KYUNGPOOK Math. J. 58(2018), 389-398
https://doi.org/10.5666/KMJ.2018.58.2.389
pISSN 1225-6951 eISSN 0454-8124

From the Eisenhart Problem to Ricci Solitons in Quaternion Space Forms

Mundalamane Manjappa Praveena
Department of Mathematics, Sri Venkateshwara College of Engineering, Bengaluru-562157, Karnataka, India
e-mail : mmpraveenamaths@gmail.com.in

Channabasappa Shanthappa Bagewadi*
Department of Mathematics, Kuvempu University, Shankaraghatta-577451, Shimoga, Karnataka, India
e-mail : prof_bagewadi@yahoo.co.in

Abstract. In this paper we obtain the condition for the existence of Ricci solitons in non-flat quaternion space form by using Eisenhart problem. Also it is proved that if (g, V, λ) is Ricci soliton then V is solenoidal if and only if it is shrinking, steady and expanding depending upon the sign of scalar curvature. Further it is shown that Ricci soliton in semi-symmetric quaternion space form depends on quaternion sectional curvature c if V is solenoidal.

1. Introduction

During 1982, Hamilton [7] made the fundamental observation that Ricci flow is an excellent tool for simplifying the structure of the manifold. It is defined for Riemannain manifolds of any dimension. It is a process which deforms the metric of a Riemannian manifold analogous to the diffusion of heat there by smoothing out the regularity in the metric. It is given by

$$\frac{\partial g(t)}{\partial t} = -2Ric(g(t)).$$

where g is Riemannian metric dependent on time t and $Ric(g)$ is Ricci tensor.

* Corresponding Author.
Received June 19, 2016; revised April 22, 2018; accepted April 25, 2018.
2010 Mathematics Subject Classification: 53C15, 53C21, 53C35, 53C55, 53C56.
Key words and phrases: quaternion Kähler manifolds, quaternion space form, parallel second order covariant tensor field, Einstein space, Ricci soliton.

Let $\tilde{\phi}_t : M \longrightarrow M$, $t \in R$ be a family of diffeomorphisms and $(\tilde{\phi}_t : t \in R)$ is a one parameter family of abelian groups called flow. It generates a vector field X_p given by

$$X_p f = \frac{df(\tilde{\phi}_t(p))}{dt}, \quad f \in C^\infty(M).$$

If Y is a vector field then $L_X Y = lim_{t\to 0}\frac{\tilde{\phi}_t^* Y - Y}{t}$ is known as Lie derivative of Y with respect to X. Ricci solitons are stationary points of the Ricci flow under $\tilde{\phi}_t : M \longrightarrow M$ in space of metrics, that is if g_0 is a metric on the codomain then $g(t) = \tilde{\phi}_t^* g_0$ the pullback of g_0, is a metric on the domain.

A Ricci soliton is a generalization of an Einstein metric and is defined on a Riemannian manifold (M, g). It is a triple (g, V, λ) with g a Riemannian metric, V a vector field generated by $\tilde{\phi}_t$ and λ a real scalar such that

$$L_V g + 2S + 2\lambda g = 0. \tag{1.1}$$

where S is a Ricci tensor of M, L_V denotes the Lie derivative operator along the vector field V. The Ricci soliton is said to be shrinking, steady and expanding when λ is negative, zero and positive respectively.

In 1923, Eisenhart [6] proved that if a positive definite Riemannian manifold (M, g) admits a second order parallel symmetric covariant tensor other than a constant multiple of the metric tensor then it is reducible. In 1925, Levy [10] obtained the necessary and sufficient conditions for the existence of such tensors. Since then, many authors both national and international have investigated the Eisenhart problem of finding symmetric and skew-symmetric parallel tensors on various spaces and obtained fruitful results. For instance, by giving a global approach based on the Ricci identity, R. Sharma [11] investigated Eisenhart problem on non-flat real and complex space forms, in 1989. Using L.P. Eisenhart problem C. Calin and M. Crasmareanu [3], C. S. Bagewadi and G. Ingalahalli [2, 8] and S. Debnath and A. Bhattacharyya [4] have studied the existence of Ricci solitons in f-Kenmotsu manifolds, α-Sasakian, Lorentzian α-Sasakian and Trans-Sasakian manifolds.

Motivated by these ideas, in this paper, we study Ricci solitons of quaternion space form which is a generalization of complex space form using L.P. Eisenhart problem. The paper is organized as follows: The section two consists of definitions, notions and basic results of quaternion space form. The third section deals with parallel symmetric second order covariant tensor and Ricci soliton in a non-flat quaternion space form. In section 4 we give semi-symmetry on quaternion space forms.

2. Preliminaries

Let M be an n$(n = 4m, m \geq 1)$-dimensional Riemannian manifold with the Riemannian metric g, then M is called a *quaternion Kaehlerian manifold* if there exists a 3-dimensional vector bundle μ consisting of tensors of type $(1, 1)$ with local basis of almost Hermitian structures J, K and L such that [1, 9, 13]

(a)

$$J^2 = K^2 = L^2 = -I, \tag{2.1}$$

$$JK = -KJ = L, \;\; KL = -LK = J, \;\; LJ = -JL = K, \tag{2.2}$$

$$g(JX, JY) = g(KX, KY) = g(LX, LY) = g(X, Y), \tag{2.3}$$

where I denoting the identity tensor of type $(1, 1)$ on M.
(b) If $\vec{\phi}$ is a cross-section of the bundle μ, then $\nabla_X\vec{\phi}$ is also a cross-section of the bundle μ, X being an arbitrary vector field on M and ∇ the Riemannian connection on M.
Further the condition **(b)** is equivalent to the following condition;
(c) There exist the local 1-forms p, q and r such that

$$\begin{aligned} \nabla_X J &= r(X)K - q(X)L, \\ \nabla_X K &= -r(X)J + p(X)L, \\ \nabla_X L &= q(X)J - p(X)K. \end{aligned}$$

Let X be a unit vector tangent to the quaternion Kaehlerian manifold M , then $X; JX; KX$ and LX form an orthonormal frame. We denote by $Q(X)$ the 4-plane spanned by them and call it the *quaternion 4-plane* determined by X. Every plane in a quaternion 4-plane is called a *quaternion plane*. The sectional curvature formed by the quaternion plane is called *quaternion sectional curvature*.

A quaternion Kaehlerian manifold is called a *quaternion-space-form* if its quaternion sectional curvatures are equal to a constant c. It is known that a quaternion Kaehlerian manifold is a quaternion-space-form if and only if its curvature tensor R is of the following form:

$$\begin{aligned} R(X,Y)Z = \frac{c}{4}[&g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY \\ &+ 2g(X,JY)JZ + g(KY,Z)KX - g(KX,Z)KY + 2g(X,KY)KZ \\ &+ g(LY,Z)LX - g(LX,Z)LY + 2g(X,LY)LZ]. \end{aligned} \tag{2.4}$$

The formulae [13]

$$R(X, Y) = R(JX, JY) = R(KX, KY) = R(LX, LY), \tag{2.5}$$

$$S(X, Y) = S(JX, JY) = S(KX, KY) = S(LX, LY), \tag{2.6}$$

$$\begin{aligned} &S(X, JY) + S(JX, Y) = 0, S(X, KY) + S(KX, Y) = 0 \;\; \textit{and} \\ &S(X, LY) + S(LX, Y) = 0, \end{aligned} \tag{2.7}$$

are well known for a quaternion Kaehler manifold where S is a Ricci tensor. If $\{e_1, e_2....e_n\}$ is an orthonormal basis for the tangent space to M then the scalar function r is given by $r = \sum_{i=1}^{n} S(e_i, e_i)$.

Symmetry of the manifold is one of the most important property among all its geometrical properties. Symmetry of the manifold basically depends on curvature tensor and the Ricci tensor of the manifold.

Definition 2.1.([5, 12]) An n-dimensional generalized complex space form is said to be *Semi-symmetric* if it satisfies

$$(R(X,Y)\cdot R)(U,V,W)=0, \quad forall\ X,\ Y\ \in\ \chi(M),$$

Definition 2.2. We shall give the concepts of symmetrization and anti-symmetri zation as below:
If ϕ is real valued p-linear function on the tangent space to M then ϕ is a covariant tensor of order p. Denote the set of all covariant tensor of order p by $\bigotimes^{\phi} T_xM$. We know that $\bigotimes^{\phi} T_xM$ is a vector space. Define two linear transformations S_p and A_p on the vector space $\bigotimes^{\phi} T_xM$ by $S_p = \sum_{\sigma\in\gamma_p}\sigma\phi$ and $A_p = \sum_{\sigma\in\gamma_p}\varepsilon\sigma\phi$ respectively, where γ_p is a permutation group on $\{1......p\}$ and ε is the signature of the permutation σ. These are called as *symmetrization and anti-symmetrization operators.*

3. Parallel Symmetric Second Order Covariant Tensor and Ricci Soliton in a Non-flat Quaternion Space Forms

Let h be a $(0,2)$-tensor which is parallel with respect to ∇ that is $\nabla h = 0$. Applying the Ricci identity [11]

$$\nabla^2 h(X,Y;Z,W) - \nabla^2 h(X,Y;W,Z) = 0, \tag{3.1}$$

we obtain the relation[11]:

$$h(R(X,Y)Z,W) + h(Z,R(X,Y)W) = 0. \tag{3.2}$$

Plugging the value of R from (2.4) in (3.2) and putting $X = W = e_i$, $1 \le i \le n$ and after simplification, we get

$$\begin{aligned} &g(Y,Z)(tr.H) - h(Y,Z) - (n+8)h(Z,Y) + g(JY,Z)tr.(HJ) + h(JY,JZ)\\ &+2h(JZ,JY) + g(KY,Z)tr.(HK) + h(KY,KZ) + 2h(KZ,KY)\\ &+g(LY,Z)tr.(HL) + h(LY,LZ) + 2h(LZ,LY) = 0 \end{aligned} \tag{3.3}$$

where H is a $(1,1)$ tensor metrically equivalent to h. Symmetrization and anti-symmetrization of (3.3) yield:

$$\begin{aligned} g(Y,Z)(tr.H) = {} & (n+9)h_s(Y,Z) - 3h_s(JY,JZ)\\ & - 3h_s(KY,KZ) - 3h_s(LY,LZ), \end{aligned} \tag{3.4}$$

$$\begin{aligned} g(JY,Z)tr.(HJ) &+ g(KY,Z)tr.(HK) + g(LY,Z)tr.(HL)\\ &= (n+7)h_a(Z,Y) - h_a(JZ,JY)\\ &\quad - h_a(KZ,KY) - h_a(LZ,LY), \end{aligned} \tag{3.5}$$

where h_s and h_a denote the symmetric and anti-symmetric parts of h.

Replacing Y and Z by JY and JZ, KY and KZ, LY and LZ respectively in (3.4) and adding the resulting equations from (3.4), we get:

$$g(Y,Z)(tr.H) = (n+6)h_s(Y,Z) - 3h_s(KY,KZ) - 3h_s(LY,LZ), \tag{3.6}$$

$$g(Y,Z)(tr.H) = (n+6)h_s(Y,Z) - 3h_s(JY,JZ) - 3h_s(LY,LZ), \tag{3.7}$$

$$g(Y,Z)(tr.H) = (n+6)h_s(Y,Z) - 3h_s(JY,JZ) - 3h_s(KY,KZ). \tag{3.8}$$

Again changing Y, Z by KY, KZ in (3.6) and adding the resultant equation to (3.6), we obtain:

$$\begin{aligned}(n+9)g(Y,Z)(tr.H) = ((n+6)^2 - 9)h_s(Y,Z) \\ - 3(n+6)h_s(LY,LZ) - 9h_s(JY,JZ).\end{aligned} \tag{3.9}$$

Multiply the equation (3.7) by -3 and adding it with (3.9) we obtain the expression:

$$g(Y,Z)(tr.H) = \frac{(n^2+9n+9)}{(n+6)}h_s(Y,Z) - \frac{(3n+9))}{(n+6)}h_s(LY,LZ). \tag{3.10}$$

Substituting Y, Z by LY, LZ in (3.10) and adding the resultant equation to (3.10), we get the relation;

$$h_s(Y,Z) = \frac{(tr.H)}{n}g(Y,Z). \tag{3.11}$$

Likewise; changing Y and Z by JY and JZ, KY and KZ, LY and LZ respectively in (3.5) and adding the resultant equations to (3.5), we obtain:

$$\begin{aligned}g(Y,JZ)tr.(HJ) + g(Y,KZ)tr.(HK) + g(Y,LZ)tr.(HL) \\ = (n+6)h_a(Y,Z) - h_a(KY,KZ) - h_a(LY,LZ),\end{aligned} \tag{3.12}$$

$$\begin{aligned}g(Y,JZ)tr.(HJ) + g(Y,KZ)tr.(HK) + g(Y,LZ)tr.(HL) \\ = (n+6)h_a(Y,Z) - h_a(JY,JZ) - h_a(LY,LZ),\end{aligned} \tag{3.13}$$

$$\begin{aligned}g(Y,JZ)tr.(HJ) + g(Y,KZ)tr.(HK) + g(Y,LZ)tr.(HL) \\ = (n+6)h_a(Y,Z) - h_a(KY,KZ) - h_a(JY,JZ).\end{aligned} \tag{3.14}$$

Again replacing Y, Z by KY, KZ in (3.12) and adding the resultant equation to (3.12), we obtain

$$\begin{aligned}(n+7)[g(Y,JZ)tr.(HJ) + g(Y,KZ)tr.(HK) + g(Y,LZ)tr.(HL)] \\ = [(n+6)^2 - 1]h_a(Y,Z) - (n+6)h_a(LY,LZ) - h_a(JY,JZ).\end{aligned} \tag{3.15}$$

Substituting (3.13) in (3.15), we obtain the expression:

$$\begin{aligned}g(Y,JZ)tr.(HJ) + g(Y,KZ)tr.(HK) + g(Y,LZ)tr.(HL) \\ = \frac{n^2+11n-29}{(n+6)}h_a(Y,Z) - \frac{(n+5)}{(n+6)}h_a(LY,LZ).\end{aligned} \tag{3.16}$$

Substituting Y, Z by LY, LZ in (3.16) and adding the resultant equation to (3.16), we get

$$h_a(Y,Z) = \frac{(n+6)}{(n^2+10n-34)}[g(Y,JZ)tr.(HJ) + g(Y,KZ)tr.(HK) + g(Y,LZ)tr.(HL)] \tag{3.17}$$

By summing (3.11) and (3.17) we obtain the expression

$$h = \frac{1}{n}(tr.H)g + \alpha[g(Y,JZ)tr.(HJ) + g(Y,KZ)tr.(HK) + g(Y,LZ)tr.(HL)], \tag{3.18}$$

where $\alpha = \frac{(n+6)}{(n^2+10n-34)}$.
The above equation implies

$$h = \left[\frac{1}{n}(tr.H)g + \alpha[tr.(HJ)\Omega_1 + tr.(HK)\Omega_2 + tr.(HL)\Omega_3]\right], \tag{3.19}$$

where $\Omega_1 = g(Y,JZ), \Omega_2 = g(Y,KZ)$ and $\Omega_3 = g(Y,LZ)$ are quaternion Kaehlerian 2-forms.

Now as H is parallel with respect to ∇, therefore $tr.H, tr.(HJ), tr.(HK)$ and $tr.(HL)$ are constants. Thus from equation (3.19) we can state the following

Theorem 3.1. *A second order parallel tensor in a non-flat quaternion space form is a linear combination(with constant coefficients) of the underlying quaternion Kaehlerian metric and quaternion Kaehlerian 2-forms.*

Corollary 3.1. *The only symmetric(anti-symmetric) parallel tensor of type* $(0,2)$ *in a non-flat quaternion space form is the quaternion Kaehlerian metric (quaternion Kaehlerian 2-forms) up to a constant multiple.*

Corollary 3.2. *A locally Ricci symmetric* ($\nabla S = 0$) *non-flat quaternion space form is an Einstein manifold.*

Proof. If $h = S$ in (3.19) then $tr.H = r$, $tr.HJ = 0$, $tr.HK = 0$ and $tr.HL = 0$ by virtue of (2.7). Equation (3.19) can be written as

$$S(Y,Z) = \frac{r}{n}g(Y,Z). \tag{3.20}$$

□

Remark 3.1. The following statements for non-flat quaternion space form are equivalent:

(1) Einstein.

(2) locally Ricci symmetric.

(3) Ricci semi-symmetric that is $R \cdot S = 0$.

Proof. The statements (1) → (2) → (3) are trivial. Now, we prove the statement (3) → (1). Here $R \cdot S = 0$ means

$$(R(X,Y) \cdot S(U,W)) = 0, \tag{3.21}$$

where X, Y, U, W are tangent vectors. This implies

$$S(R(X,Y)U,W) + S(U,R(X,Y)W) = 0. \tag{3.22}$$

Using equation (2.4) in (3.22) and putting $Y = U = e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i $(1 \le i \le n)$ we get after simplification

$$\frac{c}{4}\{nS(X,W) - rg(X,W)\} = 0. \tag{3.23}$$

If $c \neq 0$ then the above equation implies

$$S(X,W) = \frac{r}{n}g(X,W). \tag{3.24}$$

□

Therefore we conclude the following.

Lemma 3.1. *A Ricci semi-symmetric non-flat quaternion space form is an Einstein manifold provided* $c \neq 0$.

Corollary 3.3. *Suppose that on a non-flat quaternion space form, the tensor field* $L_V g + 2S$ *of type (0,2) is parallel where* V *is a given vector field. Then* (g, V) *yields a Ricci soliton if* JV, KV *and* LV *are solenoidal. In particular, if the given non-flat quaternion space form is Ricci semi-symmetric with* $L_V g$ *parallel, we have same conclusion.*

Proof. From Theorem 3.1 and Corollary 3.2, we have $\lambda = \frac{r}{n}$ as seen below:

$$\begin{aligned}(L_V g + 2S)(Y,Z) = [&\frac{1}{n} tr(L_V g + 2S)g(Y,Z) + \alpha[tr.((L_V g + 2S)J)\Omega_1(Y,Z) \\ &+ tr.((L_V g + 2S)K)\Omega_2(Y,Z) + tr.((L_V g + 2S)L)\Omega_3(Y,Z)]].\end{aligned} \tag{3.25}$$

Simplifying the above we have

$$\begin{aligned}(L_V g + 2S)(Y,Z) = [&\frac{1}{n}2(divV + r)g(Y,Z) + \alpha[2(divJV)\Omega_1(Y,Z) \\ &+ 2(tr.SJ)\Omega_1(Y,Z) + 2(divKV)\Omega_2(Y,Z) + 2(tr.SK)\Omega_2(Y,Z) \\ &+ 2(divLV)\Omega_3(Y,Z) + 2(tr.SL)\Omega_3(Y,Z)]].\end{aligned} \tag{3.26}$$

By virtue of (2.7) the above equation becomes

$$\begin{aligned}(L_V g + 2S)(Y,Z) = [&\frac{2}{n}[(divV + r)g(Y,Z) + 2\alpha[(divJV)\Omega_1(Y,Z) \\ &+ (divKV)\Omega_2(Y,Z) + (divLV)\Omega_3(Y,Z)]].\end{aligned} \tag{3.27}$$

Suppose $divJV = 0$, $divKV = 0$ and $divLV = 0$ i.e., JV, KV and LV are solenoidal and because $JV = KV = LV = iV$, $divV = 0$. Hence by (3.27)

$$(L_V g + 2S)(Y,Z) = \frac{2r}{n} g(Y,Z) = 2\lambda g(Y,Z). \tag{3.28}$$

Therefore $\lambda = \frac{r}{n}$. Thus (g, V, λ) is Ricci soliton with $\lambda = \frac{r}{n}$. □

Lemma 3.2. *Let (g, V, λ) be a Ricci soliton in a non-flat quaternion space form. Then V is solenoidal if and only if it is shrinking, steady and expanding depending upon the sign of scalar curvature.*

Proof. Using equation (3.24) in (1.1) we get

$$(L_V g)(Y,Z) + 2\frac{r}{n} g(Y,Z) + 2\lambda g(Y,Z) = 0, \tag{3.29}$$

Putting $Y = Z = e_i$ where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i $(1 \le i \le n)$, we get

$$(L_V g)(e_i, e_i) + 2\frac{r}{n} g(e_i, e_i) + 2\lambda g(e_i, e_i) = 0. \tag{3.30}$$

The above equation implies

$$divV + r + \lambda n = 0. \tag{3.31}$$

If V is solenoidal then $divV = 0$. Therefore the equation (3.31) gives

$$\lambda = \frac{-r}{n}.$$

Thus V is solenoidal if and only if $\lambda < 0, \lambda = 0$ and $\lambda > 0$. □

4. Semi-symmetric Quaternion Space Forms

Using Definition 2.1 we have

$$(R(X,Y) \cdot R)(U,Z,W) = 0. \tag{4.1}$$

This implies

$$R(X,Y)R(U,Z)W - R(R(X,Y)U,Z)W - R(U,R(X,Y)Z)W - R(U,Z)R(X,Y)W = 0. \tag{4.2}$$

Taking inner product with tangent vector T we have

$$\begin{aligned} g(R(X,Y)R(U,Z)W,T) &- g(R(R(X,Y)U,Z)W,T) - g(R(U,R(X,Y)Z)W,T) \\ &- g(R(U,Z)R(X,Y)W,T) = 0. \end{aligned} \tag{4.3}$$

Use equation (2.4) in (4.3) and simplify it by putting $X = Z = e_i$. Again putting $Y = T = e_i$ and taking summation over i $(1 \le i \le n)$ we get

$$S(U,W) = c\left(-12n^2 - 51n - \frac{131}{2}\right) g(U,W). \tag{4.4}$$

This implies,

$$S(U,W) = \beta g(U,W). \tag{4.5}$$

where

$$\beta = c\left(-12n^2 - 51n - \frac{131}{2}\right). \tag{4.6}$$

That is quaternion space form is an Einstein manifold.

Hence we can state following result;

Theorem 4.1. *If a quaternion space form satisfies $R \cdot R = 0$ then it is an Einstein manifold.*

Using equation (4.5) in (1.1), we get

$$(L_V g)(U,W) + 2\beta g(U,W) + 2\lambda g(U,W) = 0. \tag{4.7}$$

Putting $U = W = e_i$ and simplifying we get

$$divV + \beta + \lambda = 0. \tag{4.8}$$

If V is solenoidal then $divV = 0$. Therefore the equation (4.8) gives

$$\lambda = -\beta. \tag{4.9}$$

Thus we can state the following from (4.6) and (4.9)

Lemma 4.1. *If (g, V, λ) is a Ricci soliton in a quaternion space form satisfying semi symmetric condition then Ricci solitons depend on quaternion sectional curvature c provided V is solenoidal.*

References

[1] T. Adachi and S. Maeda, *Some characterizations of quaternionic space forms,* Proc. Japan Acad. Ser. A Math. Sci., **76**(2000), 168–172.

[2] C. S. Bagewadi and G. Ingalahalli, *Ricci solitons in Lorentzian α-Sasakian manifolds,* Acta Math. Acad. Paedagog. Nyh*á*zi., **28(1)**(2012), 59–68.

[3] C. Călin and M. Crasmareanu, *From the Eisenhart problem to Ricci solitons in f-Kenmotsu manifolds,* Bull. Malays. Math. Sci. Soc. (2), **33(3)**(2010), 361–368.

[4] S. Debnath and A. Bhattacharyya, *Second order parallel tensor in Trans-Sasakian manifolds and connection with ricci soliton,* Lobachevskii J. Math., **33(4)**(2012), 312–316.

[5] R. Deszcz, *On pseudosymmetric spaces,* Bull. Soc. Math. Belg. Ser. A, **44**(1992), 1–34.

[6] L. P. Eisenhart, *Symmetric tensors of the second order whose first covariant derivatives are zero,* Trans. Amer. Math, Soc., **25(2)**(1923), 297–306.

[7] R. S. Hamilton, *The Ricci flow on surfaces,* Mathematics and general relativity (Santa Cruz, CA, 1986), Contemp. Math., **71**, Amer. Math. Soc., (1988), 237–262.

[8] G. Ingalahalli and C. S. Bagewadi, *Ricci solitons in α-Sasakian manifolds,* ISRN Geometry, (2012), Article ID 421384, 1–13.

[9] S. Ishihara, *Quaternion Kählerian manifolds,* J. Differential Geometry, **9**(1974), 483–500.

[10] H. Levy, *Symmetric tensors of the second order whose covariant derivatives vanish,* Ann. Math. (2), **27(2)**(1925), 91–98.

[11] R. Sharma, *Second order parallel tensor in real and complex space forms,* Internat. J. Math. Math. Sci., **12(4)**(1989), 787–790.

[12] Z. I. Szabo, *Structure theorems on Riemannian spaces satisfying $R(X,Y)\cdot R=0$,* J. Differential Geom., **17**(1982), 531–582.

[13] K. Yano and M. Kon, *Structures on manifolds,* Series In Pure Mathematics 3, World Scientific Publishing Co., Singapore, 1984.