad -1 \le t \le 0,$$

now, one may have

$$x_2(t) \triangleq x_1(t - \frac{1}{2}),$$
$$x_3(t) \triangleq x_2(t - \frac{1}{2}) = x_1(t-1),$$

hence

$$\dot{x}_1(t) = x_3(t) + u(t),$$
$$\dot{x}_2(t) = 4x_1(t) - 4x_2(t) - x_3(t) - u(t),$$
$$\dot{x}_3(t) = 8x_2(t) - 3x_3(t) - 4x_1(t) + u(t),$$
$$x_1(0) = 1,\ x_2(0) = 1,\ x_3(0) = 1, \tag{4.7}$$

using Bezier curve, one may have

$$x_{1,bezier} = (1 - \frac{1}{2}t)^3 p1[0] + \frac{3}{2}t(1 - \frac{1}{2}t)^2 p1[1] + \frac{3}{4}t^2(1 - \frac{1}{2}t)p1[2] + \frac{1}{8}t^3 p1[3],$$
$$x_{2,bezier} = (1 - \frac{1}{2}t)^3 p2[0] + \frac{3}{2}t(1 - \frac{1}{2}t)^2 p2[1] + \frac{3}{4}t^2(1 - \frac{1}{2}t)p2[2] + \frac{1}{8}t^3 p2[3],$$
$$x_{3,bezier} = (1 - \frac{1}{2}t)^3 p3[0] + \frac{3}{2}t(1 - \frac{1}{2}t)^2 p3[1] + \frac{3}{4}t^2(1 - \frac{1}{2}t)p3[2] + \frac{1}{8}t^3 p3[3],$$
$$u_{bezier} = (1 - \frac{1}{2}t)^3 q[0] + \frac{3}{2}t(1 - \frac{1}{2}t)^2 q[1] + \frac{3}{4}t^2(1 - \frac{1}{2}t)q[2] + \frac{1}{8}t^3 q[3], \tag{4.8}$$

then one may substitute (4.8) in (4.7), and solve this system by using Maple 16 software. The optimal value of J in proposed method is 1.43068782747222, when the value of J in [10] is $J = 1.849730$. The graphs of approximated solution $u(t)$ and $x_1(t)$ are plotted in Figs. 9 and 10.

5. Conclusions

Time-delayed optimal control problems with terminal inequality constraints can be approximately solved by a combined parameter. To this end, a Páde approximation is utilized to acquire a corresponding problem without a time-delayed argument. The results obtained by the Bezier curve are in good agreement with the given exact solutions. The study shows that the method is effective technique to solve time-delayed optimal control problems, and the technique is easy to implement and computationally very attractive without sacrificing the accuracy of the solution.

Acknowledgements. The author would like to thank the anonymous reviewer of this paper for their careful reading, constructive comments and nice suggestions which has improved the paper very much.

References

[1] H. Allouche and A. Tazdayte, *Numerical solution of singular boundary value problems with logarithmic singularities by pade approximation and collocation methods*, J. Comput. Appl. Math., **311**(2017), 324–341.

[2] H. Banks, *Approximation of nonlinear functional differential equation control systems*, J. Optim. Theory Appl., **29**(1979), 383–408.

[3] M. Basin and J. Perez, *An optimal regulator for linear systems with multiple state and input delays*, Optimal Control Appl. Methods, **28**(2007), 45–57.

[4] M. Basin and J. Rodriguez-Gonzalez, *Optimal control for linear systems with multiple time delays in control input*, IEEE Trans. Automat. Control, **51(1)**(2006), 91–97.

[5] F. Ghomanjani, *A new approach for solving fractional differential-algebric equations*, J. Taibah Univ. Sci., **11**(2017), 1158-1164.

[6] F. Ghomanjani, M. H. Farahi and M. Gachpazan, *Bezier control points method to solve constrained quadratic optimal control of time varying linear systems*, Comput. Appl. Math., **31(3)**(2012), 433–456.

[7] G. L. Kharatishdi, *The maximum principle in the theory of optimal processes with time lags*, Doll. Akad, **136**(1961), 39-43.

[8] N. N. Krasovskii, *On the analytic construction of an optimal control in a system with time lags*, Prickl. Mat. Mech., **26**(1962), 39–51.

[9] J. P. LaSalle, *The time optimal control problem*, Contributions to the Theory of Nonlinear Oscillations V, Princeton Univ. Press, Princeton, N.J., (1960), 1–24.

[10] A. Nazemi and M. Mansoori, *Solving optimal control problems of the time-delayed systems by Haar wavelet*, J. Vib. Control, **22**(2016), 2657–2670.

[11] M. N. Oguztoreli, *A time optimal control problem for systems described by differential difference equations*, J. Soc. Indust. Appl. Math. Ser. A Control, **1**(1963), 290–310.

[12] D. W. Ross, *Optimal control of systems described by differential difference equations*, Ph.D. dissertiation, Dept of Elec. Enery., Stanford University, Stanford, Calif, 1968.

[13] K. Wong, D. Clements and K. Teo, *Optimal control computation for nonlinear time-lag systems*, J. Optim. Theory Appl., **47**(1985), 91–107.

[14] C. Wu, K. L. Teo, R. Li, and Y. Zhao, *Optimal control of switched systems with time delay*, Appl. Math. Lett., **19**(2006), 1062–1067.