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Quaternionic Direction Curves

Burak Şahiner
Department of Mathematics, Manisa Celal Bayar University, Manisa 45140, Turkey
e-mail : burak.sahiner@cbu.edu.tr

Abstract. In this paper, we define new quaternionic associated curves called quaternionic principal-direction curves and quaternionic principal-donor curves. We give some properties and relationships between Frenet vectors and curvatures of these curves. For spatial quaternionic curves, we give characterizations for quaternionic helices and quaternionic slant helices by means of their associated curves.

1. Introduction

Curves with a mathematical relationship between them are called associated curves. The study of associated curves is an interesting and important research area of the fundamental theory of curves. Some properties, such as Frenet vectors and curvatures of original curves, can be characterized by using their associated curves. Within this area of interest, various associated curves have been defined, such as Bertrand curve mates, Mannheim partner curves and involute-evolute curve couples, and they have been studied in different spaces, such as Euclidean spaces, Minkowski spaces, and dual spaces [1, 3, 7, 12, 14, 17].

Recently, Choi and Kim [4] introduced a new associated curve for a given curve as the integral curve of the vector field generated by the Frenet frame along it. This associated curve has been called the direction curve. Using these associated curves provides a canonical method to construct general helices and slant helices, which are widely used in various research areas in science and nature. These curves have attracted many authors to begin to study them. While non-null direction curves have been studied by Choi *et al.* [5], null curves have been studied by Qian and Kim [15] in three-dimensional Minkowski space E_1^3. Körpınar *et al.* [11] used the Bishop frame to study these curves. Macit and Düldül [13] defined a W-direction curve by using a unit Darboux vector field W for a given curve and introduced a V-direction curve associated with a curve lying on the surface. They also studied

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direction curves in four-dimensional Euclidean space.

In this paper, we give the definition of quaternionic direction curves as the quaternionic integral curve of a quaternion-valued function generated by Frenet vectors for a given quaternionic curve. Then, by taking this quaternion-valued function as the principal normal vector field of the curve, we define principal-direction and principal-donor curves for spatial quaternionic and quaternionic curves. We provide relationships between Frenet vectors and curvatures of a given quaternionic curve and its quaternionic principal-direction curve, and then obtain some properties of these curves. Moreover, for spatial quaternionic curves, we provide characterizations for quaternionic helices and quaternionic slant helices with the aid of their associated curves.

2. Preliminaries

In this section, we give a brief summary of basic concepts concerning quaternionic curves and some definitions and theorems about these curves in Euclidean 3-space E^3 and in Euclidean 4-space E^4.

A real quaternion, q, is defined as $q = a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3$ where a_i $(i = 0, 1, 2, 3)$ are real numbers, and the basis $\{e_0, e_1, e_2, e_3\}$ has the following properties:

$$e_0 = 1,\ e_i \times e_i = -1,\ (i = 1, 2, 3) \text{ and } e_i \times e_j = -e_j \times e_i = e_k,\ (1 \leq i, j, k \leq 3)$$

where (ijk) is an even permutation of (123). The algebra for quaternions is denoted by Q.

A real quaternion can also be given the form $q = s_q + v_q$ where $s_q = a_0$ and $v_q = a_1e_1 + a_2e_2 + a_3e_3$ are the scalar part and vector part of q, respectively [6]. If $q = s_q + v_q$ and $p = s_p + v_p$ are two quaternions in Q. Then the quaternion product of q and p is given by [6]

$$q \times p = s_q s_p - \langle v_q, v_p \rangle + s_q v_p + s_p v_q + v_q \wedge v_p$$

where $\langle , \rangle$ and $\wedge$ denote the inner product and vector product of E^3, respectively.

The conjugate of $q = s_q + v_q$ is defined by $\bar{q} = s_q - v_q$. The term q is called a spatial quaternion whenever $q + \bar{q} = 0$ [2]. The quaternion inner product can be defined as follows:

$$h : Q \times Q \to \mathbb{R},\ \ h\,(q, p) = \frac{1}{2}(q \times \bar{p} + p \times \bar{q}).$$

The norm of a quaternion $q = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4$ is defined as [6]

$$\|q\|^2 = h(q, q) = q \times \bar{q} = \bar{q} \times q = a_1^2 + a_2^2 + a_3^2 + a_4^2.$$

If $\|q\| = 1$, then q is called unit quaternion.

Following the basic concepts above, we can give some definitions and theorems concerning quaternionic curves.

Definition 2.1.([2]) The three-dimensional Euclidean space E^3 is identified by the space of the spatial quaternions $\{q \in Q : \ q + \bar{q} = 0\}$. Let $I = [0,1]$ be an interval in $\mathbb{R}$, $s \in I$ be a parameter and Q a set of quaternions. A curve defined as $\alpha : I \subset \mathbb{R} \to Q$, $\alpha(s) = \sum_{i=1}^{3} \alpha_i(s) e_i$ is called a *spatial quaternionic curve.*

Definition 2.2.([16]) Let $\alpha : I \subset \mathbb{R} \to Q$ be a spatial quaternionic curve and $s \in I$ be the arc-length parameter of α. The Frenet vectors and curvatures of the spatial quaternionic curve $\alpha(s)$ can be given, respectively, as follows:

$$t(s) = \alpha'(s), \ n(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}, \ b(s) = t(s) \times n(s)$$

and

$$k(s) = \|\alpha'(s) \times \alpha''(s)\|, \ r(s) = \frac{h(\alpha'(s) \times \alpha''(s), \alpha'''(s))}{\|\alpha'(s) \times \alpha''(s)\|^2},$$

where the prime denotes the derivative with respect to s, $k(s)$ and $r(s)$, which are called the curvature and torsion of the spatial quaternionic curve $\alpha(s)$, respectively.

Moreover, the following relationship between the Frenet vectors holds [8]:

$$\begin{aligned} &t(s) \times t(s) = n(s) \times n(s) = b(s) \times b(s) = -1, \\ &t(s) \times n(s) = b(s) = -n(s) \times t(s), \\ &n(s) \times b(s) = t(s) = -b(s) \times n(s), \\ &b(s) \times t(s) = n(s) = -t(s) \times b(s). \end{aligned}$$

Theorem 2.3.([2]) *Let $\alpha : I \subset \mathbb{R} \to Q$, $\alpha(s) = \sum_{i=1}^{3} \alpha_i(s) e_i$ be an arc-lengthed spatial quaternionic curve with Frenet frame $\{t, \ n, \ b\}$ and curvatures $\{k, r\}$. Then the Frenet formulae of the quaternionic curve $\alpha(s)$ can be given in matrix form as follows:*

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & r \\ 0 & -r & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}.$$

Definition 2.4.([2]) The four-dimensional Euclidean space E^4 is identified by the space of the quaternions. Let $I = [0,1]$ be an interval in $\mathbb{R}$, $s \in I$ be a parameter and Q a set of quaternions. A curve defined by $\beta : I \subset \mathbb{R} \to Q$, $\beta(s) = \sum_{i=0}^{3} \beta_i(s) e_i$ is called a *quaternionic curve.*

Definition 2.5.([16]) Let $\beta : I \subset \mathbb{R} \to Q$, $\beta(s) = \sum_{i=0}^{3} \beta_i(s) e_i$ be a quaternionic curve and $s \in I$ be the arc-length parameter of β. The Frenet vectors and curvatures of

$\beta(s)$ can be given respectively as follows:

$$\begin{aligned}
T(s) &= \beta'(s),\\
N(s) &= \frac{\beta''(s)}{\|\beta''(s)\|},\\
B_1(s) &= \eta\, B_2(s) \times T(s) \times N(s), (\eta = \pm 1)\\
B_2(s) &= \eta \frac{T(s) \times N(s) \times \beta'''(s)}{\|T(s) \times N(s) \times \beta'''(s)\|},
\end{aligned}$$

and

$$\begin{aligned}
K(s) &= \|\beta''(s)\|,\\
k(s) &= \frac{\|T(s) \times N(s) \times \beta'''(s)\|}{\|\beta''(s)\|},\\
r(s) - K(s) &= \frac{h(\beta^{(iv)}(s), B_2(s))}{\|T(s) \times N(s) \times \beta'''(s)\|},
\end{aligned}$$

where the prime denotes the derivative with respect to s, $K(s)$, $k(s)$ and $r(s)-K(s)$, which are called the *principal curvature*, *torsion* and *bitorsion* of β, respectively, and $B_2(s) \times T(s) \times N(s)$ is the ternary product of the vectors.

Theorem 2.6.([2]) *Let $\beta : I \subset \mathbb{R} \to Q$, $\beta(s) = \sum_{i=0}^{3} \beta_i(s) e_i$ be a quaternionic curve in E^4 with Frenet frame $\{T, N, B_1, B_2\}$ and curvatures $\{K,\ k,\ r-K\}$. Then the Frenet formulae for the quaternionic curve $\beta(s)$ can be given in matrix form as follows:*

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & K & 0 & 0 \\ -K & 0 & k & 0 \\ 0 & -k & 0 & (r-K) \\ 0 & 0 & -(r-K) & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}.$$

Definition 2.7.([9]) A spatial quaternionic curve α is called a *spatial quaternionic helix* if its unit tangent vector t makes a constant angle with a fixed unit quaternion U.

Theorem 2.8.([9]) *Let α be a spatial quaternionic curve with nonzero curvatures. Then α is a spatial quaternionic helix if and only if the following applies:*

$$\frac{r}{k} = constant.$$

Definition 2.9.([10]) A spatial quaternionic curve α is called the *spatial quaternionic slant helix* if its unit normal vector n makes a constant angle with a fixed unit quaternion U.

Theorem 2.10.([10]) *Let α be a spatial quaternionic curve with nonzero curvatures. Then α is a spatial quaternionic slant helix if and only if the following applies:*

$$\frac{k^2}{(r^2+k^2)^{3/2}}\left(\frac{r}{k}\right)' = \text{constant}.$$

3. Spatial Quaternionic Direction Curves

In this section, we define spatial quaternionic principal-direction and principal-donor curves in Euclidean 3-space and obtain some relationships between these curves.

For a spatial quaternionic curve $\alpha : I \to E^3$ with Frenet frame $\{t, n, b\}$, consider a quaternion valued function X given by

$$X(s) = x(s)\ t(s) + y(s)\ n(s) + z(s)\ b(s), \tag{3.1}$$

where x, y and z are differentiable functions of s which is the arc-length parameter for α. It is assumed that X is unit, i.e, the following equality holds:

$$x^2(s) + y^2(s) + z^2(s) = 1. \tag{3.2}$$

By differentiating equation (3.2), we have the following:

$$xx' + yy' + zz' = 0. \tag{3.3}$$

The spatial quaternionic curve $\bar{\alpha}(\bar{s})$ defined by $\dfrac{d\bar{\alpha}}{d\bar{s}} = X(s)$ is called the spatial quaternionic integral curve of $X(s)$. Since X is unit, it is clear that the arclength parameter $\bar{s}$ of $\bar{\alpha}$ is equal to $s + c$, where c is a constant. Without loss of generality, we can assume that $\bar{s} = s$. The spatial quaternionic curve $\bar{\alpha}$ is unique up to translation.

Now, we can give the definitions for the spatial quaternionic X-direction curve, spatial quaternionic X-donor curve, spatial quaternionic principal-direction curve and the spatial quaternionic principal-donor curve in E^3 in the following sections.

Definition 3.1. Let α be a spatial quaternionic curve in E^3 and X be a quaternion-valued function satisfying equations (3.1) and (3.2). The spatial quaternionic integral curve $\bar{\alpha} : I \to E^3$ of X is called the *spatial quaternionic X-direction curve* of α. The curve α whose spatial quaternionic X-direction curve is $\bar{\alpha}$ is called the *spatial quaternionic X-donor curve* of $\bar{\alpha}$.

Definition 3.2. Let α be a spatial quaternionic curve in E^3. The spatial quaternionic integral curve $\bar{\alpha}$ of $n(s)$ in (3.1) is called the *spatial quaternionic principal-direction curve*. In other words, a spatial quaternionic principal-direction curve is a spatial quaternionic integral curve of $X(s)$ with $x(s) = z(s) = 0,\ y(s) = 1$ in (3.1). Moreover, α is called the *spatial quaternionic principal-donor curve* of $\bar{\alpha}$.

With the aid of Definition 3.2, we can produce the following theorem:

Theorem 3.3. *Let α be a spatial quaternionic curve in E^3 with Frenet frame $\{t, n, b\}$ and curvatures $\{k, r\}$, and let $\bar{\alpha}$ be the spatial quaternionic principal-direction curve of α with Frenet frame $\{\bar{t}, \bar{n}, \bar{b}\}$ and curvatures $\{\bar{k}, \bar{r}\}$. The principal curvature $\bar{k}$ and the torsion $\bar{r}$ of the spatial quaternionic principal-direction curve $\bar{\alpha}$ of α can be given, respectively, as follows:*

$$\bar{k} = \sqrt{k^2 + r^2} \ \ \textit{and} \ \ \bar{r} = \frac{k^2}{k^2 + r^2} \left(\frac{r}{k}\right)'.$$

Proof. Since $\bar{\alpha}$ is the spatial quaternionic principal-direction curve of α, the parameter s can be used as the arc-length parameter for both the spatial quaternionic curves α and $\bar{\alpha}$. From Definition 3.2, we know that $\bar{t} = \dfrac{d\bar{\alpha}}{ds} = n$. By taking the derivative, we have $\bar{t}' = n' = -kt + rb$, where the prime indicates the derivative with respect to s. Since $\bar{t}' = \bar{k}\bar{n}$, we get $\bar{k} = \sqrt{k^2 + r^2}$. The second and third derivates of $\bar{\alpha}$ are found respectively as follows:

$$\bar{\alpha}'' = n' = -kt + rb$$

and

$$\bar{\alpha}''' = -k't - k^2 n + r'b - r^2 n$$

From Definition 2.2, we get $\bar{r} = \dfrac{k^2}{k^2 + r^2} \left(\dfrac{r}{k}\right)'$. Note that $\bar{r}$ can also be found by using the equality $\bar{r} = -h(\bar{b}', \bar{n})$. Moreover, the principal normal and binormal vector fields of $\bar{\alpha}$ can also be obtained as

$$\bar{n} = -\frac{k}{\sqrt{k^2 + r^2}} t + \frac{r}{\sqrt{k^2 + r^2}} b,$$

and

$$\bar{b} = \bar{t} \times \bar{n} = \frac{r}{\sqrt{k^2 + r^2}} t + \frac{k}{\sqrt{k^2 + r^2}} b,$$

respectively. $\square$

On the other hand, the curvatures of spatial quaternionic principal-donor curve α can be given in terms of the curvatures of $\bar{\alpha}$ as given in following theorem.

Theorem 3.4. *Let α be the spatial quaternionic curve in E^3 with Frenet frame $\{t, n, b\}$ and curvatures $\{k, r\}$ and let $\bar{\alpha}$ be the spatial quaternionic principal-direction curve of α with Frenet frame $\{\bar{t}, \bar{n}, \bar{b}\}$ and curvatures $\{\bar{k}, \bar{r}\}$. The principal curvature k and the torsion r of spatial quaternionic principal-donor curve α of $\bar{\alpha}$ can be given, respectively, as follows:*

$$k(s) = \bar{k}(s) \cos\left(\int \bar{r}(s) ds\right) \ \ \textit{and} \ \ r(s) = \bar{k}(s) \sin\left(\int \bar{r}(s) ds\right).$$

Proof. Let the ratio $\dfrac{r(s)}{k(s)} = f(s)$. Then the torsion of the spatial quaternionic principal-direction curve can be rewritten as:

$$\bar{r}(s) = \frac{f'(s)}{1+f^2(s)}.$$

This means that $\displaystyle\int \bar{r}(s)ds = \arctan(f(s))$. So we get the following:

$$f(s) = \tan\left(\int \bar{r}(s)ds\right).$$

By putting $\dfrac{r(s)}{k(s)}$ instead of $f(s)$, using trigonometric relationships, and taking into account that $\bar{k} = \sqrt{k^2+r^2}$, we have:

$$\kappa(s) = \bar{\kappa}(s)\cos\left(\int \bar{\tau}(s)ds\right)$$

and

$$\tau(s) = \bar{\kappa}(s)\sin\left(\int \bar{\tau}(s)ds\right). \qquad \square$$

From Theorem 3.3, we have the following corollary.

Corollary 3.5. *Let α be a spatial quaternionic curve in E^3 with curvatures $\{k, r\}$ and let $\bar{\alpha}$ be a spatial quaternionic principal-direction curve of α with curvatures $\{\bar{k}, \bar{r}\}$. The following relationship is satisfied:*

$$\frac{\bar{r}}{\bar{k}} = \frac{k^2}{(k^2+r^2)^{3/2}}\left(\frac{r}{k}\right)'.$$

Thus, we can produce the following theorem that can be used to construct a spatial quaternionic slant helix from a spatial quaternionic helix by using the spatial quaternionic direction curves.

Theorem 3.6. *Let α be a spatial quaternionic curve with nonzero curvatures and let $\bar{\alpha}$ be the spatial quaternionic principal-direction curve of α. The curve α is a spatial quaternionic helix if and only if $\bar{\alpha}$ is a spatial quaternionic slant helix.*

Proof. The proof is clear from Corollary 3.5, Theorem 2.10, and Theorem 2.12. $\square$

Now, we can discuss the condition where the spatial quaternionic principal-direction curve of $\bar{\alpha}$ is equal to α, while $\bar{\alpha}$ is a spatial quaternionic integral curve of (3.1).

Proposition 3.7. *Let α be a spatial quaternionic curve and $\bar{\alpha}$ be a spatial quaternionic integral curve of* (3.1). *The spatial quaternionic principal-direction curve of $\bar{\alpha}$ is equal to α if and only if the functions $x(s)$, $y(s)$, $z(s)$ in* (3.1) *are as follows:*

$$x(s)=0,\;\; y(s)=\sin\left(\int\tau(s)ds\right),\;\; z(s)=\cos\left(\int\tau(s)ds\right).$$

Proof. By differentiating (3.1) with respect to s using the Frenet formulae for the spatial quaternionic curves α and $\bar{\alpha}$, and the fact that $X=\bar{t}$, we have the following:

$$\bar{k}\bar{n}=(x'-yk)t+(y'+xk-zr)n+(z'+yr)b.$$

On the other hand, since α is the spatial quaternionic principal-direction curve of $\bar{\alpha}$, i.e., $\alpha'=t=\bar{n}$, we have the following system of differential equations

$$\left.\begin{array}{l} x'-yk=\bar{k} \\ y'+xk-zr=0 \\ z'+yr=0 \end{array}\right\}. \tag{3.4}$$

Multiplying the first, second and third equations in (3.4) by x, y and z, respectively, and adding the results, we have:

$$xx'+yy'+zz'=x\bar{k}. \tag{3.5}$$

From equations (3.3) and (3.5), we have $x=0$. By substituting the variable x in system (3.4), the following solution is obtained:

$$x(s)=0,\;\; y(s)=\sin\left(\int\tau(s)ds\right),\;\; z(s)=\cos\left(\int\tau(s)ds\right).$$ □

4. Quaternionic Direction Curves

In this section, we give definitions of the quaternionic $V-$direction and quaternionic principal-direction curves and study the relationship between a given quaternionic curve and its quaternionic principal-direction curve.

Let $\beta: I\subset\mathbb{R}\to Q$, $\beta(s)=\sum_{i=0}^{3}\beta_i(s)e_i$ be a quaternionic curve in E^4 with Frenet frame $\{T,N,B_1,B_2\}$ and curvatures $\{K,\ k,\ r-K\}$, where $e_i,\ (i=0,1,2,3)$ are quaternionic units as mentioned in the Preliminaries. Consider a quaternion valued function V in E^4 given by

$$V(s)=u_1(s)T(s)+u_2(s)N(s)+u_3(s)B_1(s)+u_4(s)B_2(s), \tag{4.1}$$

where $u_i\ (i=1,2,3,4)$ are differentiable functions of s which is the arc-length parameter of β. If we take the quaternion valued function V as unit, the following equality holds

$$u_1^2(s)+u_2^2(s)+u_3^2(s)+u_4^2(s)=1. \tag{4.2}$$

By differentiating equation (4.2), we have

$$u_1u_1' + u_2u_2' + u_3u_3' + u_4u_4' = 0. \tag{4.3}$$

The quaternionic curve $\bar{\beta}(\bar{s})$ defined by $\dfrac{d\bar{\beta}}{d\bar{s}} = V(s)$ is called the quaternionic integral curve of $V(s)$. Since $V(s)$ is unit, it is clear that the arc-length parameter of $\bar{\beta}$ is equal to $s+c$, where c is a constant. Without loss of generality, we can assume that $\bar{s} = s$. The quaternionic curve $\bar{\beta}$ is unique up to translation in E^4.

Now, we can give definitions for the quaternionic V-direction, quaternionic V-donor, quaternionic principal-direction, and quaternionic principal-donor curves.

Definition 4.1. Let $\beta : I \subset \mathbb{R} \to Q$, $\beta(s) = \sum_{i=0}^{3} \beta_i(s)e_i$ be a quaternionic curve and $V(s)$ a quaternion-valued function in E^4 satisfying equations (4.1) and (4.2). The quaternionic integral curve $\bar{\beta} : I \subset \mathbb{R} \to Q$ of $V(s)$ is called the *quaternionic V-direction curve* of β. The quaternionic curve β whose quaternionic V-direction curve is $\bar{\beta}$ is called the *quaternionic V-donor curve* of $\bar{\beta}$ in E^4.

Definition 4.2. Let $\beta : I \subset \mathbb{R} \to Q$, $\beta(s) = \sum_{i=0}^{3} \beta_i(s)e_i$ be a quaternionic curve in E^4. A quaternionic curve $\bar{\beta}$ of $N(s)$ in (4.1) is called the *quaternionic principal-direction curve* of β, and β is called the *quaternionic principal-donor curve* of $\bar{\beta}$.

By using Definition 4.2, we can produce the following theorem:

Theorem 4.3. *Let β be a quaternionic curve in E^4 with Frenet frame $\{T, N, B_1, B_2\}$ and curvatures $\{K,\ k,\ r-K\}$ and $\bar{\beta}$ be the quaternionic principal-direction curve of β with Frenet frame $\{\bar{T}, \bar{N}, \bar{B}_1, \bar{B}_2\}$ and curvatures $\{\bar{K},\ \bar{k},\ \bar{r}-\bar{K}\}$. The principal curvature $\bar{K}$, the torsion $\bar{k}$, and the bitorsion $\bar{r}-\bar{K}$ of the quaternionic principal-direction curve $\bar{\beta}$ can be given respectively as:*

$$\bar{K} = \sqrt{K^2 + k^2},$$

$$\bar{k} = \frac{\sqrt{k^4(r-K)^2 + K^2k^2(r-K)^2 + (-Kk' + kK')^2}}{K^2 + k^2},$$

$$\bar{r} - \bar{K} = \frac{\sqrt{K^2+k^2}\left\{k\left[(r-K)(-K''k + k''K - Kk(r-K)^2 + 2K'k') + (r-K)'(-2Kk' + kK')\right] - 2Kk'^2(r-K)\right\}}{k^4(r-K)^2 + K^2k^2(r-K)^2 + (-Kk' + kK')^2}$$

Proof. Since $\bar{\beta}$ is the quaternionic principal-direction curve of β, the parameter s can be used for the arc-length parameter of both quaternionic curves β and $\bar{\beta}$. Since $\bar{\beta}$ is the quaternionic principal-direction curve of β, we have $\bar{\beta}' = \bar{T} = N$. By differentiating this equality and using Frenet formulas, we get $\bar{T}' = \bar{K}\bar{N} = N' = -KT + kB_1$. So the principal curvature of $\bar{\beta}$ is obtained as $\bar{K} = \sqrt{K^2 + k^2}$. The principal normal vector field of $\bar{\beta}$ is also found to be:

$$\bar{N} = -\frac{K}{\sqrt{K^2 + k^2}}T + \frac{k}{\sqrt{K^2 + k^2}}B_1.$$

By taking the third order derivative of $\bar{\beta}$ and using Frenet formulas, we get the following:

$$\bar{\beta}''' = -K'T - (K^2 + k^2)N + k'B_1 + k(r-K)B_2.$$

By using Definition 2.5, the torsion and bitorsion of the quaternionic principal-direction curve $\bar{\beta}$ can be respectively found as follows:

$$\bar{k} = \frac{\sqrt{k^4(r-K)^2 + K^2k^2(r-K)^2 + (-Kk' + kK')^2}}{K^2 + k^2}$$

and

$$\bar{r} - \bar{K} = \frac{\sqrt{K^2+k^2}\left\{k\left[(r-K)(-K''k + k''K - Kk(r-K)^2 + 2K'k') + (r-K)'(-2Kk' + kK')\right] - 2Kk'^2(r-K)\right\}}{k^4(r-K)^2 + K^2k^2(r-K)^2 + (-Kk' + kK')^2}$$

Moreover, the first and second binormal vector fields of the quaternionic principal-direction curve $\bar{\beta}$ can be obtained respectively as follows:

$$\bar{B}_1 = \frac{1}{\sqrt{k^4(r-K)^2 + K^2k^2(r-K)^2 + (-Kk' + kK')^2}}\left[\frac{k(kK' - k'K)}{\sqrt{K^2+k^2}}T + \frac{K(kK' - k'K)}{\sqrt{K^2+k^2}}B_1 - \frac{k^3(r-K) + K^2k(r-K)}{\sqrt{K^2+k^2}}B_2\right],$$

and

$$\bar{B}_2 = \frac{1}{\sqrt{k^4(r-K)^2 + K^2k^2(r-K)^2 + (-Kk' + kK')^2}}\left[\frac{k^2(r-K)}{\sqrt{K^2+k^2}}T + \frac{Kk(r-K)}{\sqrt{K^2+k^2}}B_1 + \frac{-Kk' + kK'}{\sqrt{K^2+k^2}}B_2\right], \qquad \square$$

Now, we can discuss the condition where the quaternionic principal-direction curve of $\bar{\beta}$ is equal to β, while $\bar{\beta}$ is a quaternionic integral curve of (4.1).

Proposition 4.4. *Let β be a quaternionic curve and $\bar{\beta}$ be a quaternionic integral curve of* (4.1)*. The quaternionic principal-direction curve of $\bar{\beta}$ is equal to β if and only if $u_1 = 0$ and the following system of differential equations is satisfied:*

$$\left.\begin{aligned} u_2' &= ku_3 \\ u_3' &= -ku_2 + (r-K)u_4 \\ u_4' &= -(r-K)u_3 \end{aligned}\right\}$$

where $u_i, (i = 1,2,3,4)$ in (4.1)*.*

Proof. By differentiating (4.1) with respect to s and using Theorem 2.6, we have

$$V' = (u_1' - u_2K)T + (u_1K + u_2' - u_3k)N + (u_2k + u_3' - u_4(r-K))B_1 + (u_3(r-K) + u_4')B_2.$$

Since $\bar{\beta}$ is a quaternionic integral curve of (4.1), $\bar{\beta}'' = \bar{T}' = \bar{K}\bar{N} = V'$. On the other hand, since the quaternionic principal-direction curve of $\bar{\beta}$ is equal to β, we get $\beta' = T = \bar{N}$. Thus, we have the following system of differential equations:

$$\left.\begin{aligned} u_1' &= Ku_2 + \bar{K} \neq 0 \\ u_2' &= -Ku_1 + ku_3 \\ u_3' &= -ku_2 + (r-K)u_4 \\ u_4' &= -(r-K)u_3 \end{aligned}\right\}. \tag{4.4}$$

Multiplying the first, second, third and fourth equations in (4.4) by $u_1,\ u_2,\ u_3,\ u_4$, respectively, and adding the results, we obtain the following:

$$u_1u_1' + u_2u_2' + u_3u_3' + u_4u_4' = \bar{K}u_1. \tag{4.5}$$

From equations (4.3) and (4.5), we obtain $u_1 = 0$. Thus, system (4.4) can be rewritten as follows:

$$\left.\begin{aligned} u_2 &= -\frac{\bar{K}}{K} \neq 0 \\ u_2' &= ku_3 \\ u_3' &= -ku_2 + (r-K)u_4 \\ u_4' &= -(r-K)u_3 \end{aligned}\right\}. \tag{4.6}$$

Specifically, if the ratio $\dfrac{\bar{K}}{K}$ is known, a solution of system (4.6) can be found as follows. By changing the variable $u(s) = \int_0^s (r(s) - K(s))ds$, system (4.6) can be rewritten:

$$\left.\begin{aligned} \frac{du_2}{du} &= \frac{k}{r-K}u_3 \\ \frac{du_3}{du} &= -\frac{k}{r-K}u_2 + u_4 \\ \frac{du_4}{du} &= -u_3 \end{aligned}\right\}. \tag{4.7}$$

By differentiating the second equation and using the third equation of system (4.7), we get

$$\frac{d^2u_3}{du^2} + u_3 = \frac{d}{du}(A(u)), \tag{4.8}$$

where $A(u) = \dfrac{k\bar{K}}{(r-K)K}$. By solving differential equation (4.8), we find the following:

$$u_3 = \cos u \int A(u)\cos u du + \sin u \int A(u) \sin u du.$$

By using the third equation of system (4.7), we get

$$u_4 = -\sin u \int A(u)\cos u du + \cos u \int A(u) \sin u du.$$

Thus, we have a solution of system (4.4):

$$\left.\begin{aligned} u_1 &= 0, \\ u_2 &= -\frac{\bar{K}}{K}, \\ u_3 &= \cos u \textstyle\int A(u)\cos u du + \sin u \int A(u)\sin u du, \\ u_4 &= -\sin u \textstyle\int A(u)\cos u du + \cos u \int A(u)\sin u du, \end{aligned}\right\}$$

where $u(s) = \int_0^s (r(s) - K(s))ds$. $\square$

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