

## Cofinite Graphs and Groupoids and their Profinite Completions

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ABSTRACT. Cofinite graphs and cofinite groupoids are defined in a unified way extending the notion of cofinite group introduced by Hartley. These objects have in common an underlying structure of a directed graph endowed with a certain type of uniform structure, called a cofinite uniformity. Much of the theory of cofinite directed graphs turns out to be completely analogous to that of cofinite groups. For instance, the completion of a directed graph  $\Gamma$  with respect to a cofinite uniformity is a profinite directed graph and the cofinite structures on  $\Gamma$  determine and distinguish all the profinite directed graphs that contain  $\Gamma$  as a dense sub-directed graph. The completion of the underlying directed graph of a cofinite graph or cofinite groupoid is observed to often admit a natural structure of a profinite graph or profinite groupoid, respectively.

### 1. Introduction

Embedding an algebraic object, such as a group, ring, or module, into a projec-

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Received February 8, 2016; revised April 20, 2017; accepted April 1, 2018.

2010 Mathematics Subject Classification: 05C63, 54F65, 57M15, 20E18.

Key words and phrases: profinite graph, cofinite graph, profinite group, cofinite group, profinite groupoid, cofinite groupoid, uniform space, completion, cofinite entourage.

tive limit of well-behaved objects is a frequently used tactic in algebra and number theory. For example, in commutative algebra, polynomial rings are embedded in rings of formal power series. Or more generally, if  $R$  is any commutative ring and  $I$  is an ideal, then the  $I$ -adic completion of  $R$  is the projective limit of the inverse system of quotient rings  $R/I^n$ ,  $n \geq 0$ . The case of  $R = \mathbb{Z}$  and  $I = (p)$ , where  $p$  is a prime, yields the  $p$ -adic integers. These rings are of particular importance in number theory.

This idea is also used for constructing compactifications of (discrete) graphs and directed graphs. For example, the ends of a finitely generated group are the "boundary points" of the so called end compactification of the Cayley (directed) graph of the group. The end compactification can be constructed as a certain completion of the (directed) graph which is a projective limit of finite (directed) graphs, and thus is a profinite (directed) graph (see Example 1.2). Similarly fundamental profinite groupoids are constructed using certain completions of fundamental groupoids (see 6.5).

There is a topological approach to producing such projective limits known as *completion*. The rough idea is to impose a suitable topology on the object making it into a topological group (or ring, or module, etc.) so that Cauchy sequences (or more generally, Cauchy nets) can be defined and used to construct the completion. By choosing different such topologies, various completions are formed. In the case of a residually finite group, Hartley [2] introduced the terminology of cofinite groups for the topological group structures that can be imposed such that the completion is a profinite group. Namely, a *cofinite group* is a topological group  $G$  whose open normal subgroups of finite indexes form a neighborhood base for the identity. Our purpose in this paper is to investigate to what extent the technique of profinite completion can be extended to more general structures that may only have at most a partial multiplication; for instance, we consider graphs (in the sense of Serre [5]) and groupoids.

One feature that all the objects that will concern us here have in common is an underlying directed graph structure. (It may be quite simple, as in the case of a group, which we view as a directed graph whose vertex set consists of only the identity element.) Thus, we begin by exploring embeddings of directed graphs as dense sub-directed graphs of profinite directed graphs with the goal of developing a general theory that can be applied widely in many situations. Initially we note that, without some modification, the topological approach used in the classical situations to construct and distinguish various completions breaks down for directed graphs in general. The following easy example illustrates this point.

**Example 1.1.** Let  $\Gamma$  be the directed graph obtained by subdividing the real line at the integer points, so  $V(\Gamma) = \mathbb{Z}$ , and directing each edge in the increasing direction. Incidentally,  $\Gamma$  is the Cayley directed graph of the additive group  $\mathbb{Z}$  with respect to the generating set  $\{1\}$ . We view  $\Gamma$  as a topological directed graph with the discrete topology. (Our terminology and notation for directed graphs is fully explained in the next section.)

For each integer  $n \geq 0$ , let  $[-n, n]$  denote the connected sub-directed graph of  $\Gamma$  with vertex set  $\{0, \pm 1, \dots, \pm n\}$ . Thus  $[-n, n]$  is a combinatorial arc of length  $2n$ . There is a unique retraction  $\theta_n: \Gamma \rightarrow [-n, n]$ ; it is the simplicial map defined on vertices by  $\theta(i) = i$  for  $-n \leq i \leq n$ ,  $\theta(i) = n$  for  $i \geq n$ , and  $\theta(i) = -n$  for  $i \leq -n$ . Also form the quotient directed graph  $\Gamma_n = [-n, n]/(-n = n)$  obtained by identifying the vertex  $-n$  with the vertex  $n$ , and let  $q_n: [-n, n] \rightarrow \Gamma_n$  denote the natural quotient map. Thus  $\Gamma_n$  is a combinatorial circle of length  $2n$ . Let  $\phi_n = q_n \theta_n: \Gamma \rightarrow \Gamma_n$  denote the composite map of directed graphs.

For integers  $0 \leq m \leq n$ , let  $\theta_{mn}: [-n, n] \rightarrow [-m, m]$  be the restriction of the retraction  $\theta_m: \Gamma \rightarrow [-m, m]$ , and note that  $\theta_{mn} \theta_n = \theta_m$ . Also note that  $\theta_{mn}$  determines a map of directed graphs  $\phi_{mn}: \Gamma_n \rightarrow \Gamma_m$  on the quotient directed graphs, and  $\phi_{mn} \phi_n = \phi_m$  holds. We have constructed two inverse systems of finite directed graphs and maps of directed graphs indexed by the directed set of non-negative integers:  $([-n, n], \theta_{mn})$  and  $(\Gamma_n, \phi_{mn})$ . Denote their inverse limits by  $\widehat{\Gamma}_1 = \varprojlim \Gamma_n$  and  $\widehat{\Gamma}_2 = \varprojlim [-n, n]$ . Endow each of the finite graphs  $[-n, n]$  and  $\Gamma_n$  with the discrete topology so that  $\widehat{\Gamma}_1$  and  $\widehat{\Gamma}_2$  are compact Hausdorff topological directed graphs.

Embed  $\Gamma$  in  $\widehat{\Gamma}_1$  via the canonical map of directed graphs determined by the compatible family of maps  $\phi_n: \Gamma \rightarrow \Gamma_n$ . Then it is easy to see that  $\widehat{\Gamma}_1 = \Gamma \cup \{\infty\}$  and its underlying topological space is the one-point compactification of the discrete underlying topological space of  $\Gamma$ . Similarly,  $\Gamma$  embeds in  $\widehat{\Gamma}_2$  as a dense discrete sub-directed graph and  $\widehat{\Gamma}_2 = \Gamma \cup \{-\infty, \infty\}$  is a two-point compactification of  $\Gamma$ . Thus  $\widehat{\Gamma}_1$  and  $\widehat{\Gamma}_2$  are non-isomorphic topological directed graphs, each containing  $\Gamma$  as a dense sub-directed topological directed graph.

Thus endowing a directed graph  $\Gamma$  with a topological directed graph structure is not enough to uniquely determine a completion of  $\Gamma$ , and so something more is needed to define a suitable notion of a cofinite directed graph. Although, in the previous example, the relative topology that  $\Gamma$  inherits from each  $\widehat{\Gamma}_i$  is the same, namely the discrete topology, the uniform structures they induce on  $\Gamma$  are different.

A profinite directed graph, like any compact Hausdorff space, has a unique uniformity that induces its topology. We explore this uniform structure on a profinite directed graph in Section 3, and characterize it in terms of what we call cofinite entourages (Theorem 3.4). Then by analyzing the relative uniform structure induced on a sub-directed graph of a profinite directed graph, a natural notion of cofinite directed graph is discovered (Definition 4.1). As justification for our definition, in Section 5 we show that every cofinite graph  $\Gamma$  has a unique completion  $\overline{\Gamma}$  (Theorem 5.6), and it is a profinite directed graph (Theorem 5.10). Moreover, profinite directed graphs are precisely the compact cofinite directed graphs.

In Section 6 we conclude that the paper has some applications that we are interested in for future reference. Specifically we consider graphs (in the sense of Serre [5]) and groupoids. These objects are directed graphs with an additional structure. We require that the cofinite directed graph structures on them also respect the additional structure. In the case of a graph, it turns out that in general

the completion of a cofinite graph is a profinite graph. However for a cofinite groupoid, we need to impose a mild restriction to ensure that the completion is a groupoid. This condition automatically holds for groupoids with only finitely many identities (such as a group, which has a unique identity), and thus the completion of a cofinite groupoid with a finite vertex set is a profinite groupoid (Theorem 6.15).

We conclude the introduction by the following motivational example: The end compactification of a directed graph.

**Example 1.2.** Let  $\Gamma$  be a connected, locally finite directed graph (see [6]). For each finite sub-directed graph  $\Sigma$  of  $\Gamma$ , let  $R_\Sigma$  be the equivalence relation on  $\Gamma$  whose equivalence classes are the singleton subsets of  $E(\Sigma)$  and the components of  $\Gamma \setminus E(\Sigma)$ . That is,  $(x, y) \in R_\Sigma$  if and only if  $x = y$  or  $x$  and  $y$  belong to the same component of  $\Gamma \setminus E(\Sigma)$ . Observe that  $R_\Sigma$  is compatible with the directed graph structure and has finite index (since  $\Gamma$  is connected). Let  $I$  be the non-empty directed set of all finite sub-directed graphs of  $\Gamma$ , partially ordered by inclusion. We have the following properties:

1. If  $\Sigma_1, \Sigma_2 \in I$  and  $\Sigma_1 \subseteq \Sigma_2$ , then  $R_{\Sigma_2} \subseteq R_{\Sigma_1}$ .
2. For every  $x \in \Gamma$ , there exists  $\Sigma \in I$  such that  $R_\Sigma(x) = \{x\}$ .
3.  $\bigcap_{\Sigma \in I} R_\Sigma = \delta_\Gamma$ .

Hence the set  $\{R_\Sigma \mid \Sigma \in I\}$  is a base for a (cofinite) uniformity on  $\Gamma$ . The completion of  $\Gamma$  (endowed with this cofinite structure) is given by  $\widehat{\Gamma} = \varprojlim \Gamma_\Sigma$  and is called the *end compactification* of  $\Gamma$ . In particular, the end compactification of  $\Gamma$  is a profinite directed graph and  $\Gamma$  is a discrete open sub-directed graph of  $\widehat{\Gamma}$ .

## 2. Preliminaries

### 2.1. Co-discrete Equivalence Relations

We do not distinguish between a binary relation and its graph. Thus, by a *binary relation* on a set  $X$  we mean simply a subset  $R \subseteq X \times X$ . Given a binary relation  $R$  on a set  $X$  and  $a \in X$ , we write  $R[a] = \{x \in X \mid (a, x) \in R\}$ . More generally, for any subset  $A$  of  $X$ , we put  $R[A] = \bigcup\{R[a] \mid a \in A\}$ .

The *composition* of two binary relations  $R$  and  $S$  on a set  $X$ , is the binary relation  $SR \subseteq X \times X$  defined by:  $(x, z) \in SR$  if and only if there exists  $y \in X$  such that  $(x, y) \in R$  and  $(y, z) \in S$ . Note that composition is an associative operation on the set of binary relations on a set  $X$ .

The following lemma identifies certain equivalence relations on topological spaces that are useful in describing the uniform structure of profinite spaces. Recall that the *index* of an equivalence relation  $R$  on a set  $X$  is the cardinality of the set  $X/R$  of equivalence classes of  $X$  with respect to  $R$ .

**Lemma 2.1.** *Let  $R$  be an equivalence relation on a topological space  $X$ . Then the following conditions are equivalent:*

- (1) The equivalence relation  $R$  is an open subset of the product space  $X \times X$ .
- (2) Every equivalence class  $R[a]$  is an open subset of  $X$ .
- (3) The quotient space  $X/R$  is a discrete space.

Additionally when  $X$  is compact, such an equivalence relation  $R$  has finite index and is a clopen subset of  $X \times X$ .

*Proof.* By the corollary to Proposition 4 of [1, Section I.4.2], (1)  $\Rightarrow$  (2). Being an equivalence relation,

$$R = \bigcup_{a \in X} (R[a] \times R[a])$$

which makes it clear that (2)  $\Rightarrow$  (1). Note also that (2)  $\Leftrightarrow$  (3) since points are open in  $X/R$  if and only if equivalence classes are open in  $X$ . To show the last part, assume now that  $X$  is compact and that  $R$  is an equivalence relation on  $X$  satisfying the equivalent conditions (1)–(3). Then the quotient space  $X/R$  is compact and discrete, and thus a finite space. Consequently  $R$  is of finite indexes and it is closed in  $X \times X$  since it is the finite union of the closed sets  $R[a] \times R[a]$ , for  $a \in X$ .  $\square$

The *inverse* of  $R$  is the binary relation  $R^{-1}$  on  $X$  consisting of all pairs  $(y, x) \in X \times X$  such that  $(x, y) \in R$ . Note that the following properties hold for any binary relations  $R_1, R_2$ , and  $R_3$  on  $X$ :

1.  $(R_1 R_2) R_3 = R_1 (R_2 R_3)$ ;
2.  $(R_1 R_2)^{-1} = R_2^{-1} R_1^{-1}$ ;
3. if  $A$  is any subset of  $X$ , then  $(R_1 R_2)[A] = R_1(R_2([A]))$ .

## 2.2. Directed Graphs

A *directed graph* is a set  $\Gamma$  with a distinguished subset  $V(\Gamma)$  and two functions  $s, t: \Gamma \rightarrow V(\Gamma)$  satisfying  $s(x) = t(x) = x$  for all  $x \in V(\Gamma)$ . Each element of  $V(\Gamma)$  is called a *vertex*. We denote the complement of  $V(\Gamma)$  by  $E(\Gamma) = \Gamma \setminus V(\Gamma)$  and each  $e \in E(\Gamma)$  is called an *edge* with *source*  $s(e)$  and *target*  $t(e)$ . Observe that the vertex set  $V(\Gamma)$  is determined by the source map  $s$ , and by the target map  $t$ : it is precisely the set of fixed points of each of these maps.

### Sub-directed graphs

A subset  $\Sigma$  of a directed graph  $\Gamma$  is called a *sub-directed graph* if it contains the source and target of each of its members. Under the restrictions of the source and target maps on  $\Gamma$ , a sub-directed graph  $\Sigma$  is itself a directed graph. Note that  $V(\Sigma) = V(\Gamma) \cap \Sigma$ .

### Products of directed graphs

Let  $(\Gamma_i)_{i \in I}$  be a family of directed graphs. We make the Cartesian product  $\Gamma =$

$\prod_{i \in I} \Gamma_i$  into a directed graph in the following way. The vertex set of  $\Gamma$  is the set  $V(\Gamma) = \prod_{i \in I} V(\Gamma_i)$  of the underlying sets. The source and target maps are defined coordinate-wise: if  $x = (x_i) \in \Gamma$ , then  $s(x) = (s(x_i))$  and  $t(x) = (t(x_i))$ . The set  $\Gamma$  with this directed graph structure is called a *product directed graph*.

**Maps of directed graphs**

A *map of directed graphs* is a function  $f: \Gamma \rightarrow \Delta$  between directed graphs that preserves sources and targets. Note that if  $f: \Gamma \rightarrow \Delta$  is a map of directed graphs, then  $f(V(\Gamma)) \subseteq V(\Delta)$ . For if  $x \in V(\Gamma)$ , then  $f(x) = f(s(x)) = s(f(x))$  and hence  $f(x) \in V(\Delta)$ . However  $f$  does not necessarily map edges to edges. Compositions of maps of directed graphs are maps of directed graphs. Also note that if  $f: \Gamma \rightarrow \Delta$  is a map of directed graphs and  $\Sigma$  is a sub-directed graph of  $\Gamma$ , then  $f(\Sigma)$  is a sub-directed graph of  $\Delta$ .

**Compatible equivalence relations on directed graphs**

An equivalence relation  $R$  on the underlying set  $\Gamma$  is *compatible equivalence relation* on  $\Gamma$  if  $(s(x), s(y)) \in R$  and  $(t(x), t(y)) \in R$  whenever  $(x, y) \in R$ . In other words, an equivalence relation  $R$  on  $\Gamma$  is compatible with the directed graph structure if and only if  $R$  is a sub-directed graph of the product directed graph  $\Gamma \times \Gamma$ .

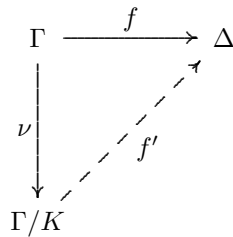
If  $f: \Gamma \rightarrow \Delta$  is a map of directed graphs, then its kernel

$$\ker f = \{(x, y) \in \Gamma \times \Gamma \mid f(x) = f(y)\}$$

is clearly a compatible equivalence relation on  $\Gamma$ .

On the other hand, if  $R$  is any compatible equivalence relation on a directed graph  $\Gamma$ , then the set of equivalence classes  $\Gamma/R$  has a unique directed graph structure such that the natural map  $\nu: \Gamma \rightarrow \Gamma/R$  is a map of directed graphs. Its source and target maps are given by  $s(R[x]) = R[s(x)]$  and  $t(R[x]) = R[t(x)]$ , and  $V(\Gamma/R) = \nu(V(\Gamma))$ . Note that in this case  $\ker \nu = R$ , and hence, every compatible equivalence relation on a directed graph  $\Gamma$  is the kernel of some map of directed graphs with domain  $\Gamma$ .

Moreover, we have this *first isomorphism theorem* for directed graphs: Let  $f: \Gamma \rightarrow \Delta$  be a map of directed graphs and let  $K = \ker f$ . Then  $K$  is a compatible equivalence relation on  $\Gamma$  and there is an injective map of directed graphs  $f': \Gamma/K \rightarrow \Delta$  given by  $K[x] \mapsto f(x)$ , which is an isomorphism when  $f$  is surjective.



The intersection of a non-empty family of compatible equivalence relations on

a directed graph  $\Gamma$  is again a compatible equivalence relation on  $\Gamma$ . Thus, given any subset  $T \subseteq \Gamma \times \Gamma$ , the intersection of all compatible equivalence relations on  $\Gamma$  containing  $T$  is the unique smallest such equivalence relation on  $\Gamma$ . It is denoted by  $T^\sharp$  and called the *compatible equivalence relation on  $\Gamma$  generated by  $T$* . Alternatively,  $T^\sharp$  can be constructed as follows. Let  $\langle T \rangle = T \cup (s \times s)[T] \cup (t \times t)[T]$ , the smallest sub-directed graph of  $\Gamma \times \Gamma$  containing  $T$ . Then  $T^\sharp = \langle T \rangle^*$ , the reflexive and transitive closure of  $\langle T \rangle$ .

### 3. The Uniform Structure of a Profinite Directed Graph

A *topological directed graph* is a directed graph  $\Gamma$  endowed with a topology such that the source and target functions  $s, t: \Gamma \rightarrow V(\Gamma)$  are continuous maps onto the subspace  $V(\Gamma)$ . We form the *category of topological directed graphs* by taking the class of objects to consist of all topological directed graphs and taking the morphisms to be all continuous maps of directed graphs. (That is, a morphism is both continuous and a map of directed graphs.) It is clear that a morphism  $f: \Gamma \rightarrow \Delta$  is an isomorphism of topological directed graphs (i.e., an isomorphism in the category of topological directed graphs) if and only if  $f$  is both a bijective map of directed graphs and a homeomorphism.

Every sub-directed graph of a topological directed graph will be regarded as a topological directed graph with the subspace topology. Likewise, if  $K$  is a compatible equivalence relation on a topological directed graph  $\Gamma$ , then the quotient directed graph  $\Gamma/K$  will always be endowed with the quotient topology making it a topological directed graph. Also given any family of topological directed graphs  $(\Gamma_i)_{i \in I}$ , we make the product of the underlying topological spaces  $\Gamma = \prod_{i \in I} \Gamma_i$  into a topological directed graph with source and target maps given by  $s(x_i) = (s(x_i))$  and  $t(x_i) = (t(x_i))$ . Then the projection maps  $\phi_i: \Gamma \rightarrow \Gamma_i$  are continuous maps of directed graphs and it is clear that  $\Gamma$  is the product in the category of topological graphs. Likewise the inverse limit of an inverse system of topological directed graphs and continuous maps of directed graphs exists in the category of topological directed graphs and its underlying topological space is the inverse limit of the corresponding inverse system of underlying spaces. (It can also be constructed as a sub-directed graph of a product of directed graphs in the usual way.)

A *profinite directed graph* is a compact, Hausdorff, totally disconnected directed graph. Equivalently, a profinite directed graph is a projective limit of finite discrete directed graphs; see, for instance, [4] or the remarks following the proof of Theorem 3.4. It is easy to see that any closed sub-directed graph of a profinite directed graph is a profinite directed graph and that products and inverse limits of profinite directed graphs are profinite directed graphs. Note that the vertex set  $V(\Gamma)$  of a profinite directed graph  $\Gamma$  is closed, since it is the fixed point set of a continuous map of a Hausdorff space to itself. On the other hand, the edge set  $E(\Gamma) = \Gamma \setminus V(\Gamma)$  need not be closed.

Recall that every compact Hausdorff space  $X$  has a unique uniformity that induces its topology. This uniformity consists of all neighborhoods of the diagonal

$\delta_X$  in  $X \times X$ ; see, for instance, [1, II.4.1, Theorem 1]. Therefore, as we may, we regard every compact Hausdorff space as a uniform space endowed with the unique uniformity inducing its topology. Our goal in this section is to characterize profinite directed graphs in terms of their unique uniform structures.

For this purpose, we make use of the following observation.

**Lemma 3.1.** *Let  $X$  be a compact, Hausdorff, totally disconnected space and let  $W$  be an entourage of  $X$ . Then there exists an equivalence relation  $R$  on  $X$  such that  $R$  is open in  $X \times X$  and  $R \subseteq W$ . In particular,  $R$  has finite index and is an entourage of  $X$ .*

*Proof.* Let  $I$  be the non-empty set of all equivalence relations  $R$  on  $X$  such that  $R$  is a clopen subset of  $X \times X$ . Let  $x, y \in X$  with  $x \neq y$ . Since  $X$  is Hausdorff and the clopen subsets of  $X$  form a base for its topology (see, for example, [7, Lemma 0.1.1(c)]), there is a clopen subset  $C$  of  $X$  with  $x \in C$  and  $y \notin C$ . The equivalence relation  $R$  that partitions  $X$  into the two equivalence classes  $C$  and  $C' = X \setminus C$ , namely

$$R = (C \times C) \cup (C' \times C'),$$

is a clopen subset of  $X \times X$  and  $(x, y) \notin R$ . It follows that the intersection of the family  $I$  is equal to the diagonal  $\delta_X$  in  $X \times X$ .

By replacing  $W$  with its interior in  $X \times X$ , we may assume that  $W$  is an open neighborhood of  $\delta_X$  in  $X \times X$ . So for each  $R \in I$ , the set  $R \setminus W$  is closed in  $X \times X$  and

$$\bigcap_{R \in I} R \setminus W \subseteq \bigcap_{R \in I} R \setminus \delta_X = \emptyset.$$

However  $X \times X$  is compact, so there exist  $R_1, \dots, R_n \in I$  for some positive integer  $n$  such that  $(R_1 \setminus W) \cap \dots \cap (R_n \setminus W) = \emptyset$ . Let  $R = R_1 \cap \dots \cap R_n$ . Then  $R$  is an equivalence relation on  $X$  which is a clopen subset of  $X \times X$  and  $R \subseteq W$ . Furthermore,  $R$  is a neighborhood of  $\delta_X$  in  $X \times X$ , and so  $R$  is an entourage of  $X$ .  $\square$

To describe the uniform structure of profinite directed graphs and their sub-directed graphs, it is convenient to make the following definition.

**Definition 3.2.**(Cofinite entourage) Let  $\Gamma$  be a directed graph endowed with a uniformity. A *cofinite entourage* of  $\Gamma$  is a compatible equivalence relation of finite indexes on  $\Gamma$  which is also an entourage of  $\Gamma$ .

It should be noted that a directed graph  $\Gamma$  endowed with a uniformity always has at least one cofinite entourage, namely  $R = \Gamma \times \Gamma$ . Here is a list of some fundamental properties of cofinite entourages that the reader can easily verify.

**Lemma 3.3.** *Let  $\Gamma$  and  $\Delta$  be directed graphs endowed with uniformities. Then their cofinite entourages have the following properties.*

- (1) *If  $R$  is a cofinite entourage of  $\Gamma$  and  $x \in \Gamma$ , then  $R[x]$  is a clopen neighborhood of  $x$  in  $\Gamma$ .*



- (2) Every cofinite entourage of  $\Gamma$  is a clopen subset of  $\Gamma \times \Gamma$ .
- (3) If  $R_1$  and  $R_2$  are cofinite entourages of  $\Gamma$ , then  $R_1 \cap R_2$  is also a cofinite entourage of  $\Gamma$ .
- (4) If  $R$  is a compatible equivalence relation on  $\Gamma$  that contains a cofinite entourage of  $\Gamma$ , then  $R$  is itself a cofinite entourage.
- (5) If  $f: \Gamma \rightarrow \Delta$  is a uniformly continuous map of directed graphs and  $S$  is a cofinite entourage of  $\Delta$ , then  $(f \times f)^{-1}[S]$  is a cofinite entourage of  $\Gamma$ .
- (6) If  $\Sigma$  is a sub-directed graph of  $\Gamma$  and  $R$  is a cofinite entourage of  $\Gamma$ , then the restriction  $R \cap (\Sigma \times \Sigma)$  is a cofinite entourage on  $\Sigma$ .

We are now ready to give our characterization of profinite directed graphs in terms of their uniform structures.

**Theorem 3.4.** *Let  $\Gamma$  be a directed graph endowed with a compact Hausdorff topology. Then  $\Gamma$  is a profinite directed graph if and only if the set of cofinite entourages of  $\Gamma$  is a base for the uniformity on  $\Gamma$ .*

*Proof.* Suppose first that  $\Gamma$  is a profinite directed graph and let  $W$  be an entourage of  $\Gamma$ . By Lemma 3.1, there exists an equivalence relation  $R$  on  $\Gamma$  such that  $R$  is open in  $X \times X$  and  $R \subseteq W$ . Let

$$S = R \cap (s \times s)^{-1}[R] \cap (t \times t)^{-1}[R].$$

Then  $S$  is an equivalence relation on  $\Gamma$  and it is open in  $X \times X$ . We claim that  $S$  is also a subgraph of  $\Gamma \times \Gamma$ , and thus is a compatible equivalence relation on  $\Gamma$ . For if  $(x, y) \in S$ , then

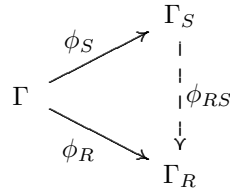
- (i)  $s(x, y) = (s(x), s(y)) \in R$  since  $(x, y) \in (s \times s)^{-1}[R]$ ,
- (ii)  $s(x, y) \in (s \times s)^{-1}[R]$  since  $s(s(x), s(y)) = (s(x), s(y)) \in R$ , and
- (iii)  $s(x, y) \in (t \times t)^{-1}[R]$  since  $t(s(x), s(y)) = (s(x), s(y)) \in R$ .

Hence  $s(x, y) \in S$ . Similarly  $t(x, y) \in S$ , and therefore  $S$  is compatible. Furthermore, it is of finite indexes by Lemma 2.1. Hence  $S$  is a cofinite entourage of  $\Gamma$  such that  $S \subseteq W$ . It follows that the set of all cofinite entourages of  $\Gamma$  is a base for the uniformity on  $\Gamma$ .

Conversely, assume now that the non-empty set, say  $I$ , of all cofinite entourages of  $\Gamma$  is a base for the uniformity on  $\Gamma$ . For each  $R \in I$ , the quotient  $\Gamma_R = \Gamma/R$  is compact and discrete, and thus a (finite) profinite directed graph. Denote the quotient map  $\Gamma \rightarrow \Gamma_R$  by  $x \mapsto x_R$ . The map  $\Gamma \rightarrow \prod_{R \in I} \Gamma_R$  given by  $x \mapsto (x_R)$  is a continuous map of directed graphs which is injective (as its kernel is  $\bigcap_{R \in I} R = \delta_\Gamma$  since  $I$  is a non-empty base for the uniformity on  $\Gamma$  and  $\Gamma$  is Hausdorff). Being an injective continuous map of a compact space to a Hausdorff space, this map is also a topological embedding. Thus  $\Gamma$  is isomorphic to a closed sub-directed graph

of the profinite directed graph  $\prod_{R \in I} \Gamma_R$  and therefore is itself a profinite directed graph.  $\square$

We observe now that Theorem 3.4 provides a canonical realization of a profinite directed graph  $\Gamma$  as a projective limit of finite directed graphs. Let  $I$  be the set of all cofinite entourages of  $\Gamma$ , partially ordered by reverse inclusion:  $R_1 \leq R_2$  if and only if  $R_1 \supseteq R_2$ . Since the intersection of two cofinite entourages is a cofinite entourage, it follows that  $I$  is a directed set. For each  $R \in I$ , the quotient  $\Gamma_R = \Gamma/R$  is compact and discrete, and thus a finite directed graph. Denote the natural quotient map by  $\phi_R: \Gamma \rightarrow \Gamma_R$ . Note that for all  $R \leq S$  in  $I$ , there exists a unique map of directed graphs  $\phi_{RS}: \Gamma_S \rightarrow \Gamma_R$  such that  $\phi_{RS}\phi_S = \phi_R$ .



Moreover,  $(\Gamma_R, \phi_{RS})_{R \leq S \in I}$  is an inverse system of finite discrete directed graphs. We claim that  $(\Gamma, \phi_R)_{R \in I}$  is its inverse limit. By the commutative diagrams above,  $(\phi_R: \Gamma \rightarrow \Gamma/R)_{R \in I}$  is a compatible family of morphisms and so determines a morphism  $\phi: \Gamma \rightarrow \varprojlim \Gamma_R$ . Since each  $\phi_R$  is surjective, the image  $\phi(\Gamma)$  is a dense subset of  $\varprojlim \Gamma_R$ . But  $\Gamma$  is compact and  $\varprojlim \Gamma_R$  is Hausdorff, and so  $\phi(\Gamma)$  is closed. It follows that  $\phi$  is surjective. Furthermore,  $I$  is a base for the uniformity on  $\Gamma$  (by Theorem 3.4) and  $\Gamma$  is Hausdorff, so

$$\ker \phi = \bigcap_{R \in I} \ker \phi_R = \bigcap_{R \in I} R = \delta_\Gamma.$$

Therefore  $\phi$  is also injective, and hence an isomorphism of topological directed graphs. This establishes our claim.

#### 4. Cofinite Directed Graphs

Let  $\Gamma$  be a profinite directed graph and let  $\Sigma$  be a sub-directed graph endowed with the induced uniformity. Then  $\Sigma$  is Hausdorff and the family of all  $R \cap (\Sigma \times \Sigma)$  as  $R$  runs through all cofinite entourages of  $\Gamma$ , is a fundamental system of entourages of  $\Sigma$ . Since each  $R \cap (\Sigma \times \Sigma)$  is a cofinite entourage of the uniform sub-directed graph  $\Sigma$ , it follows that every uniform sub-directed graph  $\Sigma$  of a profinite directed graph  $\Gamma$  has a fundamental system of entourages consisting of cofinite entourages of  $\Sigma$ . In light of this observation, we make the following definition.

**Definition 4.1.**(Cofinite directed graph) A *cofinite directed graph* is a directed graph  $\Delta$  endowed with a Hausdorff uniformity that has a base consisting of cofinite entourages of  $\Delta$ .

Note that by virtue of Theorem 3.4, profinite directed graphs are precisely the compact cofinite directed graphs. Also note that the definition of cofinite directed graphs does not assume that the source and target maps are uniformly continuous, or even continuous. However, we next observe that this is automatically true.

**Theorem 4.2.** *Let  $\Gamma$  be a cofinite directed graph. Then the source and target maps  $s, t: \Gamma \rightarrow V(\Gamma)$  are uniformly continuous maps onto the uniform subspace  $V(\Gamma)$ . In particular,  $\Gamma$  is a Hausdorff topological graph and  $V(\Gamma)$  is a closed subset of  $\Gamma$ .*

*Proof.* Let  $R$  be a cofinite entourage of  $\Gamma$ . Then

$$(s \times s)[R] \subseteq R \cap [V(\Gamma) \times V(\Gamma)] \text{ and } (t \times t)[R] \subseteq R \cap [V(\Gamma) \times V(\Gamma)]$$

and thus  $s, t$  are uniformly continuous since the sets  $R \cap [V(\Gamma) \times V(\Gamma)]$ , as  $R$  runs through all cofinite entourages of  $\Gamma$ , form a base for the relative uniformity on  $V(\Gamma)$ . The last part follows since uniformly continuous maps are continuous and the vertex set of a Hausdorff topological directed graph is closed.  $\square$

We always endow a cofinite directed graph with the topology induced by its uniformity, and as we just observed, this makes it into a Hausdorff topological directed graph. This topology can be characterized as follows.

Let  $\Gamma$  be a cofinite directed graph and let  $I$  be a non-empty set of cofinite entourages of  $\Gamma$  forming a base for the uniformity on  $\Gamma$ . We denote the closure of a subset  $A$  of  $\Gamma$  (in its uniform topology) by  $\overline{A}$ , and note that it is given by

$$\overline{A} = \bigcap_{R \in I} R[A]$$

and each  $R[A]$  is a clopen neighborhood of  $A$  in  $\Gamma$ ; see, for instance, Proposition II.1.2.2 and its corollary of [1]. Similarly, if  $W \subseteq \Gamma \times \Gamma$ , then its closure  $\overline{W}$  in the product space  $\Gamma \times \Gamma$  is given by

$$\overline{W} = \bigcap_{R \in I} (R \times R)[W] = \bigcap_{R \in I} RWR$$

and each  $RWR$  is a clopen neighborhood of  $W$  in  $\Gamma \times \Gamma$ . See [1, Chapter II] or [3, Chapter 6] for more on the basic properties of uniform topologies.

As we noted above, every sub-directed graph of a profinite directed graph is a cofinite directed graph when it is endowed with its relative uniformity. On the other hand, cofinite directed graphs can be constructed from scratch as follows.

Let  $\Gamma$  be any abstract directed graph and let  $I$  be a filter base of compatible equivalence relations of finite indexes on  $\Gamma$ . (That is,  $I$  is a non-empty set with the property: if  $R_1, R_2 \in I$ , then there exists  $R_3 \in I$  such that  $R_3 \subseteq R_1 \cap R_2$ .)

Then  $I$  is a base for a uniformity on  $\Gamma$ ; see for instance [3, Theorem 6.2]. Thus  $\Gamma$  endowed with this uniformity is a cofinite directed graph provided that the induced topology is Hausdorff. Moreover, it is obvious that all of the various cofinite structures that can be put on  $\Gamma$  arise in this way.

Amongst all uniform structures of this type that can be put on  $\Gamma$ , there is a unique maximal one, namely that in which  $I$  consists of *all* compatible equivalence relations of finite indexes on  $\Gamma$ . We next observe that  $\Gamma$  endowed with the finest uniformity is Hausdorff, and hence is a cofinite directed graph.

**Proposition 4.3.** *Let  $\Gamma$  be an abstract directed graph endowed with the unique finest uniformity having a base consisting of compatible equivalence relations of finite indexes on  $\Gamma$ . Then  $\Gamma$  is a discrete cofinite directed graph.*

*Proof.* Let  $y \in V(\Gamma)$ . Denote by  $\Delta$  the finite (abstract) directed graph with  $V(\Delta) = \{a, b\}$  and  $E(\Delta) = \{e, f, g, \bar{g}\}$ , where  $s(e) = t(e) = a$ ,  $s(f) = t(f) = b$ ,  $s(g) = a$ ,  $t(g) = b$ ,  $s(\bar{g}) = b$  and  $t(\bar{g}) = a$ . There is a unique map of directed graphs  $\phi: \Gamma \rightarrow \Delta$  such that  $\phi(y) = a$  and  $\phi(x) = b$  for all vertices  $x \neq y$ . The kernel  $R = \ker \phi$  is a compatible equivalence relation of finite indexes on  $\Gamma$  such that  $R[y] = \{y\}$ .

Now let  $y \in E(\Gamma)$  and let  $\Delta = \{a, e\}$  be the finite directed graph with  $V(\Delta) = \{a\}$ . Define a map of directed graphs  $\phi: \Gamma \rightarrow \Delta$  by  $\phi(y) = e$  and  $\phi(x) = a$  for all  $x \neq y$ . Then the kernel  $R = \ker \phi$  is a compatible equivalence relation of finite indexes on  $\Gamma$  such that  $R[y] = \{y\}$ .

We see that every singleton subset  $\{y\}$  of  $\Gamma$  is open in the topology induced by the uniformity on  $\Gamma$ , and hence this topology is discrete. In particular,  $\Gamma$  is Hausdorff, and so a cofinite directed graph.  $\square$

It would be interesting to know which other filter bases  $I$  of compatible equivalence relations of finite indexes on an abstract directed graph  $\Gamma$ , besides the largest one, make  $\Gamma$  into a cofinite directed graph. The following result helps with this.

**Proposition 4.4.** *If  $\Gamma$  is an abstract directed graph and  $I$  is a non-empty filter base of compatible equivalence relations of finite indexes on  $\Gamma$ , then the following conditions on  $\Gamma$  endowed with the uniformity generated by  $I$  are equivalent:*

- (1)  $\Gamma$  is Hausdorff, and so  $\Gamma$  is a cofinite directed graph;
- (2)  $\Gamma$  is totally disconnected;
- (3)  $\bigcap_{R \in I} R = \delta_\Gamma$ , the diagonal in  $\Gamma \times \Gamma$ .

*Proof.* (1)  $\Rightarrow$  (2): Let  $x \in \Gamma$  and let  $C$  be the component of  $\Gamma$  containing  $x$ . For all  $R \in I$ , note that  $C \subseteq R[x]$  since  $R[x]$  is a clopen set containing  $x$ . Furthermore, when  $\Gamma$  is Hausdorff,  $\bigcap R[x] = \{x\}$ , and thus  $C = \{x\}$  and  $\Gamma$  is totally disconnected.

(2)  $\Rightarrow$  (3): Let  $x, y \in \Gamma$  with  $x \neq y$ . When  $\Gamma$  is totally disconnected, there exists an open set  $U$  of  $\Gamma$  with  $x \in U$  and  $y \notin U$  since otherwise the subset  $\{x, y\}$  would be connected. Since  $U$  is open, there exists  $R \in I$  such that  $R[x] \subseteq U$ . Then  $y \notin R[x]$  so that  $(x, y) \notin R$ , and it follows that  $\bigcap R = \delta_\Gamma$ .

(3)  $\Rightarrow$  (1): We have that  $\bar{\delta}_\Gamma = \bigcap R \delta_\Gamma R = \bigcap R$  since  $\delta_\Gamma \subseteq R$  and  $R^2 = R$  as each  $R \in I$  is an equivalence relation. Thus, when (3) holds,  $\delta_\Gamma$  is closed in  $\Gamma \times \Gamma$  and thus  $\Gamma$  is Hausdorff.  $\square$

Consequently, we see that every cofinite directed graph  $\Gamma$  is totally disconnected and that the intersection of all of its cofinite entourages is equal to the diagonal  $\delta_\Gamma$  of  $\Gamma \times \Gamma$ .

## 5. Completions of Cofinite Directed Graphs

In this section we define completions of cofinite directed graphs analogously to the way that completions of cofinite groups are defined in [2]. We show that every cofinite directed graph  $\Gamma$  has a completion  $\bar{\Gamma}$  (using a standard construction), and that it is unique up to a unique isomorphism extending the identity map on  $\Gamma$ . It turns out to be precisely the Hausdorff completion of its underlying uniform space (Corollary 5.7). Further generalizing the situation for cofinite groups [2], we observe that the completion  $\bar{\Gamma}$  is a profinite directed graph and that its cofinite entourages are precisely the closures  $\bar{R}$  in  $\bar{\Gamma} \times \bar{\Gamma}$  of the cofinite entourages  $R$  of  $\Gamma$ . We begin with an analogue of [2, Theorem 2.1].

**Theorem 5.1.** *Let  $\Gamma$  be a cofinite directed graph which is embedded as a dense uniform sub-directed graph of a compact Hausdorff topological directed graph  $\bar{\Gamma}$ . If  $\Delta$  is any compact Hausdorff topological directed graph and  $f: \Gamma \rightarrow \Delta$  is a uniformly continuous map of directed graphs, then there exists a unique continuous map of directed graphs  $\bar{f}: \bar{\Gamma} \rightarrow \Delta$  extending  $f$ .*

*Proof.* Being compact and Hausdorff,  $\Delta$  is a complete uniform space. So there exists a unique uniformly continuous map  $\bar{f}: \bar{\Gamma} \rightarrow \Delta$  such that  $\bar{f}|_\Gamma = f$ ; see, for example, [1, II.3.6, Theorem 2]. It remains to verify that  $\bar{f}$  is a map of directed graphs. Let  $B = \{x \in \bar{\Gamma} \mid \bar{f}(s(x)) = s(\bar{f}(x))\}$ . Then  $B$  is closed in  $\bar{\Gamma}$  since  $\bar{f} \circ s$  and  $s \circ \bar{f}$  are continuous maps from  $\bar{\Gamma}$  to  $\Delta$  and  $\Delta$  is Hausdorff. However,  $\Gamma \subseteq B$  and  $\Gamma$  is dense in  $\bar{\Gamma}$ . Thus  $B = \bar{\Gamma}$  and hence  $\bar{f}$  preserves sources. Similarly,  $\bar{f}$  preserves targets and thus is a map of directed graphs.  $\square$

**Corollary 5.2.** *In the situation of the theorem,  $\bar{f}(\bar{\Gamma}) = \overline{f(\Gamma)}$ .*

*Proof.* Since  $\bar{f}$  is continuous,  $\bar{f}(\bar{\Gamma}) \subseteq \overline{f(\Gamma)} = \overline{f(\Gamma)}$ . On the other hand,  $f(\Gamma) \subseteq \bar{f}(\bar{\Gamma})$  and  $\bar{f}(\bar{\Gamma})$  is closed, since it is a compact subset of the Hausdorff space  $\Delta$ . Thus  $\overline{f(\Gamma)} \subseteq \bar{f}(\bar{\Gamma})$ .  $\square$

Based on the theorem above, we make the following definition.

**Definition 5.3.** (Completion) Let  $\Gamma$  be a cofinite directed graph. Then any compact Hausdorff topological directed graph  $\bar{\Gamma}$  that contains  $\Gamma$  as a dense uniform sub-directed graph is called a *completion* of  $\Gamma$ .

As an immediate consequence of Theorem 5.1, we have the following result on uniqueness of completions.

**Corollary 5.4.** *If  $\bar{\Gamma}_1$  and  $\bar{\Gamma}_2$  are completions of a cofinite directed graph  $\Gamma$ , then there is a unique isomorphism of topological directed graphs  $\bar{i}: \bar{\Gamma}_1 \rightarrow \bar{\Gamma}_2$  extending the identity map on  $\Gamma$ .*

We turn now to the question of existence of completions. Let  $\Gamma$  be a cofinite directed graph and let  $I$  be any cofinal set of cofinite entourages of  $\Gamma$ . Note that  $(I, \leq)$ , where  $\leq = \supseteq$  is the reverse of inclusion, is a directed set. Indeed  $\leq$  is a partial order on  $I$ , and since  $I$  is cofinal, if  $R, S \in I$ , then there exists  $T \in I$  such that  $T \subseteq R \cap S$ , and so  $T \geq R$  and  $T \geq S$ .

For each  $R \in I$ , put  $\Gamma_R = \Gamma/R$  (a finite discrete topological graph) and denote the quotient map  $\Gamma \rightarrow \Gamma_R$  by  $x \mapsto x_R$ . For  $R, S \in I$  with  $R \leq S$ , observe that the identity map on  $\Gamma$  determines a map of directed graphs  $\phi_{RS}: \Gamma_S \rightarrow \Gamma_R$  given by  $\phi_{RS}(x_S) = x_R$ ; it is well defined since  $S \subseteq R$  and continuous since  $\Gamma_S$  is discrete. These maps have the properties:

- (i)  $\phi_{RR} = \text{id}_{\Gamma_R}$  for all  $R \in I$ ; and
- (ii)  $\phi_{RS}\phi_{ST} = \phi_{RT}$  for all  $R \leq S \leq T$  in  $I$ .

Hence we have an inverse system of (finite discrete) topological directed graphs and (surjective) continuous maps of directed graphs  $(\Gamma_R, \phi_{RS})_{R \leq S \in I}$ , indexed by the directed set  $I$ .

Let  $\widehat{\Gamma} = \varprojlim \Gamma_R$  be the inverse limit of this inverse system (in the category of topological directed graphs) and denote the projection maps by  $\phi_R: \widehat{\Gamma} \rightarrow \Gamma_R$ . Then  $\widehat{\Gamma}$  is a topological directed graph and there is a canonical map of directed graphs  $i: \Gamma \rightarrow \widehat{\Gamma}$  given by  $i(x) = (x_R)_{R \in I}$ . Moreover,  $\widehat{\Gamma}$  is compact and Hausdorff since its underlying space is an inverse limit of compact Hausdorff spaces. Thus, as usual, we regard  $\widehat{\Gamma}$  as a uniform space with the unique uniformity compatible with its topology. It should be noted that the pre-images under the projection maps  $\phi_R: \widehat{\Gamma} \rightarrow \Gamma_R$  of basis entourages of  $\Gamma_R$  form a base for the uniformity on  $\widehat{\Gamma}$  (see, for instance, [1]). However, being finite and discrete,  $\{\delta_{\Gamma_R}\}$  is a base for the uniformity on  $\Gamma_R$ ; and  $(\phi_R \times \phi_R)^{-1}[\delta_{\Gamma_R}] = \ker \phi_R$ . Hence, the set of all  $\ker \phi_R$ , as  $R$  runs through  $I$ , is a base for the uniformity on  $\widehat{\Gamma}$ .

With this setup, we make the following observation.

**Lemma 5.5.** *If  $\Gamma$  is a cofinite directed graph, then  $\widehat{\Gamma} = \varprojlim \Gamma_R$  is a compact Hausdorff topological directed graph and the canonical map  $i: \Gamma \rightarrow \widehat{\Gamma}$  is a uniform embedding whose image  $i(\Gamma)$  is a dense sub-directed graph of  $\widehat{\Gamma}$ .*

*Proof.* For each  $R \in I$ , note that  $(i \times i)^{-1}[\ker \phi_R] = R$  and  $R$  is an entourage of  $\Gamma$ . Since the family of all  $\ker \phi_R$ , where  $R$  ranges over  $I$ , is a base for the uniformity on  $\widehat{\Gamma}$ , we see that  $i$  is uniformly continuous.

Since  $\Gamma$  is Hausdorff,  $\ker i = \bigcap_{R \in I} R = \delta_\Gamma$  by Proposition 4.4. Thus  $i$  is injective. Further observe that  $(i \times i)[R] = \ker \phi_R \cap [i(\Gamma) \times i(\Gamma)]$  for all  $R \in I$  (a base for the uniformity on  $\Gamma$ ). Hence  $i$  is a uniform isomorphism from  $\Gamma$  onto the uniform subspace  $i(\Gamma)$  of  $\widehat{\Gamma}$ . That is,  $i$  is a uniform embedding.

Finally, note that for each  $R \in I$ , the image  $\phi_R(i(\Gamma)) = \Gamma_R$ . Therefore, since the underlying space of  $\widehat{\Gamma}$  is the inverse limit of the underlying spaces of the  $\Gamma_R$ , it follows that  $i(\Gamma)$  is dense in  $\widehat{\Gamma}$ . □

Combining these observations we obtain the following result.

**Theorem 5.6.**(Existence and uniqueness of completions) *Every cofinite directed graph  $\Gamma$  has a completion, and it is unique up to an unique isomorphism of topological directed graphs extending the identity map on  $\Gamma$ .*

**Corollary 5.7.** *If  $\Gamma$  is a cofinite directed graph, then its completion  $\bar{\Gamma}$  is equal to the Hausdorff completion of the underlying uniform space of  $\Gamma$ .*

*Proof.* Since a compact Hausdorff space (endowed with its unique compatible uniformity) is complete, this follows from [1, II.3.6, Theorem 2].  $\square$

In the definition of the completion of a cofinite directed graph  $\Gamma$ , we did not require that  $\bar{\Gamma}$  is a cofinite directed graph. However, it turns out that this is automatically true. In proving this and other things, we modify our notation as follows.

From now on, if  $A$  is a subset of a cofinite directed graph  $\Gamma$ , then  $\bar{A}$  will denote the closure of  $A$  in the completion  $\bar{\Gamma}$ . Thus the closure of  $A$  in  $\Gamma$  is given by  $\bar{A} \cap \Gamma$ . Likewise, if  $W \subset \Gamma \times \Gamma$ , then  $\bar{W}$  will denote the closure of  $W$  in the product space  $\bar{\Gamma} \times \bar{\Gamma}$ ; and thus  $\bar{W} \cap (\Gamma \times \Gamma)$  is its closure in  $\Gamma \times \Gamma$ . (That is, a bar over a set will mean its closure in the largest available space.) Note that this convention is consistent with denoting the completion of  $\Gamma$  by  $\bar{\Gamma}$  since  $\Gamma$  is dense in  $\bar{\Gamma}$ . It is also consistent with denoting the unique extension given by Theorem 5.1 of a uniformly continuous map  $f$  from  $\Gamma$  to a compact Hausdorff topological directed graph  $\Delta$  by  $\bar{f}: \bar{\Gamma} \rightarrow \Delta$  since  $\bar{f}$  is the closure in  $\bar{\Gamma} \times \Delta$  of the subset  $f \subseteq \Gamma \times \Delta$ .

We first prove a couple of lemmas.

**Lemma 5.8.** *If  $\Gamma$  is a cofinite directed graph and  $\Sigma$  is a sub-directed graph of  $\Gamma$ , then  $\bar{\Sigma}$  is a sub-directed graph of  $\bar{\Gamma}$  and  $V(\bar{\Sigma}) = \overline{V(\Sigma)}$ . In particular,  $V(\bar{\Gamma}) = \overline{V(\Gamma)}$ .*

*Proof.* Since  $s(\Sigma) \subseteq \Sigma$  and the source map is continuous, it follows that  $s(\bar{\Sigma}) \subseteq \bar{\Sigma}$ . Similarly,  $t(\bar{\Sigma}) \subseteq \bar{\Sigma}$ , and so  $\bar{\Sigma}$  is a subgraph of  $\bar{\Gamma}$ . Its vertex set  $V(\bar{\Sigma}) = \bar{\Sigma} \cap V(\bar{\Gamma})$  is a closed set containing  $V(\Sigma)$ , and thus  $\overline{V(\Sigma)} \subseteq V(\bar{\Sigma})$ . On the other hand,  $V(\bar{\Sigma}) = s(\bar{\Sigma}) \subseteq \overline{s(\Sigma)} = \overline{V(\Sigma)}$ . Hence  $V(\bar{\Sigma}) = \overline{V(\Sigma)}$ .  $\square$

It follows from this lemma that if  $\Gamma$  is a cofinite directed graph and  $\Sigma$  is a sub-directed graph (endowed with the relative uniformity), then  $\bar{\Sigma}$  (endowed with the relative uniformity) is the completion of  $\Sigma$ .

**Lemma 5.9.** *Let  $\Gamma$  be a cofinite directed graph and let  $\bar{\Gamma}$  be its completion. If  $R$  is a cofinite entourage of  $\Gamma$ , then its closure  $\bar{R}$  in  $\bar{\Gamma} \times \bar{\Gamma}$  is a cofinite entourage of  $\bar{\Gamma}$  and  $\bar{R} \cap (\Gamma \times \Gamma) = R$ .*

*Proof.* The natural map  $\nu: \Gamma \rightarrow \Gamma/R$  is a uniformly continuous map of directed graphs, and so by Theorem 5.1, it extends to a uniformly continuous map of directed graphs  $\bar{\nu}: \bar{\Gamma} \rightarrow \Gamma/R$ . Note that  $\ker \bar{\nu}$  is a cofinite entourage of  $\bar{\Gamma}$  since  $\Gamma/R$  is discrete. Also note that  $\ker \bar{\nu} \cap (\Gamma \times \Gamma) = \ker \nu = R$ . To complete the proof, it suffices to show that  $\ker \bar{\nu} = \bar{R}$ .

First note that  $R = \ker \nu \subseteq \ker \bar{\nu}$  and  $\ker \bar{\nu}$  is closed in  $\bar{\Gamma} \times \bar{\Gamma}$ . Thus  $\bar{R} \subseteq \ker \bar{\nu}$ . Conversely, let  $(x, y) \in \ker \bar{\nu}$  and let  $W$  be an open neighborhood of  $(x, y)$  in  $\bar{\Gamma} \times \bar{\Gamma}$ . Then

$$W \cap R = W \cap \ker \nu = W \cap \ker \bar{\nu} \cap (\Gamma \times \Gamma) \neq \emptyset$$

since  $W \cap \ker \bar{\nu}$  is a non empty open subset of  $\bar{\Gamma} \times \bar{\Gamma}$  and  $\Gamma \times \Gamma$  is dense in  $\bar{\Gamma} \times \bar{\Gamma}$ . So  $(x, y) \in \bar{R}$ , and thus the opposite inclusion  $\ker \bar{\nu} \subseteq \bar{R}$  also holds.  $\square$

**Theorem 5.10.** *Let  $\Gamma$  be a cofinite directed graph. Then its completion  $\bar{\Gamma}$  is a profinite directed graph and the cofinite entourages of  $\bar{\Gamma}$  are precisely all the  $\bar{R}$ , where  $R$  runs through all cofinite entourages of  $\Gamma$ .*

*Proof.* Let  $U$  be an entourage of  $\bar{\Gamma}$ . Choose a closed entourage  $W$  such that  $W \subseteq U$ ; see, for example, [1, II.1.2, Corollary 2]. Since  $W \cap (\Gamma \times \Gamma)$  is an entourage of  $\Gamma$ , there exists a cofinite entourage  $R$  of  $\Gamma$  such that  $R \subseteq W \cap (\Gamma \times \Gamma)$ . By Lemma 5.9,  $\bar{R}$  is a cofinite entourage of  $\bar{\Gamma}$  and  $\bar{R} \subseteq W \subseteq U$  since  $W$  is closed. It follows that the cofinite entourages of  $\bar{\Gamma}$  form a fundamental system of entourages of  $\bar{\Gamma}$ . Therefore  $\bar{\Gamma}$  is a profinite directed graph by Theorem 3.4.

Now let  $S$  be any cofinite entourage of  $\bar{\Gamma}$  and put  $R = S \cap (\Gamma \times \Gamma)$ . Then  $R$  is a cofinite entourage of  $\Gamma$  and it is dense in  $S$ , since  $S$  is a non-empty open subset of  $\bar{\Gamma} \times \bar{\Gamma}$  and  $\Gamma \times \Gamma$  is dense in  $\bar{\Gamma} \times \bar{\Gamma}$ . Thus,  $S = \bar{R} \cap S = \bar{R}$  since  $S$  is also closed. Conversely, if  $R$  is a cofinite entourage of  $\Gamma$ , then  $\bar{R}$  is a cofinite entourage of  $\bar{\Gamma}$  by Lemma 5.9.  $\square$

**Corollary 5.11.** *If  $\Gamma$  is a cofinite directed graph and  $R$  is a cofinite entourage of  $\Gamma$ , then the inclusion map  $\Gamma \hookrightarrow \bar{\Gamma}$  induces an isomorphism of topological directed graphs  $\Gamma/R \rightarrow \bar{\Gamma}/\bar{R}$ .*

*Proof.* Let  $f: \Gamma \rightarrow \bar{\Gamma}/\bar{R}$  be the map of directed graphs given by  $f(x) = \bar{R}[x]$ , i.e., the map induced by the inclusion map  $\Gamma \hookrightarrow \bar{\Gamma}$ . Note that  $f$  is onto since  $\Gamma$  is dense in  $\bar{\Gamma}$ , and  $\ker f = \bar{R} \cap (\Gamma \times \Gamma) = R$  by Lemma 5.9. Thus, by the first isomorphism theorem for directed graphs, there is an isomorphism of directed graphs  $f': \Gamma/R \rightarrow \bar{\Gamma}/\bar{R}$  such that the following diagram commutes:

$$\begin{array}{ccc}
 \Gamma & \xrightarrow{f} & \bar{\Gamma}/\bar{R} \\
 \downarrow q & \nearrow f' & \\
 \Gamma/R & & 
 \end{array}$$

and  $f'$  is also a homeomorphism since  $\Gamma/R$  and  $\bar{\Gamma}/\bar{R}$  are discrete.  $\square$

**Example 5.12.** Let  $\Gamma$  be a topological directed graph. Suppose that  $\Gamma$  is completely regular so that it is embedded in its Stone-Ćech compactification  $\beta(\Gamma)$ ; we will regard  $\Gamma \subseteq \beta(\Gamma)$ . We claim that  $\beta(\Gamma)$  has a unique topological directed graph structure such that  $\Gamma$  is a sub-directed graph. By the universal property of Stone-Ćech compactifications, the source and target maps on  $\Gamma$  extend to continuous maps  $s, t: \beta(\Gamma) \rightarrow \beta(\Gamma)$  and the extensions are unique. Note that  $s(s(x)) = s(x)$



and  $t(t(x)) = t(x)$  for all  $x \in \beta(\Gamma)$  since these equations hold for all  $x$  in the dense subspace  $\Gamma$ . It is also easy to check that the fixed point sets of  $s$  and  $t$  in  $\beta(\Gamma)$  are both equal to  $\overline{V(\Gamma)}$  (the closure of  $V(\Gamma)$  in  $\beta(\Gamma)$ ). Thus  $\beta(\Gamma)$  equipped with the unique continuous extensions of the source and target maps of  $\Gamma$  is a topological directed graph with  $V(\beta(\Gamma)) = \overline{V(\Gamma)}$ , and our claim follows.

**Theorem 5.13.** *Let  $\Gamma$  be a directed graph endowed with its maximal cofinite uniformity. Then  $\beta(\Gamma)$  is the completion of  $\Gamma$ .*

*Proof.* By Proposition 4.3,  $\Gamma$  is discrete. So  $\beta(\Gamma)$  is a totally disconnected, compact Hausdorff space containing  $\Gamma$  as a subspace. (It is well known that the Stone–Čech compactification of a discrete space is totally disconnected.) Therefore, equipped with the unique topological directed graph structure such that  $\Gamma$  is a sub-directed graph (as constructed above),  $\beta(\Gamma)$  is a profinite directed graph.

Applying Theorem 3.4, we see that the set of all cofinite entourages of  $\beta(\Gamma)$  is a base for its uniformity and given any cofinite entourage  $S$  of  $\beta(\Gamma)$ , note that  $S \cap (\Gamma \times \Gamma)$  is a cofinite entourage of  $\Gamma$  (since  $\Gamma$  has the maximal cofinite uniformity). Thus the inclusion map  $\Gamma \hookrightarrow \beta(\Gamma)$  is uniformly continuous.

On the other hand, suppose that  $R$  is any cofinite entourage of  $\Gamma$  (i.e., any compatible equivalence relation of finite indexes on  $\Gamma$ ). Then  $\Gamma/R$  is a finite discrete directed graph (and thus a compact Hausdorff space), so the quotient map  $\phi_R: \Gamma \rightarrow \Gamma/R$  extends to a continuous map  $\phi'_R: \beta(\Gamma) \rightarrow \Gamma/R$  which is a map of directed graphs (since the restriction to the dense sub-directed graph  $\Gamma$  is a map of directed graphs). Observe that  $\ker \phi'_R$  is a cofinite entourage of  $\beta(\Gamma)$  and  $\ker \phi'_R \cap (\Gamma \times \Gamma) = R$ . It follows that  $\Gamma$  is uniformly embedded in  $\beta(\Gamma)$ , and hence  $\beta(\Gamma)$  is a compact Hausdorff topological directed graph that contains  $\Gamma$  as a dense uniform sub-directed graph. Consequently  $\beta(\Gamma)$  is the completion of  $\Gamma$ .  $\square$

## 6. Applications

### 6.1. Cofinite Spaces

A *cofinite space* is a cofinite directed graph  $X$  with  $V(X) = X$ . In this case, the source and target maps are equal to the identity map on  $X$ , and so are irrelevant. The completion  $\overline{X}$  of a cofinite space  $X$  is also a cofinite space since  $V(\overline{X}) = \overline{V(X)} = \overline{X}$  (by Lemma 5.8), and it is compact. Thus  $\overline{X}$  is a profinite space (i.e., a compact Hausdorff totally disconnected space).

### 6.2. Cofinite Graphs

By a graph we mean a graph in the sense of Serre [5] and Stallings [6]. Thus a *graph* is a set  $\Gamma$  equipped with a *source map*  $s: \Gamma \rightarrow \Gamma$  and an *inversion map*  $^{-1}: \Gamma \rightarrow \Gamma$  such that for all  $x \in \Gamma$ ,

$$s(s(x)) = s(x), (x^{-1})^{-1} = x, \text{ and } s(x) = x \Leftrightarrow x^{-1} = x.$$

The common fixed point set of these two maps is called the *vertex set* of  $\Gamma$  and is

denoted by  $V(\Gamma) = \{x \in \Gamma \mid s(x) = x\} = \{x \in \Gamma \mid x^{-1} = x\}$ . For  $x \in \Gamma$ , the vertex  $s(x)$  is called the *source* of  $x$  and we define the *target* of  $x$  to be the vertex  $t(x) = s(x^{-1})$ . We call  $x^{-1}$  the *inverse* of  $x$ . The set  $E(\Gamma) = \Gamma \setminus V(\Gamma)$  is called the *edge set* of  $\Gamma$ . In other words, a graph is directed graph equipped with an involution that interchanges sources and targets and whose fixed point set is the vertex set.

A *map of graphs* is a function  $f: \Gamma \rightarrow \Delta$  between graphs that preserves sources and inverses. It then follows that targets are also preserved, and thus  $f$  is also a map of directed graphs.

Let  $R$  be an equivalence relation on a graph  $\Gamma$ . We say that  $R$  is a *graph equivalence relation* if  $\Gamma/R$  admits a structure of a graph such that the natural map  $\nu: \Gamma \rightarrow \Gamma/R$  is a map of graphs. In this case, it is clear that the graph structure on  $\Gamma/R$  is unique and given by:  $s(R[x]) = R[s(x)]$ ,  $t(R[x]) = R[t(x)]$ ,  $R[x]^{-1} = R[x^{-1}]$ , and  $V(\Gamma/R) = \nu(V(\Gamma))$ . It is also easy to see that  $R$  is a graph equivalence relation if and only if these two conditions hold:

- (1) if  $(x, y) \in R$ , then  $(s(x), s(y)) \in R$  and  $(x^{-1}, y^{-1}) \in R$ ; and
- (2)  $(x, s(x)) \in R$  if and only if  $(x, x^{-1}) \in R$ .

**Definition 6.1.**(Cofinite graph) Let  $\Gamma$  be a graph endowed with a uniformity. A *cofinite graph entourage* of  $\Gamma$  is a graph equivalence relation  $R$  of finite indexes on  $\Gamma$  which is also an entourage of  $\Gamma$ . We say that  $\Gamma$  is a *cofinite graph* if it is Hausdorff and its uniformity has a base consisting of cofinite graph entourages of  $\Gamma$ .

In particular, a cofinite graph  $\Gamma$  is a cofinite directed graph. So by Lemma 4.2, the source and target maps of a cofinite graph  $\Gamma$  are uniformly continuous, and by a similar argument, its involution is also uniformly continuous. Thus  $\Gamma$  is a Hausdorff topological graph (i.e., a graph endowed with a topology such that the source, target, and inversion maps are all continuous). Furthermore, by Theorem 5.10, its completion  $\bar{\Gamma}$  as a cofinite directed graph is a profinite directed graph. We show next that there is a unique involution on  $\bar{\Gamma}$  making it into a cofinite graph such that  $\Gamma$  is a subgraph (i.e., a sub-directed graph closed under the involution).

**Theorem 6.2.** *If  $\Gamma$  is a cofinite graph, then its completion  $\bar{\Gamma}$  has a unique involution making it into a cofinite graph such that  $\Gamma$  is a subgraph. Moreover, the cofinite graph entourages of  $\bar{\Gamma}$  are precisely the closures  $\bar{R}$  in  $\bar{\Gamma} \times \bar{\Gamma}$  of all cofinite graph entourages  $R$  of  $\Gamma$ .*

*Proof.* Let  $\bar{\Gamma}^{\text{Reverse}}$  be the topological directed graph with the same underlying topological space as  $\bar{\Gamma}$  but with source and target maps interchanged. Then the map  $\Gamma \rightarrow \bar{\Gamma}^{\text{Reverse}}$  given by  $x \mapsto x^{-1}$  is a map of directed graphs. Applying Theorem 5.1, we see that this map has a unique continuous extension to a map of directed graphs  $\bar{\Gamma} \rightarrow \bar{\Gamma}^{\text{Reverse}}$ . We denote this extension, regarded as a map from  $\bar{\Gamma}$  to itself (i.e., by composing with the identity map  $\bar{\Gamma}^{\text{Reverse}} \rightarrow \bar{\Gamma}$ ), also by  $x \mapsto x^{-1}$ . It is clearly the unique continuous extension to  $\bar{\Gamma}$  of the involution on  $\Gamma$  and satisfies the

property:  $(x^{-1})^{-1} = x$  for all  $x \in \bar{\Gamma}$  (since this property holds on the dense subset  $\Gamma$ ).

It remains to show that the fixed set  $A = \{x \in \bar{\Gamma} \mid x = x^{-1}\}$  of this involution is equal to  $V(\bar{\Gamma})$ . Note first that  $A$  is a closed subset of  $\bar{\Gamma}$  containing  $V(\Gamma)$ , and thus  $V(\bar{\Gamma}) = \bar{V(\Gamma)} \subseteq A$  (using Lemma 5.8). To see the opposite inclusion, let  $x \in A$  and let  $R$  be a cofinite graph entourage of  $\Gamma$ . Then the natural map  $\nu: \Gamma \rightarrow \Gamma/R$  is a map of graphs, and in particular is involution-preserving. Since  $\Gamma$  is dense in  $\bar{\Gamma}$ , it is clear that the unique continuous extension  $\bar{\nu}: \bar{\Gamma} \rightarrow \Gamma/R$  is also involution-preserving. Thus  $\bar{\nu}(x) = \bar{\nu}(x^{-1}) = \bar{\nu}(x)^{-1}$ . However  $\Gamma/R$  is a graph, and so  $\bar{\nu}(x)$  is a vertex of  $\Gamma/R$ . Since  $\bar{R} = \ker \bar{\nu}$  (by the proof of Lemma 5.9) and the vertex set of  $\Gamma/R$  is  $\nu(V(\Gamma))$ , it follows that  $x \in \bar{R}[V(\Gamma)]$ . However the set of closures  $\bar{R}$  of all cofinite graph entourages  $R$  of  $\Gamma$  form a base for the uniformity on  $\bar{\Gamma}$ . Consequently  $x \in V(\bar{\Gamma})$ , and so  $x \in V(\bar{\Gamma})$  by Lemma 5.8.

To establish the second part, it suffices (by Theorem 5.10) to show that for each cofinite entourage  $R$  of  $\Gamma$ ,  $R$  is a graph equivalence relation on  $\Gamma$  if and only if its closure  $\bar{R}$  is a graph equivalence relation on  $\bar{\Gamma}$ . However, this is an immediate consequence of Corollary 5.11 since it is clear that the induced isomorphism of directed graphs  $\Gamma/R \rightarrow \bar{\Gamma}/\bar{R}$  is also involution-preserving. Thus  $\nu: \Gamma \rightarrow \Gamma/R$  is a map of graphs if and only if  $\bar{\nu}: \bar{\Gamma} \rightarrow \bar{\Gamma}/\bar{R}$  is a map of graphs.  $\square$

**Corollary 6.3.** *If  $\Gamma$  is a cofinite graph, then its completion  $\bar{\Gamma}$  is a compact, Hausdorff, totally disconnected topological graph.*

*Proof.* By Theorem 6.2,  $\bar{\Gamma}$  is a cofinite graph, and thus a topological graph. Moreover, it is compact, Hausdorff, and totally disconnected by Theorem 5.10.  $\square$

**Remark.** It is easy to see that a cofinite graph  $\Gamma$  is compact if and only if it is isomorphic to the inverse limit of an inverse system of finite discrete topological graphs and continuous maps of graphs. Such cofinite graphs are called *profinite graphs*. A somewhat more complicated fact is that profinite graphs are the same thing as compact, Hausdorff, totally disconnected topological graphs.

### 6.3. Cofinite Groupoids

Above we saw that the completions of cofinite spaces and cofinite graphs are profinite spaces and profinite graphs, respectively. We will see that the situation for cofinite groupoids is more complicated in general.

To begin with, we review some basic groupoid theory. Recall that a groupoid is a small category in which every arrow is invertible. To be more precise, a *groupoid* is a directed graph  $G$  equipped with a partial binary operation (denoted by juxtaposition) satisfying these conditions: for all  $g, h, k \in G$ ,

1.  $gh$  is defined if and only if  $t(g) = s(h)$ , and in this case  $s(gh) = s(g)$  and  $t(gh) = t(h)$ ;
2. if  $t(g) = s(h)$  and  $t(h) = s(k)$ , then  $(gh)k = g(hk)$ ;
3.  $s(g)g = g$  and  $gt(g) = g$ ;

4. there exists  $g^{-1} \in G$  such that  $s(g^{-1}) = t(g)$ ,  $t(g^{-1}) = s(g)$  and  $gg^{-1} = s(g)$ ,  $g^{-1}g = t(g)$ .

Customarily the set of composable pairs in a groupoid  $G$  is denoted

$$G^{(2)} = \{(g, h) \in G \times G \mid t(g) = s(h)\}.$$

Then the partial binary operation on  $G$  is a function  $G^{(2)} \rightarrow G$ , denoted by  $(g, h) \mapsto gh$ . Likewise, we write  $G^{(3)}$  for the set of composable triples in  $G$ . Then the associativity property (Axiom 2) can be stated as: if  $(g, h, k) \in G^{(3)}$ , then  $(gh)k = g(hk)$ . Also note that there is an identity at each vertex  $v$  in a groupoid which we are taking to be  $v$  itself; by a standard argument, it is unique. Similarly, the inverse  $g^{-1}$  of an element  $g$  satisfying Axiom 4 is unique.

A groupoid  $G$  with a single vertex is nothing other than a group, and in this case,  $V(G) = \{1\}$  and  $G^{(2)} = G \times G$ .

Given any groupoid  $G$  and vertices  $x, y \in V(G)$ , we denote by  $G(x, y)$  the (possibly empty) set of all arrows from  $x$  to  $y$  in  $G$ . Note in particular that when  $x = y$ , the set  $G(x, x)$  is a subgroupoid of  $G$  with a single vertex, and hence is a group. The groups  $G(x, x)$ , for  $x \in V(G)$ , are called the *vertex groups* or *local groups* of  $G$ .

If  $G$  and  $H$  are groupoids, a *homomorphism*  $\phi: G \rightarrow H$  is a function with the property: if  $(g_1, g_2) \in G^{(2)}$ , then  $(\phi(g_1), \phi(g_2)) \in H^{(2)}$  and  $\phi(g_1g_2) = \phi(g_1)\phi(g_2)$ . In this situation, it is easy to check that  $\phi$  is also a map of directed graphs.

A subset  $H$  of a groupoid  $G$  is a *subgroupoid* of  $G$  if:

- (1)  $H$  is a sub-directed graph of  $G$ ;
- (2) if  $h_1, h_2 \in H$  and  $(h_1, h_2) \in G^{(2)}$ , then  $h_1h_2 \in H$ ; and
- (3)  $h^{-1} \in H$  whenever  $h \in H$ .

In this case,  $H$  is itself a groupoid under the restrictions of the operations on  $G$ .

The *product groupoid* of groupoids  $G_1$  and  $G_2$  is the directed graph  $G_1 \times G_2$  with partial binary operation  $(G_1 \times G_2)^{(2)} \rightarrow G_1 \times G_2$  defined as follows. If  $(g_1, g_2), (h_1, h_2) \in G_1 \times G_2$  and  $t(g_1, g_2) = s(h_1, h_2)$ , then  $(g_1, g_2)(h_1, h_2) = (g_1h_1, g_2h_2)$ . The product of an arbitrary family of groupoids is defined similarly.

An equivalence relation  $\rho$  on a groupoid  $G$  is called a *congruence* if it is also a subgroupoid of the product groupoid  $G \times G$ . That is, a congruence on  $G$  is an equivalence relation  $\rho$  on  $G$  with the properties:

- (1) if  $(g, h) \in \rho$ , then  $(s(g), s(h)) \in \rho$  and  $(t(g), t(h)) \in \rho$ ;
- (2) if  $(g_1, g_2), (h_1, h_2) \in \rho$  and  $t(g_1, g_2) = s(h_1, h_2)$ , then  $(g_1h_1, g_2h_2) \in \rho$ ; and
- (3) if  $(g, h) \in \rho$ , then  $(g^{-1}, h^{-1}) \in \rho$ .

However, in this situation, the quotient of the underlying directed graph  $G/\rho$  need not admit a structure of a groupoid such that the natural map  $\nu: G \rightarrow G/\rho$  is a homomorphism. For instance, suppose  $G$  has distinct vertices  $x, y, z, w$  such that  $G(x, y)$  and  $G(z, w)$  are non-empty but  $G(x, w) = \emptyset$ . Let  $\rho$  be the congruence on  $G$  that identifies  $y$  to  $z$ . Then  $G/\rho$  has arrows from  $\nu(x)$  to  $\nu(y) = \nu(z)$  and arrows from  $\nu(y) = \nu(z)$  to  $\nu(w)$ , but no arrows from  $\nu(x)$  to  $\nu(w)$ . Hence  $G/\rho$  does not admit any structure of a groupoid at all.

To avoid this difficulty, we restrict our attention to a certain type of congruences that behave better. To describe these, we make use of the following notation. If  $\Gamma$  is any directed graph (for example, a groupoid) and  $n$  is a positive integer, let  $\Gamma^{(n)}$  denote the set of consecutive  $n$ -tuples  $(x_1, \dots, x_n)$  of elements of  $\Gamma$  (i.e.,  $t(x_i) = s(x_{i+1})$  for all  $1 \leq i \leq n - 1$ ). Given any map of directed graphs  $f: \Gamma \rightarrow \Delta$ , we use the same symbol to denote the induced map on the sets of consecutive  $n$ -tuples  $f: \Gamma^{(n)} \rightarrow \Delta^{(n)}$  given by  $f(x_1, \dots, x_n) = (f(x_1), \dots, f(x_n))$ . Let  $\mathcal{G}$  denote the category whose class of objects consists of all directed graphs and whose morphisms are all maps of directed graphs  $f: \Gamma \rightarrow \Delta$  such that  $f(\Gamma^{(3)}) = \Delta^{(3)}$ . We call these morphisms  $\mathcal{G}$ -maps. Note that if  $f: \Gamma \rightarrow \Delta$  is a  $\mathcal{G}$ -map, then  $f$  is surjective and also  $f(\Gamma^{(2)}) = \Delta^{(2)}$ . Finally, we say that a homomorphism of groupoids  $f: G \rightarrow H$  is a  $\mathcal{G}$ -homomorphism if its underlying map of directed graphs is a  $\mathcal{G}$ -map.

By a  $\mathcal{G}$ -congruence on a groupoid  $G$ , we mean a congruence  $\rho$  on  $G$  whose natural homomorphism  $\nu: G \rightarrow G/\rho$  is a  $\mathcal{G}$ -map. In this situation, the quotient  $G/\rho$  of the underlying directed graph of  $G$  admits a unique groupoid structure such that  $\nu$  is a  $\mathcal{G}$ -homomorphism. The inversion map and partial product on  $G/\rho$  are given by  $\nu(g)^{-1} = \nu(g^{-1})$  and  $\nu(g_1)\nu(g_2) = \nu(g_1g_2)$  for all  $g \in G$  and  $(g_1, g_2) \in G^{(2)}$ , which are well-defined since  $\rho$  is a congruence and  $\nu(G^{(2)}) = (G/\rho)^{(2)}$ . Associativity of this partial product on  $G/\rho$  carries over from  $G$  since  $\nu(G^{(3)}) = (G/\rho)^{(3)}$ .

**Definition 6.4.** Let  $G$  be a groupoid endowed with a uniformity. A *cofinite congruence* on  $G$  is a  $\mathcal{G}$ -congruence  $\rho$  of finite indexes on  $G$  such that  $\rho$  is also an entourage of  $G$ . We say that  $G$  is a *cofinite groupoid* if it is Hausdorff and the set of cofinite congruences on  $G$  is a base for its uniformity.

Note that a cofinite groupoid has an underlying cofinite directed graph structure, and thus has uniformly continuous source and target maps by Lemma 4.2. We observe next that the same goes for the involution and partial product maps.

**Lemma 6.5.** *Let  $G$  be a cofinite groupoid. Then the involution map  $G \rightarrow G, g \mapsto g^{-1}$ , and the partial product  $G^{(2)} \rightarrow G, (g, h) \mapsto gh$ , are uniformly continuous maps.*

*Proof.* Since the cofinite congruences  $\rho$  on  $G$  form a base for its uniformity, and  $(g^{-1}, h^{-1}) \in \rho$  whenever  $(g, h) \in \rho$ , it follows that the involution map  $g \mapsto g^{-1}$  is uniformly continuous. Also given any cofinite congruence  $\rho$  on  $G$ ,  $(\rho \times \rho) \cap G^{(2)}$  is an entourage of  $G^{(2)}$  such that: for all  $(g_1, h_1), (g_2, h_2) \in G^{(2)}$ , if  $(g_1, g_2) \in \rho$  and  $(h_1, h_2) \in \rho$ , then  $(g_1h_1, g_2h_2) \in \rho$ . Thus the partial product  $G^{(2)} \rightarrow G$  is a uniformly continuous map. □

In particular, it follows from Lemma 6.5 that a cofinite groupoid is a Hausdorff topological groupoid. Recall that a *topological groupoid* is a groupoid  $G$  endowed with a topology such that the maps  $G \rightarrow G, g \mapsto g^{-1}$  and  $G^{(2)} \rightarrow G, (g, h) \mapsto gh$ , are continuous (where  $G^{(2)}$  is endowed with the subspace topology of the product space  $G \times G$ ). Note that in this situation, the source and target maps are also continuous since  $s(g) = gg^{-1}$  and  $t(g) = g^{-1}g$ . Hence a topological groupoid has an underlying topological directed graph structure. The *category of topological groupoids* is the category whose objects are topological groupoids and whose morphisms are continuous homomorphisms of groupoids.

Every subgroupoid  $H$  of a topological groupoid  $G$  is given the subspace topology and thus becomes a topological groupoid. The product of a family  $(G_i)_{i \in I}$  of topological groupoids exists in the category of topological groupoids and can be constructed by taking the product  $\prod G_i$  of the underlying topological spaces and giving it the groupoid structure described above. Similarly, the inverse limit of an inverse system of topological groupoids and continuous homomorphisms exists in the category of topological groupoids, and its underlying topological space is the inverse limit of the corresponding inverse system of underlying spaces.

It is clear that each vertex group of a topological groupoid is a topological group. For cofinite groupoids we can say even more.

**Proposition 6.6.** *If  $G$  is a cofinite groupoid and  $x \in V(G)$ , then the vertex group  $G(x, x)$  is a cofinite group.*

*Proof.* Since  $G$  is Hausdorff,  $G(x, x)$  is also Hausdorff.

The set  $\{\rho[x] \cap G(x, x) \mid \rho \text{ is a cofinite congruence on } G\}$  is a base of open neighborhoods for the identity  $x$  in  $G(x, x)$ . However, if  $\rho$  is a cofinite congruence on  $G$ , then  $\rho \cap [G(x, x) \times G(x, x)]$  is a congruence of finite indexes on the group  $G(x, x)$  and  $\rho[x] \cap G(x, x)$  is the congruence class of the identity. It follows that each  $\rho[x] \cap G(x, x)$  is a normal subgroup of finite indexes in  $G(x, x)$ . Hence the identity in  $G(x, x)$  has a neighborhood base consisting of open normal subgroups of finite indexes in  $G(x, x)$ , and hence is a cofinite group.  $\square$

Note that every cofinite groupoid  $G$  has an underlying cofinite directed graph structure. We let  $\overline{G}$  denote the completion of the underlying cofinite directed graph of  $G$ . We make use of the following lemma below.

**Lemma 6.7.** *Let  $G$  be any cofinite groupoid. Then  $G^{(2)}$  is dense in  $\overline{G}^{(2)}$  (as subspaces of the product space  $\overline{G} \times \overline{G}$ ).*

*Proof.* Let  $\rho$  be a cofinite congruence on  $\Gamma$  and let  $\nu: G \rightarrow G/\rho$  be the natural map. Let  $\overline{\nu}: \overline{G} \rightarrow G/\rho$  be the unique continuous extension of  $\nu$ . Since  $\rho$  is a  $\mathcal{G}$ -congruence,  $\overline{\nu}(G^{(2)}) = (G/\rho)^{(2)}$  and thus  $\overline{\nu}[\overline{G}^{(2)}] = (\overline{G}/\overline{\rho})^{(2)}$  also holds (by Corollary 5.11). Since  $\overline{\rho} = \ker \overline{\nu}$  (by the proof of Lemma 5.9), it follows that  $\overline{G}^{(2)} \subseteq (\overline{\rho} \times \overline{\rho})[G^{(2)}] = \bigcup_{(g,h) \in G^{(2)}} (\overline{\rho}[g] \times \overline{\rho}[h])$ . Let  $I$  be the non-empty set of all cofinite congruence on  $\Gamma$ . Then using Theorem 5.10, we see that the set  $\{\overline{\rho} \times \overline{\rho} \mid \rho \in I\}$  is a base for the product uniformity on  $\overline{G} \times \overline{G}$ . Thus  $\overline{G}^{(2)} \subseteq \bigcap_{\rho \in I} (\overline{\rho} \times \overline{\rho})[G^{(2)}] = \overline{G}^{(2)}$ . It follows

that  $G^{(2)}$  is dense in  $\overline{G}^{(2)}$ . □

**Theorem 6.8.** *Let  $G$  be a cofinite groupoid. Then  $\overline{G}$  admits a unique topological groupoid structure such that  $G$  is a uniform subgroupoid. Moreover, equipped with this structure,  $\overline{G}$  is a profinite groupoid.*

*Proof.* Construct  $\overline{G}$  using the standard construction of Lemma 5.5 with  $I$  equal to the set of all cofinite congruences  $\rho$  on  $G$ . Since  $I$  is a base for the uniformity on  $G$ , it is a cofinal set of cofinite entourages of the underlying cofinite graph of  $G$ . Thus  $\overline{G} = \varprojlim_{\rho \in I} G/\rho$  is the completion of the underlying cofinite directed graph of  $G$ . For each  $\rho \in I$ , equip  $G/\rho$  with its unique groupoid structure such that the natural map  $G \rightarrow G/\rho$  is a homomorphism. Then each  $G/\rho$  is a discrete topological groupoid. Moreover, by the definition of the operations on  $G/\rho$ , it is clear that if  $\rho, \sigma \in I$  with  $\rho \leq \sigma$ , then  $\phi_{\rho\sigma}: G/\sigma \rightarrow G/\rho$  is a homomorphism of groupoids. Now  $\overline{G}$  is a profinite groupoid and the canonical map  $G \hookrightarrow \overline{G}$  is a uniform embedding and a homomorphism. Uniqueness of this topological groupoid structure on  $\overline{G}$  follows from  $G$  being dense in  $\overline{G}$  and  $G^{(2)}$  being dense in  $\overline{G}^{(2)}$  (by Lemma 6.7). □

The following corollaries of Theorem 6.8 are worth noting. The first is an immediate consequence.

**Corollary 6.9.** *If  $G$  is a compact cofinite groupoid, then  $G$  is a profinite groupoid.*

**Corollary 6.10.** *Let  $G$  be a cofinite groupoid and equip  $\overline{G}$  with its unique topological groupoid structure such that  $G$  is a uniform subgroupoid. If  $\rho$  is any cofinite congruence on  $G$ , then  $\overline{\rho}$  is a cofinite congruence on  $\overline{G}$ . Moreover, all cofinite congruences on  $\overline{G}$  are of this form.*

*Proof.* By the proof of Theorem 5.9,  $\overline{\rho} = \ker \overline{\nu}$ , where  $\overline{\nu}: \overline{G} \rightarrow G/\rho$  is the unique continuous extension of the natural map  $\nu: G \rightarrow G/\rho$ . Endow  $G/\rho$  with the unique groupoid structure such that  $\nu$  is a homomorphism. Then applying Lemma 6.7, we see that  $\overline{\nu}$  is also a homomorphism. Therefore  $\overline{\rho} = \ker \overline{\nu}$  is a congruence on  $\overline{G}$ . Let  $\eta: \overline{G} \rightarrow \overline{G}/\overline{\rho}$  be the natural map. Since  $\nu$  is a  $\mathcal{G}$ -map, it follows from Corollary 5.11 that  $\eta$  is also a  $\mathcal{G}$ -map and thus  $\overline{\rho}$  is a  $\mathcal{G}$ -congruence. However  $\overline{\rho}$  is also an entourage of  $\overline{G}$  by Lemma 5.9, so it is cofinite congruence on  $\overline{G}$ .

Conversely, let  $\sigma$  be any cofinite congruence on  $\overline{G}$ . Then the restriction  $\rho = \sigma \cap (G \times G)$  is a cofinite congruence on  $G$ , and as in the proof of Lemma 5.9,  $\overline{\rho} = \sigma$ . □

**Corollary 6.11.** *Let  $G$  be a cofinite groupoid and equip  $\overline{G}$  with its unique topological groupoid structure such that  $G$  is a uniform subgroupoid. If  $H$  is a compact Hausdorff topological groupoid and  $\phi: G \rightarrow H$  is a uniformly continuous homomorphism, then the unique continuous map  $\overline{\phi}: \overline{G} \rightarrow H$  extending  $\phi$  is a homomorphism of groupoids.*

*Proof.* The set  $C = \{(g, h) \in \overline{G}^{(2)} \mid \overline{\phi}(gh) = \overline{\phi}(g)\overline{\phi}(h)\}$  is a closed subset of  $\overline{G}^{(2)}$  containing  $G^{(2)}$ . So, by Lemma 6.7,  $C = \overline{G}^{(2)}$  and hence  $\overline{\phi}$  is a homomorphism. □

#### 6.4. Cofinite Groupoids with Finite Vertex Sets

We will see that cofinite groupoids with only finitely many vertices behave very much like cofinite groups in many regards. To start with, we make a convenient definition.

We say that a congruence  $\rho$  on a groupoid  $G$  is *rigid* if its restriction to  $V(G)$  is trivial; i.e.,  $\rho \cap [V(G) \times V(G)] = \delta_{V(G)}$ . In this case, if  $(g, h) \in \rho$ , then  $s(g) = s(h)$  and  $t(g) = t(h)$ . In particular, when  $\rho$  is rigid, the natural map  $\nu: G \rightarrow G/\rho$  has the property that  $\nu[(G^{(3)})] = (G/\rho)^{(3)}$ , and so (as in the proof of Theorem 6.8)  $G/\rho$  has a unique groupoid structure such that  $\nu: G \rightarrow G/\rho$  is a homomorphism (and it is bijective on the vertex sets).

There is an equivalent alternative approach to rigid congruences using normal subgroups. Notice that if  $\rho$  is a rigid congruence on a groupoid  $G$ , then for each  $x \in V(G)$ , the congruence class  $\rho[x]$  is a normal subgroup of the vertex group  $G(x, x)$  and this family  $(\rho[x])_{x \in V(G)}$  is *coherent* in the following sense: if  $g \in G$ , then  $\rho[s(g)]^g = \rho[t(g)]$  (where  $\rho[s(g)]^g = g^{-1}\rho[s(g)]g$ ). Conversely, if  $(N_x)_{x \in V(G)}$  is any coherent family of normal subgroups of  $G$ , then there is a unique rigid congruence  $\rho$  on  $G$  such that  $\rho[x] = N_x$  for all  $x \in V(G)$ ; it is defined by  $(g, h) \in \rho$  if and only if  $s(g) = s(h)$ ,  $t(g) = t(h)$ , and  $gh^{-1} \in N_{s(g)}$  (or equivalently,  $g^{-1}h \in N_{t(g)}$ ).

In order for a groupoid to have rigid congruences of finite indexes, its vertex set must be finite. And in this case, we make the following very useful observation.

**Lemma 6.12.** *Let  $G$  be a cofinite groupoid with a finite vertex set. Then the set of rigid cofinite congruences on  $G$  is a base for its uniformity.*

*Proof.* We are assuming that  $G$  is non-empty as the assertion is trivial in the case where  $G$  is empty. Since  $G$  is Hausdorff, its finite vertex set  $V(G)$  is a discrete subset of  $G$ . So for each  $x \in V(G)$ , there exists a cofinite congruence  $\rho_x$  on  $G$  such that  $\rho_x[x] \cap V(G) = \{x\}$ . Then the finite intersection  $\sigma = \bigcap_{x \in V(G)} \rho_x$  is a cofinite congruence on  $G$  and it is rigid. Consequently, the set of all  $\rho \cap \sigma$ , where  $\rho$  runs through all cofinite congruences on  $G$ , is a base for the uniformity on  $G$  consisting of rigid cofinite congruences.  $\square$

Extending a property of cofinite groups, we next observe that every cofinite groupoid with only finitely many vertices is completely determined by its topological groupoid structure.

**Proposition 6.13.** *Let  $G$  be a topological groupoid with a finite vertex set. Then  $G$  has a uniformity making it into a cofinite groupoid and inducing its topology if and only if each  $G(x, y)$  is an open subset of  $G$  and each vertex group  $G(x, x)$  is a cofinite group, for  $x, y \in V(G)$ . Moreover, in this case, such a uniformity on  $G$  is unique.*

*Proof.* Suppose that the  $G(x, y)$  are open and that the  $G(x, x)$  are cofinite groups. By taking one component at a time, we may assume that the underlying directed graph of  $G$  is connected; i.e., that  $G(x, y)$  is non-empty for all  $x, y \in V(G)$ . Fix  $x \in V(G)$ . Then for each open normal subgroup  $N$  of  $G(x, x)$ , we claim that



there is a unique rigid congruence of finite indexes  $\rho_N$  on  $G$  such that  $\rho_N[x] = N$ . We construct  $\rho_N$  by forming a coherent family of normal subgroups  $N_y \triangleleft G(y, y)$ ,  $y \in V(G)$ . For each  $y \in V(G)$ , choose  $g \in G(x, y)$  and put  $N_y = N^g$ . Note that  $N_y$  does not depend on the choice of  $g$  since  $N$  is a normal subgroup of  $G(x, x)$ . Let  $\rho_N$  be the corresponding rigid congruence on  $G$  with  $\rho[y] = N_y$  for each vertex  $y$ . In particular,  $\rho[x] = N_x = N$  and it is clear that  $\rho_N$  is the unique such congruence on  $G$ . Furthermore,  $\rho_N$  has finite index. Indeed, if we make a choice of  $g_y \in G(x, y)$  for each  $y \in V(G)$ , then every congruence class is of the form  $\rho_N[g_y^{-1}gg_z] = g_y^{-1}\rho_N[g]g_z = g_y^{-1}(gN)g_z$  for some  $y, z \in V(G)$  and  $g \in G(x, x)$ . Thus, if  $G$  has  $n$  vertices, then the index of  $\rho_N$  is  $n^2$  times the index of  $N$  in  $G(x, x)$ . For later use, also note that each congruence class of  $\rho_N$  is an open subset of  $G$  since the cosets  $gN$  are open in  $G(x, x)$  and we are assuming that each  $G(x, y)$  is open in  $G$ .

Note that  $B = \{\rho_N \mid N \text{ is an open normal subgroup of } G(x, x)\}$  is a base for a uniformity on  $G$ . Indeed, if  $N_1$  and  $N_2$  are open normal subgroups of  $G(x, x)$ , then so is  $N_1 \cap N_2$  and  $\rho_{N_1 \cap N_2} = \rho_{N_1} \cap \rho_{N_2}$ . We claim that this uniformity induces the topology on  $G$ . Let  $U$  be an open subset of  $G$  and let  $g \in U$ . Put  $y = s(g)$  and  $z = t(g)$  so that  $g \in G(y, z)$ . Choose  $g_y \in G(x, y)$  and  $g_z \in G(x, z)$ . Then  $g_y g g_z^{-1} \in G(x, x)$ . Since we are assuming that  $G(y, z)$  is open in  $G$ , we may assume that  $U \subseteq G(y, z)$  and so  $g_y U g_z^{-1}$  is an open subset of  $G(x, x)$  containing  $g_y g g_z^{-1}$ . However  $G(x, x)$  is a cofinite group, and so there exists an open normal subgroup  $N$  such that  $g_y g g_z^{-1} N \subseteq g_y U g_z^{-1}$ . Since  $g_y g g_z^{-1} N = \rho_N[g_y g g_z^{-1}] = g_y \rho_N[g] g_z^{-1}$ , it follows that  $\rho_N[g] \subseteq U$  and hence  $U$  is open in the uniform topology generated by our base  $B$ . On the other hand, as previously noted, every congruence class  $\rho_N[g]$ , where  $N$  is an open normal subgroup of  $G(x, x)$  and  $g \in G$ , is open in  $G$ . Hence the uniform topology generated by  $B$  is equal to the topology on  $G$ , as claimed.

We have seen that the topology on  $G$  is generated by a uniformity with a base consisting of congruences of finite indexes on  $G$ . It remains to check that  $G$  is Hausdorff. But this follows easily by noting that  $G$  is the disjoint union of the open subsets  $G(y, z)$ ,  $y, z \in V(G)$ , and each  $G(y, z)$  is homeomorphic to  $G(x, x)$  which is Hausdorff (being a cofinite group).

Conversely assume that  $G$  has a cofinite groupoid structure. By Proposition 6.6, each vertex group  $G(x, x)$  is a cofinite group. By Lemma 6.12, there exists a rigid cofinite congruence  $\sigma$  on  $G$ . Then for all  $g \in G(x, y)$ , we see that  $\sigma[g] \subseteq G(x, y)$ , and so each  $G(x, y)$  is open in  $G$ .

For the last part, consider any cofinite groupoid structure on  $G$  that induces its topology. By Lemma 6.12, this uniformity on  $G$  has a base, say  $C$ , consisting of rigid congruences of finite indexes on  $G$ . Given any  $\rho \in C$  and  $x \in V(G)$ , the congruence class  $N = \rho[x]$  is an open normal subgroup of  $G(x, x)$ . By the uniqueness of such a rigid congruence,  $\rho_N = \rho \subseteq \rho$  and  $\rho_N \in B$ . On the other hand, given any open normal subgroup  $N$  of  $G(x, x)$ , there exists  $\rho \in C$  such that  $\rho[x] \subseteq N$  (since  $N$  is also open in  $G$  and  $x \in N$ ). Now  $\rho$  and  $\rho_N$  are rigid congruences with  $\rho[x] \subseteq \rho_N[x]$ , and thus  $\rho \subseteq \rho_N$ . It follows that  $C$  and  $B$  are bases for the same uniformity on  $G$ , and hence  $G$  has a unique cofinite groupoid structure that induces its topology.  $\square$

Viewing groups as groupoids with a single vertex, we obtain the following result as an immediate corollary.

**Proposition 6.14.** *A Hausdorff topological group  $G$  is a cofinite group if and only if there is a uniformity on  $G$  with a base consisting of congruences of finite indexes on  $G$  that induces the topology on  $G$ . Moreover, in this case, such a uniformity on  $G$  is unique.*

The following is a refinement of Theorem 6.8 for cofinite groupoids with finite vertex sets.

**Theorem 6.15.** *Let  $G$  be a cofinite groupoid with a finite vertex set. Then  $V(\overline{G}) = V(G)$  and  $\overline{G}$  admits a unique cofinite groupoid structure such that  $G$  is a uniform subgroupoid. Moreover, endowed with this structure,  $\overline{G}$  is a profinite groupoid and the rigid cofinite congruences on  $\overline{G}$  consist precisely of the closures  $\overline{\rho}$  of all rigid cofinite congruences  $\rho$  on  $G$ .*

*Proof.* By Lemma 5.8,  $V(\overline{G}) = \overline{V(G)} = V(G)$  (since a finite subset of a Hausdorff space is closed). The rest of the result follows from Theorem 6.8 and Corollary 6.10 since when  $V(\overline{G}) = V(G)$ , it is clear that a cofinite congruence  $\rho$  on  $G$  is rigid if and only if its closure  $\overline{\rho}$  is rigid.  $\square$

## 6.5. The Fundamental Profinite Groupoid of a Profinite Graph

We conclude by briefly sketching an application of the theory of completions of cofinite groupoids which we explore in greater detail in a subsequent paper. Initially, let  $\Gamma$  be a finite discrete graph and let  $\pi_1(\Gamma)$  be its fundamental groupoid (see, for instance, [6]). Endow  $\pi_1(\Gamma)$  with the uniformity having a base consisting of all rigid congruences of finite indexes on  $\pi_1(\Gamma)$ . Note that for each vertex  $v$  of  $\Gamma$ , the vertex group based at  $v$  is the fundamental group  $\pi_1(\Gamma, v)$ , which is a free group. It is a well-known fact that free groups are residually finite. So, by the correspondence between coherent families of normal subgroups of finite indexes of the vertex groups and rigid congruences on  $\pi_1(\Gamma)$ , it follows that the  $\pi_1(\Gamma)$  is Hausdorff (i.e., the intersection of all congruences of finite indexes is identity congruence on  $\pi_1(\Gamma)$ ) and thus a cofinite groupoid. (This is clearly the unique maximal cofinite groupoid structure on  $\pi_1(\Gamma)$ .) We call the completion of  $\pi_1(\Gamma)$ , endowed with the unique cofinite groupoid structure such that  $\pi_1(\Gamma)$  is a uniform subgroupoid, the *profinite fundamental groupoid* of  $\Gamma$  and denote it by

$$\widehat{\pi}_1(\Gamma) = \overline{\pi_1(\Gamma)}.$$

Although this construction only requires the vertex set of  $\Gamma$  to be finite, we will only be applying it to finite graphs in what follows.

By Theorem 6.15,  $\widehat{\pi}_1(\Gamma)$  is a profinite groupoid whose vertex set is  $V(\pi_1(\Gamma)) = V(\Gamma)$ .

Next let  $f: \Gamma \rightarrow \Delta$  be a map of finite graphs. We use the same symbol to denote the induced homomorphism  $f: \pi_1(\Gamma) \rightarrow \pi_1(\Delta)$ ; see [6] for more details on

the fundamental groupoid functor. Then, by Corollary 6.11, the unique continuous extension of  $f$  is a homomorphism of groupoids which we denote by

$$\widehat{f}: \widehat{\pi}_1(\Gamma) \rightarrow \widehat{\pi}_1(\Delta).$$

Moreover, if  $f: \Gamma \rightarrow \Delta$  and  $g: \Delta \rightarrow \Omega$  are maps of finite graphs, then it is clear that  $\widehat{gf} = \widehat{g}\widehat{f}$ . It is also clear that the identity map on  $\Gamma$  induces the identity homomorphism in  $\widehat{\pi}_1(\Gamma)$ . Thus we have defined a functor from the category of finite discrete graphs and maps of graphs to the category of profinite groupoids and continuous homomorphisms. We extend this functor to the entire category of profinite graphs and continuous maps of graphs as follows.

Let  $\Gamma$  be any profinite graph and let  $I$  be the set of all cofinite entourages of  $\Gamma$ . By Theorem 3.4,  $I$  is a base for the uniformity on  $\Gamma$  and it follows that  $\Gamma = \varprojlim \Gamma_R$ , the projective limit of the inverse system  $(\Gamma_R, \phi_{RS})$  of finite discrete graphs and surjective maps of graphs indexed by the directed set  $I$  (partially ordered by reverse inclusion). Applying our profinite fundamental groupoid functor to this inverse system yields an inverse system of profinite groupoids and continuous homomorphisms  $(\widehat{\pi}_1(\Gamma_R), \widehat{\phi}_{RS})$  indexed by the directed set  $I$ . We define the *profinite fundamental groupoid* of  $\Gamma$  to be the inverse limit (in the category of topological groupoids) of this system and denote it by

$$\widehat{\pi}_1(\Gamma) = \varprojlim \widehat{\pi}_1(\Gamma_R).$$

It should be noted that when  $\Gamma$  is a finite graph, this latter definition of  $\widehat{\pi}_1(\Gamma)$  agrees with the former one above since in this case the singleton set  $\{\delta_\Gamma\}$  is a cofinal subset of the directed set  $I$ , and thus  $\varprojlim \widehat{\pi}_1(\Gamma_R) = \widehat{\pi}_1(\Gamma/\delta_\Gamma)$  and  $\Gamma/\delta_\Gamma = \Gamma$ . Moreover, it is easy to see that an inverse limit of profinite groupoids is also a profinite groupoid. Thus the profinite fundamental groupoid of any profinite graph is in general a profinite groupoid.

Turning to induced homomorphisms, we sketch a proof of the following result.

**Theorem 6.16.** *Let  $\Gamma, \Delta$  be profinite graphs and let  $f: \Gamma \rightarrow \Delta$  be a continuous map of graphs. Then there is a unique continuous map  $\widehat{f}: \widehat{\pi}_1(\Gamma) \rightarrow \widehat{\pi}_1(\Delta)$  such that for each pair  $R, S$  of cofinite graph entourages of  $\Gamma, \Delta$  respectively with  $f(R) \subseteq S$ , the diagram*

$$\begin{array}{ccc} \widehat{\pi}_1(\Gamma) & \xrightarrow{\widehat{f}} & \widehat{\pi}_1(\Delta) \\ \phi_R \downarrow & & \downarrow \psi_S \\ \widehat{\pi}_1(\Gamma_R) & \xrightarrow{\widehat{f}_{SR}} & \widehat{\pi}_1(\Delta_S) \end{array}$$

*is commutative. Moreover,  $\widehat{f}$  is a homomorphism of groupoids.*

Let  $I, J$  be the set of all cofinite graph entourages of  $\Gamma, \Delta$  respectively. Denote the canonical projection maps by  $\phi_R: \widehat{\pi}_1(\Gamma) \rightarrow \widehat{\pi}_1(\Gamma/R)$  and  $\psi_S: \widehat{\pi}_1(\Delta) \rightarrow \widehat{\pi}_1(\Delta/S)$ , for each  $R \in I$  and  $S \in J$ .

**Step 1.** For each  $S \in J$ , there is a unique map  $\widehat{f}_S: \widehat{\pi}_1(\Gamma) \rightarrow \widehat{\pi}_1(\Delta/S)$  such that if  $R \in I$  and  $f(R) \subseteq S$ , then  $\widehat{f}_S = \widehat{f}_{SR} \widehat{\phi}_R$ . Moreover, being the composition of two continuous homomorphisms,  $\widehat{f}_S$  is also a continuous homomorphism.

Since  $f$  is uniformly continuous, we can choose  $R \in I$  such that  $f(R) \subseteq S$ . Thus it suffices to show that if  $R_1, R_2 \in I$  and  $f(R_i) \subseteq S$ , then  $\widehat{f}_{SR_1} \widehat{\phi}_{R_1} = \widehat{f}_{SR_2} \widehat{\phi}_{R_2}$ . To see this, let  $R = R_1 \cap R_2$ . Note that  $R \in I$  with  $R_i \leq R$ , and so  $\widehat{f}_{SR_1} \widehat{\phi}_{R_1} = \widehat{f}_{SR_1} \widehat{\phi}_{R_1 R} \widehat{\phi}_R$ . However, by the functorial properties of homomorphisms induced by maps of finite graphs,  $\widehat{f}_{SR_1} \widehat{\phi}_{R_1 R} = \widehat{f}_{SR_1} \widehat{\phi}_{R_1 R} = \widehat{f}_{SR}$ . Thus  $\widehat{f}_{SR_1} \widehat{\phi}_{R_1} = \widehat{f}_{SR} \widehat{\phi}_R$ . Similarly  $\widehat{f}_{SR_2} \widehat{\phi}_{R_2} = \widehat{f}_{SR} \widehat{\phi}_R$ , and step 1 is completed.

**Step 2.** The family of maps  $(\widehat{f}_S)_{S \in J}$  is compatible with the inverse system  $(\widehat{\pi}_1(\Delta/S), \widehat{\psi}_{S_1 S_2})$  and thus determines a unique continuous homomorphism  $\widehat{f}: \widehat{\pi}_1(\Gamma) \rightarrow \widehat{\pi}_1(\Delta)$  such that  $\psi_S \widehat{f} = \widehat{f}_S$  for all  $S \in J$ .

Let  $S_1 \leq S_2$  in  $J$ . Choose  $R \in I$  such that  $f(R) \subseteq S_1 \cap S_2$ . Then  $\widehat{\psi}_{S_1 S_2} \widehat{f}_{S_2} = \widehat{\psi}_{S_1 S_2} \widehat{f}_{S_2 R} \widehat{\phi}_R$  (by Step 1 since  $f(R) \subseteq S_2$ ). However, by the functorial properties of homomorphisms induced by maps of finite graphs,  $\widehat{\psi}_{S_1 S_2} \widehat{f}_{S_2 R} = \widehat{\psi}_{S_1 S_2} \widehat{f}_{S_2 R} = \widehat{f}_{S_1 R}$ . Thus  $\widehat{\psi}_{S_1 S_2} \widehat{f}_{S_2} = \widehat{f}_{S_1 R} \widehat{\phi}_R = \widehat{f}_{S_1}$  (by Step 1 since also  $f(R) \subseteq S_1$ ).

Now by Steps 1 and 2, the homomorphism  $\widehat{f}: \widehat{\pi}_1(\Gamma) \rightarrow \widehat{\pi}_1(\Delta)$  has the desired property.

It is now an easy exercise to show that the functorial properties hold in general for induced homomorphisms:

1. If  $i: \Gamma \rightarrow \Gamma$  is the identity map on a profinite graph  $\Gamma$ , then  $\widehat{i}$  is the identity homomorphism on  $\widehat{\pi}_1(\Gamma)$ .
2. If  $\Gamma, \Delta$ , and  $\Omega$  be profinite graphs and  $\Gamma \xrightarrow{f} \Delta \xrightarrow{g} \Omega$  are uniformly continuous maps of graphs, then  $\widehat{gf} = \widehat{g}\widehat{f}$ .

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