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Super Quasi-Einstein Manifolds with Applications to General Relativity

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Abstract. The object of the present paper is to study super quasi-Einstein manifolds. Some geometric properties of super quasi-Einstein manifolds have been studied. We also discuss $S(QE)_4$ spacetime with space-matter tensor and some properties related to it. Finally, we construct an example of a super quasi-Einstein spacetime.

1. Introduction

A Riemannian or semi-Riemannian manifold (M^n, g), $n = dimM \geq 2$, is said to be an Einstein manifold if the following condition

$$S = \frac{r}{n} g \tag{1.1}$$

holds on M, where S and r denote the Ricci tensor and the scalar curvature of (M^n, g) respectively. According to ([6], p.432), (1.1) is called the Einstein metric condition. Einstein manifolds play an important role in Riemannian Geometry as well as in general theory of relativity. Also Einstein manifolds form a natural subclass of various classes of Riemannian manifolds by a curvature condition impossed on their Ricci tensor ([6],p.432-433). For instance, every Einstein manifold belongs to the class of Riemannian manifolds (M^n, g) realizing the following relation:

$$S(X,Y) = ag(X,Y) + bA(X)A(Y), \tag{1.2}$$

where a, $b \in \mathbb{R}$ and A is a non-zero 1-form such that

$$g(X,U) = A(X), \tag{1.3}$$

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for all vector fields X. Moreover, different structures on Einstein manifolds have been studied by several authors. In 1993, Tamassy and Binh [41] studied weakly symmetric structures on Einstein manifolds (see also [28]).

A non-flat Riemannian or semi-Riemannian manifold (M^n, g) $(n > 2)$ is defined to be a quasi-Einstein manifold if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition (1.2).

It is to be noted that Chaki and Maity [9] also introduced the notoin of quasi-Einstein manifolds in a different way. They have taken a, b are scalars and the vector field U metrically equivalent to the 1-form A as a unit vector field. Such an n-dimensional manifold is denoted by $(QE)_n$. Quasi-Einstein manifolds have been studied by several authors such as De and Ghosh [13], De and De [12] and De, Ghosh and Binh [23], Bejan [5] and many others.

Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. For instance, the Robertson-Walker spacetimes are quasi-Einstein manifolds. So quasi-Einstein manifolds have some importance in the general theory of relativity.

Quasi-Einstein manifolds have been generalized by several authors in several ways such as generalized quasi-Einstein manifolds [7, 14, 40], super quasi-Einstein manifolds [8, 21, 34] and many others.

As a generalization of quasi-Einstein manifold Chaki introduced the notion of a super quasi-Einstein manifold in [8]. According to him a non-flat Riemannian or semi-Riemannian manifold (M^n, g) $(n > 2)$ is called a super quasi-Einstein manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$\begin{aligned} S(X,Y) &= ag(X,Y) + bA(X)A(Y) \\ &\quad + c[A(X)B(Y) + A(Y)B(X)] + dD(X,Y), \end{aligned} \tag{1.4}$$

where a, b, c, d are scalars of which $b \neq 0$, $c \neq 0$, $d \neq 0$ and A, B are non-zero 1-forms such that

$$g(X,U) = A(X), \;\; g(X,V) = B(X), \tag{1.5}$$

U, V being mutually orthogonal unit vector fields, D is a symmetric (0,2) tensor with zero trace which satisfies the condition

$$D(X,U) = 0, \tag{1.6}$$

for all X. In such a case a, b, c, d are called the associated scalars, A, B are called the associated main and auxiliary 1-forms, U, V are called the main and auxiliary generators and D is called the associated tensor of the manifold. Such an n-dimensional manifold is denoted by the symbol $S(QE)_n$. If $c = 0$, then the manifold reduces to a pseudo quasi-Einstein manifold studied by Shaikh [39]. In [8] Chaki proved that a viscous fluid spacetime admitting heat flux and satisfying Einstein's equation without cosmological constant is a 4-dimensional semi-Riemannian super quasi-Einstein manifold.

A Riemannian or semi-Riemannian manifold of quasi-constant curvature was given by Chen and Yano [10] as a conformally flat manifold with the curvature tensor $\tilde{R}$ of type $(0,4)$ satisfied the condition

$$\begin{aligned}\tilde{R}(X,Y,Z,W) = a[&g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\\ &+ b[g(X,W)A(Y)A(Z) - g(X,Z)A(Y)A(W)\\ &+ g(Y,Z)A(X)A(W) - g(Y,W)A(X)A(Z)],\end{aligned} \tag{1.7}$$

where $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$, R is the curvature tensor of type $(1,3)$, and a, b are scalar functions of which $b \neq 0$ and A is a non-zero 1-form defined by

$$g(X,U) = A(X), \tag{1.8}$$

for all X and U being a unit vector field.

It can be easily seen that if the curvature tensor $\tilde{R}$ of the form (1.7), then the manifold is conformally flat. On the other hand, Gh.Vranceanu [42] defined the notion of almost constant curvature by the same expression (1.7). Later A. L. Mocanu [31] pointed out that the manifold introduced by Chen and Yano and the manifold introduced by Gh. Vranceanu are same. Hence a Riemannian or semi-Riemannian manifold is said to be *of quasi-constant curvature* if the curvature tensor $\tilde{R}$ satisfied the relation (1.7). Such a manifold is denoted by $(QC)_n$.

Subsequently Chaki [8] generalized the notion of $(QC)_n$, called a manifold of super quasi-constant curvature which is needed for the study of a $S(QE)_n$ and is defined as follows:

A non-flat Riemannian or semi-Riemannian manifold (M^n, g) $(n \geq 3)$ shall be called a manifold of super quasi-constant curvature if its curvature tensor $\tilde{R}$ of type $(0,4)$ satisfies the condition

$$\begin{aligned}\tilde{R}(X,Y,Z,W) = p[&g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\\ &+ q[g(Y,Z)A(X)A(W) + g(X,W)A(Y)A(Z)\\ &- g(X,Z)A(Y)A(W) - g(Y,W)A(X)A(Z)]\\ &+ s[g(Y,Z)\{A(X)B(W) + A(W)B(X)\}\\ &+ g(X,W)\{A(Y)B(Z) + B(Y)A(Z)\}\\ &- g(X,Z)\{A(Y)B(W) + B(Y)A(W)\}\\ &- g(Y,W)\{A(X)B(Z) + B(X)A(Z)\}]\\ &\ t[g(Y,Z)D(X,W) + g(X,W)D(Y,Z)\\ &- g(X,Z)D(Y,W) - g(Y,W)D(X,Z)],\end{aligned} \tag{1.9}$$

where p, q, s, t are scalars of which $p \neq 0, q \neq 0, s \neq 0, t \neq 0$, A, B are two non-zero 1-forms defined in (1.5), U, V being two unit vector fields such that $g(U,V) = 0$ and D is a symmetric tensor of type (0,2) defined in (1.6). Such an n-dimensional manifold shall be denoted by $S(QC)_n$. If in (1.9), $s = t = 0$ then the manifold becomes a manifold of quasi-constant curvature. This justifies the name manifold of super quasi-constant curvature.

The notion of quasi-conformal curvature tensor was given by Yano and Sawaki [43] and is defined as follows:

$$\begin{aligned} C^\star(X,Y)Z =& a_1 R(X,Y)Z + b_1[S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY] \\ &- \frac{r}{n}[\frac{a_1}{n-1} + 2b_1][g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{1.10}$$

where a_1 and b_1 are constants, R is the curvature tensor of type (1,3), S is the Ricci tensor of type (0,2), Q is the Ricci operator and r is the scalar curvature of the manifold.

If $a_1 = 1$ and $b_1 = -\frac{1}{n-2}$, then (1.10) reduces to the conformal curvature tensor C. Thus the conformal curvature tensor C is a particular case of the tensor $C^\star$. For this reason $C^\star$ is called the quasi-conformal curvature tensor. A Riemannian or a semi-Riemannian manifold is called quasi-conformally flat if $C^\star = 0$ for $n > 3$. It is known [3] that a quasi-conformally flat manifold is either conformally flat if $a_1 \neq 0$ or Einstein if $a_1 = 0$ and $b_1 \neq 0$. Since they give no restrictions for manifolds if $a_1 = 0$ and $b_1 = 0$, it is essential for us to consider the case of $a_1 \neq 0$ or $b_1 \neq 0$. The quasi-conformal curvature tensor have been studied by various authors in various ways such as Amur and Maralabhavi [3], De and Sarkar [19], De and Matsuyama [18], De, Jun and Gazi [16], Guha [24], Özgür and Sular [35], Mantica and Suh [29] and many others.

The spacetime of general relativity and cosmology is regarded as a connected 4-dimensional semi-Riemannian manifold (M^4, g) with Lorentzian metric g with signature $(-,+,+,+)$. The geometry of Lorentz manifold begins with the study of causal character of vectors of the manifold. It is due to this causality that Lorentz manifold becomes a convenient choice for the study of general relativity. Indeed by basing its study on Lorentzian manifold the general theory of relativity opens the way to the study of global questions about it [4, 11, 22, 25, 26] and many others. Also several authors studied spacetimes in different way such as [17, 20, 30, 44] and many others.

The present paper is organized as follows:

After introduction in Section 2 we study $S(QE)_n$ admitting quasi-conformal curvature tensor. In Section 3, it is shown that if the generators U and V are Killing vector fields, then the super quasi-Einstein manifold satisfies cyclic parallel Ricci tensor under certain condition. In the next two sections we consider $S(QE)_n$ with generators U and V both as concurrent and recurrent vector fields. We study $S(QE)_4$ spacetimes in Section 6. Finally, we construct an example of a super quasi-Einstein spacetime.

2. $S(QE)_n$ admitting Quasi-conformal Curvature Tensor

A $S(QE)_n$ $(n > 3)$ is not, in general a $S(QC)_n$. In this section we first consider a quasi-conformally flat $S(QE)_n$ $(n > 3)$ and show that such a $S(QE)_n$ is a $S(QC)_n$.

From (1.10) it follows that in a quasi-conformally flat Riemannian or semi-Riemannian manifold (M^n, g) $(n > 3)$, the curvature tensor $\tilde{R}$ of type $(0, 4)$ has the following form:

$$\begin{aligned} a_1\tilde{R}(X,Y,Z,W) = &-b_1[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ &+ g(Y,Z)S(X,W) - g(X,Z)S(Y,W)] \\ &+ \frac{r}{n}(\frac{a_1}{n-1} + 2b_1)[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \end{aligned} \tag{2.1}$$

Using (1.4) in (2.1) we obtain

$$\begin{aligned} \tilde{R}(X,Y,Z,W) =&\alpha[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + \beta[g(Y,Z)A(X)A(W) \\ &+ g(X,W)A(Y)A(Z) - g(X,Z)A(Y)A(W) - g(Y,W)A(X)A(Z)] \\ &+ \gamma[g(Y,Z)\{A(X)B(W) + A(W)B(X)\} + g(X,W)\{A(Y)B(Z) \\ &+ B(Y)A(Z)\} - g(X,Z)\{A(Y)B(W) + B(Y)A(W)\} \\ &- g(Y,W)\{A(X)B(Z) + B(X)A(Z)\}] \\ &+ \delta[g(Y,Z)D(X,W) + g(X,W)D(Y,Z) \\ &- g(X,Z)D(Y,W) - g(Y,W)D(X,Z)], \end{aligned} \tag{2.2}$$

where $\alpha = \frac{a_1 r + 2(n-1)(r-na)b_1}{n(n-1)a_1}$, $\beta = -\frac{bb_1}{a_1}$, $\gamma = -\frac{cb_1}{a_1}$ and $\delta = -\frac{db_1}{a_1}$.
In virtue of (1.9) it follows from (2.2) that the manifold under consideration is a $S(QC)_n$. Thus we can state the following:

Theorem 2.1. *Every quasi-conformally flat $S(QE)_n$ $(n > 3)$ is a $S(QC)_n$.*

Now we look for sufficient condition in order that a $S(QE)_n$ $(n > 3)$ may be quasi-conformally conservative. Quasi-conformal curvature tensor is said to be conservative if the divergence of $C^\star$ vanishes, i.e., $divC^\star = 0$.

In a $S(QE)_n$ if the associated scalars a, b, c and d are constant, then contracting (1.4) we have

$$r = an + b,$$

which implies that the scalar curvature r is constant, i.e., $dr(X) = 0$, for all X. Using $dr(X) = 0$ we obtain from (1.10) that

$$\begin{aligned} (\nabla_W C^\star)(X,Y,Z) =&a_1(\nabla_W R)(X,Y)Z + b_1[(\nabla_W S)(Y,Z)X - (\nabla_W S)(X,Z) \\ &+ g(Y,Z)(\nabla_W Q)(X) - g(X,Z)(\nabla_W Q)(Y)]. \end{aligned} \tag{2.3}$$

We know that $(divR)(X,Y,Z) = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)$ and from (1.4) we obtain

$$
\begin{aligned}
(\nabla_X S)(Y,Z) =& b[(\nabla_X A)(Y)A(Z) + A(Y)(\nabla_X A)(Z)\\
&+ c[(\nabla_X A)(Y)B(Z) + A(Y)(\nabla_X B)(Z)\\
&+ (\nabla_X B)(Y)A(Z) + (\nabla_X A)(Z)B(Y)]\\
&+ d(\nabla_X D)(Y,Z),
\end{aligned}
\tag{2.4}
$$

since a, b, c and d are constants.
Contracting (2.3) and using (2.4) we obtain

$$
\begin{aligned}
(divC^{\star})(X,Y,Z) =& (a_1+b_1)[b\{(\nabla_X A)(Y)A(Z) + A(Y)(\nabla_X A)(Z)\\
&- (\nabla_Y A)(X)A(Z) - A(X)(\nabla_Y A)(Z)\}\\
&+ c\{(\nabla_X A)(Y)B(Z) + A(Y)(\nabla_X B)(Z) + (\nabla_X B)(Y)A(Z)\\
&+ (\nabla_X A)(Z)B(Y) - (\nabla_Y A)(X)B(Z) - A(X)(\nabla_Y B)(Z)\\
&- (\nabla_Y B)(X)A(Z) - (\nabla_Y A)(Z)B(X)\}\\
&+ d\{(\nabla_X D)(Y,Z) - (\nabla_Y D)(X,Z)\}].
\end{aligned}
\tag{2.5}
$$

Imposing the condition that the generators U and V of the manifold are parallel vector fields gives $\nabla_X U = 0$ and $\nabla_X V = 0$. Hence

$$g(\nabla_X U, Y) = 0, \;\; i.e., \; (\nabla_X A)(Y) = 0.$$

and

$$g(\nabla_X V, Y) = 0, \;\; i.e., \; (\nabla_X B)(Y) = 0.$$

Therefore from (2.5) it follows that

$$(divC^{\star})(X,Y,Z) = (a_1+b_1)d[(\nabla_X D)(Y,Z) - (\nabla_Y D)(X,Z)].$$

Thus we can state the following:

Theorem 2.1. *If in a $S(QE)_n$ the associated scalars are constants and the generators U and V of the manifold are parallel vector fields, then the divergence free quasi-conformal curvature tensor and D is of Codazzi type are equivalent.*

3. The Generators U and V as Killing Vector Fields

In this section let us consider the generators U and V of the manifold are Killing vector fields. Then we have

$$(\pounds_U g)(X,Y) = 0 \tag{3.1}$$

and

$$(\pounds_V g)(X,Y) = 0, \tag{3.2}$$

where $\pounds$ denotes the Lie derivative.
From (3.1) and (3.2) it follows that

$$g(\nabla_X U, Y) + g(X, \nabla_Y U) = 0 \tag{3.3}$$

and

$$g(\nabla_X V, Y) + g(X, \nabla_Y V) = 0. \tag{3.4}$$

Since $g(\nabla_X U, Y) = (\nabla_X A)(Y)$ and $g(\nabla_X V, Y) = (\nabla_X B)(Y)$, we obtain from (3.3) and (3.4) that

$$(\nabla_X A)(Y) + (\nabla_Y A)(X) = 0 \tag{3.5}$$

and

$$(\nabla_X B)(Y) + (\nabla_Y B)(X) = 0, \tag{3.6}$$

for all X, Y.
Similarly, we have

$$(\nabla_X A)(Z) + (\nabla_Z A)(X) = 0, \tag{3.7}$$

$$(\nabla_Z A)(Y) + (\nabla_Y A)(Z) = 0, \tag{3.8}$$

$$(\nabla_X B)(Z) + (\nabla_Z B)(X) = 0, \tag{3.9}$$

$$(\nabla_Z B)(Y) + (\nabla_Y B)(Z) = 0, \tag{3.10}$$

for all X, Y, Z.
We also assume that the associated scalars are constants. Then from (1.4) we have

$$\begin{aligned}(\nabla_Z S)(X,Y) =& b[(\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y)]\\ &+ c[(\nabla_Z A)(X)B(Y) + A(X)(\nabla_Z B)(Y)\\ &+ (\nabla_Z B)(X)A(Y) + B(X)(\nabla_Z A)(Y)]\\ &+ d(\nabla_Z D)(X,Y).\end{aligned} \tag{3.11}$$

Using (3.11) we obtain

$$\begin{aligned}&(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y)\\ &= b[\{(\nabla_X A)(Y) + (\nabla_Y A)(X)\}A(Z) + \{(\nabla_X A)(Z) + (\nabla_Z A)(X)\}A(Y)\\ &\quad + \{(\nabla_Y A)(Z) + (\nabla_Z A)(Y)\}A(X)] + c[\{(\nabla_X B)(Y)\\ &\quad + (\nabla_Y B)(X)\}A(Z) + \{(\nabla_X B)(Z) + (\nabla_Z B)(X)\}A(Y)\\ &\quad + \{(\nabla_Y B)(Z) + (\nabla_Z B)(Y)\}A(X) + \{(\nabla_X A)(Y)\\ &\quad + (\nabla_Y A)(X)\}B(Z) + \{(\nabla_X A)(Z) + (\nabla_Z A)(X)\}B(Y)\\ &\quad + \{(\nabla_Y A)(Z) + (\nabla_Z A)(Y)\}B(X)]\\ &\quad + d[(\nabla_X D)(Y,Z) + (\nabla_Y D)(Z,X) + (\nabla_Z D)(X,Y)].\end{aligned} \tag{3.12}$$

By virtue of (3.5)−(3.10) we obtain from (3.12) that

$$\begin{aligned}&(\nabla_X S)(Y,Z)+(\nabla_Y S)(Z,X)+(\nabla_Z S)(X,Y)\\&=d[(\nabla_X D)(Y,Z)+(\nabla_Y D)(Z,X)+(\nabla_Z D)(X,Y)].\end{aligned}$$

Thus we can state the following theorem:

Theorem 3.1. *If the generators of a $S(QE)_n$ are Killing vector fields and the associated scalars are constants, then the manifold satisfies cyclic parallel Ricci tensor iff the associated tensor D is cyclic parallel.*

4. The Generators U and V as Concurrent Vector Fields

A vector field ξ is said to be concurrent if [38]

$$\nabla_X \xi = \rho X, \tag{4.1}$$

where ρ is a non-zero constant. If $\rho = 0$, the vector field reduces to a parallel vector field.

In this section we consider the vector fields U and V corresponding to the associated 1-forms A and B respectively are concurrent. Then

$$(\nabla_X A)(Y) = \alpha g(X,Y) \tag{4.2}$$

and

$$(\nabla_X B)(Y) = \beta g(X,Y), \tag{4.3}$$

where α and β are non-zero constants.

Now using (4.1) and (4.2) in (3.11) we get

$$\begin{aligned}(\nabla_Z S)(X,Y) =& b[\alpha g(X,Z)A(Y)+\alpha g(Y,Z)A(X)]\\&+c[\beta g(Y,Z)A(X)+\alpha g(X,Z)B(Y)\\&+\alpha g(Y,Z)B(X)+\beta g(X,Z)A(Y)]\\&+d(\nabla_Z D)(X,Y).\end{aligned} \tag{4.4}$$

Contracting (4.4) over X and Y, we obtain that

$$dr(Z) = 2(b\alpha + c\beta)A(Z) + 2c\alpha B(Z), \tag{4.5}$$

where r is the scalar curvature of the manifold.

Now contracting (1.4) over X and Y we obtain that $r = an + b$. Since $a, b \in \mathbb{R}$, we obtain that $dr(X) = 0$, for all X.

Thus, equation (4.5) yields

$$(b\alpha + c\beta)A(Z) + c\alpha B(Z) = 0. \tag{4.6}$$

Since α and β are non-zero constants, using (4.6) in (1.4), we finally obtain

$$S(X,Y) = ag(X,Y) - \frac{2c\beta + b\alpha}{\alpha}A(X)A(Y) + dD(X,Y). \tag{4.7}$$

Thus the manifold reduces to a pseudo quasi-Einstein manifold. Hence we can state the following:

Theorem 4.1. *If the associated vector fields of a $S(QE)_n$ are concurrent vector fields and the associated scalars are constants, then the manifold reduces to a pseudo quasi-Einstein manifold.*

5. The Generators U and V as Recurrent Vector Fields

A vector fields ξ corresponding to the associated 1-form η is said to be recurrent if [38]

$$(\nabla_X \eta)(Y) = \psi(X)\eta(Y), \tag{5.1}$$

where ψ is a non-zero 1-form.

In this section we suppose that the generators U and V corresponding to the associated 1-forms A and B are recurrent. Then we have

$$(\nabla_X A)(Y) = \lambda(X)A(Y) \tag{5.2}$$

and

$$(\nabla_X B)(Y) = \mu(X)B(Y), \tag{5.3}$$

where λ and μ are non-zero 1-forms.

A non-flat Riemannian or semi-Riemannian manifold (M^n, g) $(n > 2)$ is called a generalized Ricci recurrent manifold [15] if its Ricci tensor S of type (0,2) satisfies the condition

$$(\nabla_X S)(Y,Z) = \gamma(X)S(Y,Z) + \delta(X)g(Y,Z), \tag{5.4}$$

where γ and δ are non-zero 1-forms. If $\delta = 0$, then the manifold reduces to a Ricci recurrent manifold[36].

Now, using (5.2) and (5.3) in (3.11) we get

$$\begin{aligned}(\nabla_Z S)(X,Y) =& 2b\lambda(Z)A(X)A(Y) + c[\mu(Z)A(X)B(Y) + \lambda(Z)A(X)B(Y) \\ &+ \lambda(Z)A(Y)B(X) + \mu(Z)B(X)A(Y)] + d(\nabla_Z D)(X,Y).\end{aligned} \tag{5.5}$$

We assume that the 1-forms λ and μ are equal, i.e.,

$$\lambda(Z) = \mu(Z), \tag{5.6}$$

for all Z. Then we obtain from (5.5) and (5.6) that

$$\begin{aligned}(\nabla_Z S)(X,Y) =& 2b\lambda(Z)A(X)A(Y)\\ &+ 2c\lambda(Z)[A(X)B(Y) + B(X)A(Y)] + d(\nabla_Z D)(X,Y).\end{aligned} \tag{5.7}$$

Using (1.4) and (5.7) we have

$$(\nabla_Z S)(X,Y) = \alpha_1(Z)S(X,Y) + \alpha_2(Z)g(X,Y) + d(\nabla_Z D)(X,Y),$$

where $\alpha_1(Z) = 2\lambda(Z)$ and $\alpha_2(Z) = -2a\lambda(Z)$.
Thus we can state the following:

Theorem 5.1. *If the generators of a $S(QE)_n$ corresponding to the associated 1-forms are recurrent with the same vector of recurrence and the associated scalars are constants with the additional condition D is covariant constant, then the manifold is a generalized Ricci recurrent manifold.*

6. $S(QE)_4$ Spacetimes

In a smooth manifold (M^n, g) Petrov [37] introduced a tensor $\tilde{P}$ of type $(0,4)$ and defined by the following:

$$\tilde{P} = \tilde{R} + \frac{\kappa}{2} g \wedge T - \sigma G, \tag{6.1}$$

where $\tilde{R}$ is the curvature tensor of type $(0,4)$, T is the energy momentum tensor of type $(0,2)$, κ is the gravitational constant, σ is the energy density, G is a tensor of type (0,4) given by

$$G(X,Y,Z,W) = g(Y,Z)g(X,W) - g(X,Z)g(Y,W), \tag{6.2}$$

for all $X, Y, Z, W \in \chi(M)$ and Kulkarni-Nomizu product $E \wedge F$ of two (0,2) tensors E and F is defined by

$$\begin{aligned}(E \wedge F)(X,Y,Z,W) =& E(Y,Z)F(X,W) + E(X,W)F(Y,Z)\\ &- E(X,Z)F(Y,W) - E(Y,W)F(X,Z),\end{aligned} \tag{6.3}$$

where $X, Y, Z, W \in \chi(M)$. The tensor $\tilde{P}$ is known as the space-matter tensor of type (0,4) of the manifold M. The space-matter tensor have been studied by Ahsan and Siddiqui [1, 2] and many others.

In this section we first study $M(QE)_4$ spacetime with vanishing space-matter tensor. Now equation (6.1) can also be written as

$$\begin{aligned}\tilde{P}(X,Y,Z,W) =& \tilde{R}(X,Y,Z,W) + \frac{\kappa}{2}[g(Y,Z)T(X,W) + g(X,W)T(Y,Z)\\ &- g(X,Z)T(Y,W) - g(Y,W)T(X,Z)]\\ &- \sigma[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)].\end{aligned} \tag{6.4}$$

If $\tilde{P} = 0$, then (6.4) yields

$$\begin{aligned}\tilde{R}(X,Y,Z,W) = &-\frac{\kappa}{2}[g(Y,Z)T(X,W) + g(X,W)T(Y,Z) \\ &- g(X,Z)T(Y,W) - g(Y,W)T(X,Z)] \\ &+ \sigma[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \end{aligned} \tag{6.5}$$

The Einstein's field equation without cosmological constant is given by [32, 33]

$$S(X,Y) - \frac{r}{2}g(X,Y) = \kappa T(X,Y), \tag{6.6}$$

where κ is the gravitational constant and r is the scalar curvature of the spacetime.

Now using (1.4) and Einstein's field equation (6.6) in (6.5) we obtain

$$\begin{aligned}\tilde{R}(X,Y,Z,W) =&(\sigma - a + \frac{r}{2})[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &- \frac{b}{2}[g(Y,Z)A(X)A(W) + g(X,W)A(Y)A(Z) \\ &- g(X,Z)A(Y)A(W) - g(Y,W)A(X)A(Z)] \\ &- \frac{c}{2}[g(Y,Z)\{A(X)B(W) + A(W)B(X)\} \\ &+ g(X,W)\{A(Y)B(Z) + A(Z)B(Y)\} \\ &- g(X,Z)\{A(Y)B(W) + A(W)B(Y)\} \\ &- g(Y,W)\{A(X)B(Z) + A(Z)B(X)\}] \\ &- \frac{d}{2}[g(Y,Z)D(X,W) + g(X,W)D(Y,Z) \\ &- g(X,Z)D(Y,W) - g(Y,W)D(X,Z)]. \end{aligned} \tag{6.7}$$

In virtue of (1.9) it follows from (6.7) that the manifold under consideration is a manifold of super quasi-constant curvature.

Thus we can state the following:

Theorem 6.1. *A $S(QE)_4$ spacetime obeying Einstein's field equation with vanishing space-matter tensor is a spacetime of super quasi-constant curvature.*

But it has been proved by Shaikh[39] that if a viscous fluid pseudo quasi-Einstein spacetime obeys Einstein's field equation with a cosmological constant, then none of the energy density and isotropic pressure of the fluid can be a constant.

In view of Theorem 4.1 and the result of Shaikh leads to the following theorem:

Theorem 6.2. *If the associated vector fields of a viscous fluid $S(QE)_4$ spacetime are concurrent vector fields and the associated scalars are constants and obeys Einstein's field equation with a cosmological constant, then none of the energy density and isotropic pressure of the fluid can be a constant.*

This result has significant agreement with the recent day observational truth that energy density and isotropic pressure can not be a constant.

Here we discuss the form of energy momentum tensor [27] as follows:

$$T_{ij} = (p+\mu)v_i v_j + p g_{ij} + dP_{ij}, \tag{6.8}$$

where v_i is the velocity of the fluid (a timelike unit vector field) and P_{ij} is the anisotropic stress tensor: symmetric, traceless and such that $P_{ij}v^j = 0$. We omit the possibility of heat transfer that adds a term $q_i v_j + q_j v_i$ to the tensor T_{ij}, with $q^i v_i = 0$. This term breaks the property that v^j is an eigenvector of T_{ij}. In Theorem 4.1 we prove that if the associated vector fields of a $S(QE)_n$ are concurrent vector fields and the associated scalars are constants, then the manifold reduces to a pseudo quasi-Einstein manifold.

Hence we conclude that if the associated vector fields of a $S(QE)_4$ spacetime are concurrent vector fields and the associated scalars are constants, then the spacetime represents imperfect fluid spacetime.

7. Example of a $S(QE)_4$ Spacetimes

In this section we prove the existence of a $S(QE)_4$ spacetime by constructing a non-trivial concrete example.

We consider a Lorentzian manifold (M^4, g) endowed with the Lorentzian metric g given by

$$ds^2 = g_{ij}dx^i dx^j = (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^2)^2(dx^3)^2 - (dx^4)^2, \tag{7.1}$$

where $i, j = 1, 2, 3, 4$.

The only non-vanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor are

$$\Gamma^1_{22} = -x^1,\ \ \Gamma^2_{33} = -\frac{x^2}{(x^1)^2},\ \ \Gamma^2_{12} = \frac{1}{x^1},\ \ \Gamma^3_{23} = \frac{1}{x^2},$$

$$R_{1332} = -\frac{x^2}{x^1},\ \ S_{12} = -\frac{1}{x^1 x^2}.$$

We take the scalars a, b, c and d as follows:

$$a = \frac{1}{x^1},\ b = -\frac{4}{x^1},\ c = x^1 x^2,\ d = -\frac{1}{(x^1)^2}.$$

We choose the 1-forms as follows:

$$A_i(x) = \begin{cases} 0, & \text{for i=1,2,3} \\ 1, & \text{for i=4} \end{cases}$$

and

$$B_i(x) = \begin{cases} \frac{x^1}{\sqrt{2}}, & \text{for i=2} \\ \frac{x^2}{\sqrt{2}}, & \text{for i=3} \\ 0, & \text{for i=1,4} \end{cases}$$

at any point $x \in M$.

We take the associated tensor as follows:

$$D_{ij}(x) = \begin{cases} 1, & \text{for i=j=1,3} \\ -2, & \text{for i=j=2} \\ \frac{x^1}{x^2}, & \text{for i=1, j=2} \\ 0, & \text{otherwise} \end{cases}$$

at any point $x \in M$. From (1.4) we have

$$S_{12} = ag_{12} + bA_1A_2 + c[A_1B_2 + A_2B_1] + dD_{12} \tag{7.2}$$

It can be easily prove that the equation (7.2) is true. Clearly, the trace of the $(0, 2)$ tensor D is zero.

We shall now show that the 1-forms are unit and orthogonal.

Here,

$$g^{ij}A_iA_j = -1, \;\; g^{ij}B_iB_j = 1, \;\; g^{ij}A_iB_j = 0.$$

So, the manifold under consideration is a $S(QE)_4$ spacetime.

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