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NONRELATIVISTIC LIMIT OF CHERN-SIMONS GAUGED FIELD EQUATIONS

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Abstract. We study the nonrelativistic limit of the Chern-Simons-Dirac system on $\mathbb{R}^{1+2}$. As the light speed c goes to infinity, we first prove that there exists an uniform existence interval $[0,T]$ for the family of solutions ψ^c corresponding to the initial data for the Dirac spinor ψ_0^c which is bounded in H^s for $\frac{1}{2} < s < 1$. Next we show that if the initial data ψ_0^c converges to a spinor with one of upper or lower component zero in H^s, then the Dirac spinor field converges, modulo a phase correction, to a solution of a linear Schrödinger equation in $C([0,T];H^{s'})$ for $s' < s$.

1. Introduction

In this paper we study the nonrelativistic limit of the Chern-Simons-Dirac equations (CSD) on the Minkowski space $\mathbb{R}^{1+2}$. The (CSD) systems were introduced in [6], [12] to deal with the electromagnetic phenomenon in planar domains. Also see [7], [8] for more physical features. The Lagrangian density for the system is given

$$\mathcal{L} = \frac{1}{\kappa}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} + i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi. \tag{1.1}$$

Here, the real valued 1-form $A_\mu \in \mathbb{R}$ is to be interpreted as defining a connection; accordingly, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the curvature 2-form, and $D_\mu\psi = \partial_\mu\psi - i\frac{A_\mu}{c}\psi$ is the associated covariant derivative. The spinor field ψ is represented by a column vector with 2 complex components, $\psi : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{C}^2$. The notation $\psi^\dagger$ refers to the conjugate transpose of ψ. The totally skew symmetric tensor $\epsilon_{\mu\nu\lambda}$ is given by $\epsilon^{012} = 1$, $\kappa > 0$ is the Chern-Simons coupling constant and $m > 0$ is the mass of the spinor field ψ. For the nonrelativistic limit we consider the regime where both parameters grow linearly in c. In what follows we let

$$\kappa = m = c.$$

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The Dirac gamma matrices γ^μ are $\mathbb{C}$-valued 2×2 matrices satisfying

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2(\eta^{-1})^{\mu\nu}\,\mathbf{I}_{2\times 2}\,, \tag{1.2}$$

where η is the three dimensional Minkowski metric with signature $(-1,1,1)$. We will use the Minkowski metric to raise and lower the indices. The standard representations of γ^μ are given by

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

Greek indices, such as μ, ν, refer to $0,1,2$, whereas Latin indices i,j,k,l to the spatial indices $1,2$. Moreover we will adopt the Einstein summation convention of summing up repeated upper and lower indices.

The Euler-Lagrange equation for the Lagrangian density (1.1) reads that

$$\begin{aligned} i\gamma^\mu D_\mu\psi &= c\psi, \\ \frac{1}{2}\epsilon^{\mu\nu\rho}F_{\nu\rho} &= -J^\mu, \end{aligned} \tag{1.3}$$

where $J^\mu = \psi^\dagger\gamma^0\gamma^\mu\psi$ is a current density.

The equation (1.3) is invariant under transforms

$$A_\mu \to A_\mu + \partial_\mu\chi, \quad \psi \to \psi e^{i\frac{\chi}{c}},$$

where $\chi \in \mathbb{R}$ is a smooth real valued function, which is referred to the gauge invariance in the literature. In this paper we choose the Coulomb gauge condition

$$\partial_i A_i = 0.$$

Due to (1.2) and $D_\mu D_\nu - D_\nu D_\mu = -\frac{i}{c}F_{\mu\nu}$ we have

$$(i\gamma^\mu D_\mu)^2 = \eta^{\mu\nu}D_\mu D_\nu + \frac{i}{2c}\gamma^\mu\gamma^\nu F_{\mu\nu}.$$

So the first equation in (1.3) is written as

$$(i\gamma^\mu D_\mu - cI)^2\psi = \eta^{\mu\nu}D_\mu D_\nu\psi + \frac{i}{2c}\gamma^\mu\gamma^\nu F_{\mu\nu}\psi - c^2\psi = 0. \tag{1.4}$$

By the Coulomb gauge condition, (1.4) leads the Klein-Gordon equation

$$\begin{aligned} &\frac{1}{c^2}\partial_t\partial_t\psi - \partial_j\partial_j\psi + c^2\psi \\ &= \frac{2i}{c}A^\alpha\partial_\alpha\psi + i\frac{1}{c^2}\partial_t A_0\psi + \frac{1}{c^2}A_\alpha A^\alpha\psi - \frac{i}{2c}\epsilon_{\alpha\beta\mu}J^\mu\gamma^\alpha\gamma^\beta\psi, \end{aligned} \tag{1.5}$$

where relativistic coordinates $x^0 = ct$ and $\partial_0 = \frac{1}{c}\partial_t$ is used. With the Coulomb gauge condition, A_μ satisfies the following elliptic equations

$$\begin{aligned} \Delta A_j &= \epsilon_{ji}\partial_i J_0, \\ \Delta A_0 &= 2\epsilon_{ik}\partial_i J^k, \end{aligned} \tag{1.6}$$

where ϵ_{ij} is antisymmetric, that is, $\epsilon_{ij} = -\epsilon_{ji}$, and $\epsilon_{12} = 1$. It is often convenient to use the original α^i, β formulation of the Dirac operator. Define

$$\beta := \gamma^0, \quad \alpha^1 := \gamma^0\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha^2 := \gamma^0\gamma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

and compute

$$\beta\mathcal{D}\psi = i\frac{1}{c}\partial_t\psi + i\alpha^j\partial_j\psi - c\beta\psi.$$

Multiplying γ^0 to the both hand sides of the (1.3), we have

$$i\partial_t\psi + ic\alpha^j\partial_j\psi - c^2\beta\psi + A_0\psi - \alpha^j A_j\psi = 0. \tag{1.7}$$

Recently the local and global well-posedness of Chern-Simons system has been investigated in [9], [10]. In particular, local well-posedness of the Chern-Simons-Dirac equations with the Lorenz gauge condition $\partial_\mu A^\mu = 0$ for initial data in $H^s(s > \frac{1}{4})$ was shown by Huh and Oh [10]. This improves the earlier work of Huh [9] where local well-posedness was obtained for $s > \frac{1}{2}$ in the Coulomb gauge, $s > \frac{5}{8}$ in the Lorenz gauge, and $s > \frac{3}{4}$ in the temporal gauge. Bournaves showed in [3] that the Chern-Simons-Dirac system with the Coulomb gauge is locally well-posed for $s > \frac{1}{4}$. This extends the results of [10] from the Lorenz gauge to the Coulomb gauge.

For the nonrelativistic limit problem on the (CSD) equation we refer to the following references. The behavior of solutions of the Dirac-Maxwell system in the nonrelative limit $c \to \infty$ has been studied in [2] by Bechouche et al., where c is speed of light. They proved that the solutions of Dirac-Maxwell system converges in $C([0,T] : H^1)$ to a solution of a Schrödinger-Poisson system. The nonrelativistic limit from Klein-Gordon-Maxwell to Schrödinger-Poisson was shown in [1], [11]. The semi-nonrelativistic limit of Chern-Simons-Higgs system is studied in [4] by Chae and Huh.

Notations: $H^s = H^s(\mathbb{R}^2)$ denotes the usual Sobolev spaces such that

$$\|f\|^2_{H^s(\mathbb{R}^2)} = \|f\|^2_{L^2(\mathbb{R}^2)} + \|\Lambda^s f\|^2_{L^2}, \quad \Lambda^s = (-\Delta)^{\frac{s}{2}}.$$

The bracket $\langle\cdot\rangle$ denotes $\sqrt{1 + |\cdot|^2}$ such that

$$\langle\nabla\rangle = \sqrt{1 + |\nabla|^2}, \quad \langle\xi\rangle = \sqrt{1 + |\xi|^2}.$$

Throughout the paper ψ, $\psi_\pm$, $\tilde{\psi}_\pm$ denote $\mathbb{C}^2$ valued spinors. The upper and lower components of spinors come with superscripts such as $\psi^\pm$, which is $\mathbb{C}$ valued. We often suppress the c dependence of A_μ, ψ for simplicity.

The followings are our main results.

Theorem 1. *Let ψ_0^c be a $\mathbb{C}^2$- valued sequence in H^s for $\frac{1}{2} < s < 1$, namely, $\psi_0^c = \begin{pmatrix} \psi_0^{c+} \\ \psi_0^{c-} \end{pmatrix}$ with $\psi_0^{c\pm} \in H^s$. If $\|\psi_0^c\|_{H^s}$ is bounded, there exists a uniform existence interval $[0,T]$ such that the Chern-Simons-Drirac equation* (1.6) *and* (1.7) *has a unique local solution in $C([0,T]; H^s)$ with the initial data ψ_0^c.*

In Theorem 2 we establish the nonrelativistic limif of (CSD) when the initial data is asymptotically separated or, in other words, one of the component of ψ_0^c converges to zero as c is going to infinity.

Theorem 2. *Let ψ_0^c satisfy the same condition in Theorem 1. Moreover ψ_0^c converges to* (i) $\begin{pmatrix} v_0^+ \\ 0 \end{pmatrix}$ *or converges to* (ii) $\begin{pmatrix} 0 \\ v_0^- \end{pmatrix}$. *Let ψ^c be the solution of the* (*CSD*) *constructed in Theorem 1 and $v_\pm$ satisfy the the linear Schrödinger equation*

$$i\partial_t v_\pm \mp \frac{1}{2}\Delta v_\pm = 0 \tag{1.8}$$

with $v_+(\cdot, 0) = v_0^+$, $v_-(\cdot, 0) = v_0^-$. Let M be a pseudodiffrential operator given by $M = \sqrt{c^4 - c^2\Delta}$ and $\tilde{\psi}_\pm$ be defined by

$$\tilde{\psi}_\pm = \frac{1}{2}\left(\psi \pm iM^{-1}(\psi_t - iA_0\psi)\right).$$

Then it holds that

$$\tilde{\psi}_+ \to \begin{pmatrix} e^{-itc^2} v_+ \\ 0 \end{pmatrix} \text{ and } \tilde{\psi}_- \to \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } C([0,T]; H^{s'}) \tag{1.9}$$

for (i) *and*

$$\tilde{\psi}_+ \to \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \tilde{\psi}_- \to \begin{pmatrix} 0 \\ e^{itc^2} v_- \end{pmatrix} \text{ in } C([0,T]; H^{s'}) \tag{1.10}$$

for (ii) *with $\frac{1}{2} < s' < s$ as c is going to infinity.*

Remark 1.
(1) The convergence of the Dirac spinor to the linear Schrödinger equation can be explained roughly as follows. After the highly oscillating part $e^{\pm ic^2 t}$ subtracted, the left hand side of (1.5) converges to the linear Schrödinger part, whereas the terms in right hand side has enough $\frac{1}{c}$ factors to vanish except $2iA_0\partial_t\psi$. By α^i, β formulation we write the A_0 -equation in (1.6) as

$$\Delta A_0 = 2\partial_1(\psi^\dagger\alpha^2\psi) - 2\partial_2(\psi^\dagger\alpha^1\psi).$$

Reflecting on the initial assumptions on ψ_0^c, we may assume, for a moment, ψ^c be of the form of $\begin{pmatrix} \psi^+ \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ \psi^- \end{pmatrix}$ for some functions $\psi^\pm$. Then due to skew symmetry of α^1 and α^2, it holds that

$$\psi^\dagger\alpha^2\psi = \psi^\dagger\alpha^1\psi = 0,$$

hence $A_0 = 0$, which implies that $e^{\pm ic^2 t}\psi$ converges to $\begin{pmatrix} v_+ \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ v_- \end{pmatrix}$ respectively, where $v_\pm$ is the spinor solution of (1.8). We shall prove this heuristic argument in Section 4.

(2) The convergence of the associated gauge field A^c_μ follows immediately by Theorem 2 and the estimate (3.9) in Section 3; A^c_0 converges to zero in $C([0,T] : W^{1,\frac{1}{1-s'}})$ in both cases and (A^c_1, A^c_2) converges to (u,w) in $C([0,T] : W^{1,\frac{1}{1-s'}})$ with $\frac{1}{2} < s' < s$, where (u,w) satisfies

$$\Delta u = -\partial_2 |v_+|^2, \quad \Delta w = \partial_1 |v_+|^2$$

for (i) and

$$\Delta u = -\partial_2 |v_-|^2, \quad \Delta w = \partial_1 |v_-|^2$$

for (ii).

2. Preliminary

In this section we discuss the Duhamel formulae for two formulations of the Dirac field equations, the first order equation (1.7) and the Klein-Gordon equation (1.5).

Let us start with (1.7),

$$i\partial_t \psi + ic\alpha^j \partial_j \psi - c^2 \beta \psi = F, \quad \psi(x,0) = \psi_0, \tag{2.1}$$

where $F = -A_0 \psi + \partial^j A_j \psi$. Let $Q^c(\nabla) = \beta I - \frac{i}{c} \alpha^k \partial_j$ then (2.1) reads

$$i\partial_t \psi - c^2 Q^c(\nabla)\psi = F. \tag{2.2}$$

The 2×2 matrix $Q^c(\xi) = \beta + \frac{1}{c}\alpha^j \xi_j$ has two eigenvalues

$$\pm \lambda^c(\xi) = \pm \sqrt{1 + \frac{|\xi|^2}{c^2}}$$

with the corresponding one dimensional eigenspace

$$V_\pm = \langle (\xi_1 - i\xi_2, -c(1 \mp \lambda^c(\xi)) \rangle.$$

The projection $\Pi^c_\pm(\xi)$ to $V_\pm$ satisfies $\Pi^c_+(\xi) + \Pi^c_-(\xi) = I$ and $\Pi^c_+(\xi) - \Pi^c_-(\xi) = Q^c(\xi)/\langle \xi/c \rangle$, so it is computed as

$$\Pi^c_\pm(\xi) = \frac{1}{2}\left(I \pm \frac{Q^c(\xi)}{\langle \xi/c \rangle}\right) = \frac{1}{2}\left(I \pm \frac{\beta + \frac{1}{c}\alpha^j \xi_j}{\sqrt{1 + \frac{|\xi|^2}{c^2}}}\right). \tag{2.3}$$

In turn $Q^c(\nabla)$ is decomposed by

$$Q^c(\nabla) = Q(\nabla/c) = \langle \nabla/c \rangle \Pi^c_+(\nabla) - \langle \nabla/c \rangle \Pi^c_-(\nabla), \tag{2.4}$$

hence, if we let $L(t)\psi_0$ the solution of the free Dirac equation ($F = 0$ in (2.2)) with the initial data ψ_0, we have

$$L(t)\psi_0 := e^{ic^2 \langle \nabla/c \rangle t} \Pi^c_+(\nabla)\psi_0 + e^{-ic^2 \langle \nabla/c \rangle t} \Pi^c_-(\nabla)\psi_0. \tag{2.5}$$

By Duhamel principle we write the solution of (2.1) by

$$\psi(x,t) = L(t)\psi_0 + \int_0^t L(t-s)F(s). \tag{2.6}$$

Note that by the Taylor expansion it holds formally

$$c^2\langle\nabla/c^2\rangle \sim c^2\left(1+\frac{|\nabla|^2}{2c^2}\right).$$

On the other hands Bechouhe et al. ([1,2]) observed that taking the limit $c\to\infty$ to symbols $\Pi^c_\pm$ yields

$$\Pi^c_+(\xi)\to\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad \Pi^c_-(\xi)\to\begin{pmatrix}0&0\\0&1\end{pmatrix} \tag{2.7}$$

pointwise in ξ. Let $\psi_0=\begin{pmatrix}\psi_0^+\\ \psi_0^-\end{pmatrix}$. Due to the above two observations we predict that $L(t)\psi_0$ approaches to two Schrödinger waves multiplied by $e^{\pm ic^2t}$, the highly oscilating parts in c, as c going to infinity;

$$L(t)\psi_0\to e^{ic^2t}e^{-it\frac{\Delta}{2}}\begin{pmatrix}\psi_0^+\\0\end{pmatrix}+e^{-ic^2t}e^{it\frac{\Delta}{2}}\begin{pmatrix}0\\ \psi_0^-\end{pmatrix}.$$

The separations of $L(t)\psi_0$ to the upper and lower spinor is reflected in Theorem 2, which will be revisited in Section 4.

Next we turn to the Klein-Gordon equation (1.5),

$$\frac{1}{c^2}\partial_t\partial_t\psi-\partial_j\partial_j\psi+c^2\psi=G,\quad \psi(x,0)=\psi_0,\quad \partial_t\psi(x,0)=\psi_1,$$

where

$$G=\frac{2i}{c}A^\alpha\partial_\alpha\psi+i\frac{1}{c^2}\partial_tA_0\psi+\frac{1}{c^2}A_\alpha A^\alpha\psi-\frac{i}{2c}\epsilon_{\alpha\beta\mu}J^\mu\gamma^\alpha\gamma^\beta\psi.$$

It is useful to write the Klein-Gordon equation as a first order system

$$\frac{\partial}{\partial t}\begin{pmatrix}\psi\\ \psi_t\end{pmatrix}=\begin{pmatrix}0&I\\ c^2\Delta-c^4I&0\end{pmatrix}\begin{pmatrix}\psi\\ \psi_t\end{pmatrix}+\begin{pmatrix}0\\G\end{pmatrix}, \tag{2.8}$$

and diagonalize the 2×2 (4×4 in fact) symbol by conjugating with the Fourier transform. Let

$$M=\sqrt{c^4-c^2\Delta}=c^2\langle\nabla/c\rangle.$$

The 2×2 matrix has two eigenvalues, $\lambda_\pm=\mp iM$. The projection to the eigenspace corresponding to $\lambda_\pm$ are

$$P_\pm=\frac{1}{2}\begin{pmatrix}I&\pm iM^{-1}\\ \mp iM&I\end{pmatrix}$$

respectively. We are then lead to the transform $(\psi,\psi_t)\to(\psi_+,\psi_-)$ and $(0,G)\to(G_+,G_-)$, where

$$\psi_\pm=P_\pm\psi:=\frac{1}{2}\left(\psi\pm iM^{-1}\psi_t\right),\quad G_\pm=\pm\frac{i}{2M}G.$$

We obtain the following diagonal first order system

$$\frac{\partial}{\partial t}\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}=\begin{pmatrix}-iM&0\\0&iM\end{pmatrix}\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}+\begin{pmatrix}G_+\\G_-\end{pmatrix}$$

or equivalently, the following pair of *half-wave equations*

$$(\partial_t \pm iM)\psi_\pm = \pm \frac{i}{2M} G. \tag{2.9}$$

The half-waves $\psi_\pm$ are given by the Duhamel's formula

$$\psi_\pm = e^{\pm itM}\psi_{\pm,0} \pm \frac{i}{2}\int_0^t e^{\pm i(t-s)M}\frac{G}{M}.$$

The non relativistic limit of the linear Klein-Gordon equation is the free Schrödinger equation as formally seen below. Let us start with the symbol M. As previous, it holds formally

$$M \sim c^2\left(1 + \frac{|\nabla|^2}{2c^2}\right), \tag{2.10}$$

namely,

$$M - c^2 \sim -\frac{\Delta}{2}.$$

Taking only the homogeneous part of (2.9), we would write

$$\partial_t\psi_\pm \pm ic^2\psi_\pm = \mp i(M - c^2)\psi_\pm,$$

equivalently,

$$\partial_t\left(e^{\pm ic^2t}\psi_\pm\right) = \mp i(M - c^2)e^{\pm ic^2t}\psi_\pm.$$

Note that formal limit of $c \to \infty$ yields to

$$i\partial_t\left(e^{\pm ic^2t}\psi_\pm\right) = \mp\frac{1}{2}\Delta\left(e^{\pm ic^2t}\psi_\pm\right). \tag{2.11}$$

For the proofs of Theorem 2 we rely on the half-wave formulation of the (CSD),

$$\partial_t\left(e^{\pm ic^2t}\psi_\pm\right) = \mp i(M - c^2)e^{\pm ic^2t}\psi_\pm \pm \frac{i}{2}e^{\pm ic^2t}M^{-1}G.$$

Let us look on the symbol $M - c^2$ and M^{-1} further than (2.10). We have

$$M^{-1}(\xi) = \frac{1}{c^2\sqrt{1 + \frac{|\xi|^2}{c^2}}}, \tag{2.12}$$

$$M(\xi) - c^2 = \frac{|\xi|^2}{1 + \sqrt{1 + \frac{|\xi|^2}{c^2}}} \sim \begin{cases} |\xi|^2/2 \text{ for } |\xi| \ll c, \\ c|\xi| \text{ for } |\xi| \gg c \end{cases} \tag{2.13}$$

by the taylor expansion in $|\xi|/c \ll 1$ and the observation

$$M(\xi) - c^2 = c|\xi|\frac{\frac{|\xi|}{c}}{1 + \sqrt{1 + \frac{|\xi|^2}{c^2}}} \sim c|\xi| \text{ in } |\xi|/c \gg 1.$$

We introduce the following bounds for operators M^{-1} and $M - c^2$. All items can be proved by Plancherel's theorem. We omit the proof.

Lemma 3 (Lemma 3 in [1])**.** *For all $s \in \mathbb{R}$ the following estimates hold.*

(1) $\|M^{-1}\|_{L(H^s,H^s)} = O(1/c^2)$.
(2) $\|M^{-1}\|_{L(H^s,H^{s+1-\delta})} = O(1/c^{1+\delta}), \quad 0 \le \delta \le 1$.
(3) $\|M - c^2\|_{L(H^{s+1},H^s)} = O(c)$.
(4) $\|M - c^2\|_{L(H^{s+2},H^s)} = O(1)$.

We introduce the high and low frequency parts of functions by projection to $\{\xi||\xi| \gtrsim c\}$ and $\{\xi||\xi| \lesssim c\}$. Let χ be a smooth cut-off function on $\mathbb{R}^2$ such that $\chi(\xi) = 1$ for $|\xi| \le 1$ and $\chi(\xi) = 0$ for $|\xi| \ge 2$. We split functions $f(x)$ into low and high frequencies using $\chi(\xi/c)$:

$$f = f_l + f_h, \quad \hat{f}_l(\xi) = \chi(\xi/c)\hat{f}(\xi), \quad \hat{f}_h(\xi) = 1 - \hat{f}_l(\xi).$$

Lemma 4. *The following estimates hold on* $\mathbb{R}^2$.
(1) $\|\frac{M}{c^2} f_l\|_{L^p} \lesssim \|f_l\|_{L^p}$ *for* $1 \le p \le \infty$.
(2) $\|f_l\|_{H^{1+\epsilon}} \lesssim c^\epsilon \|f_l\|_{H^1}$ *for* $\epsilon > 0$.
(3) $\|\frac{M}{c^2} f_h\|_{L^2} \lesssim \frac{1}{c}\|f_h\|_{\dot{H}^1}$.
(4) $\|f_h\|_{L^2} \lesssim \frac{1}{c}\|f_h\|_{H^1}$.

Proof. $\frac{M}{c^2} f_l = \omega_c * f_l$ where $\hat{\omega}_c(\xi) = (1 + |\frac{\xi}{c}|^2)^{\frac{1}{2}}\chi(\frac{\xi}{2c})$, then $\|\omega_c\|_{L^1} = O(1)$ uniformly in c. By Young's inequality we have the first item. The rest items are by Plancherel's theorem. $\square$

3. Existence of an uniform local existence interval

In section 3 we show that there exists $T > 0$ independent of c such that the Chern-Simons-Dirac system

$$\begin{aligned} &i\partial_t\psi + ic\alpha^j\partial_j\psi - c^2\beta\psi + A_0\psi - \alpha^j A_j\psi = 0, \\ &\Delta A_0 = 2\partial_1(\psi^\dagger\alpha^2\psi) - 2\partial_2(\psi^\dagger\alpha^1\psi), \\ &\Delta A_1 = -\partial_2(\psi^\dagger\psi), \\ &\Delta A_2 = -\partial_1(\psi^\dagger\psi) \end{aligned} \tag{3.1}$$

has a local solution up to $[0, T]$. As in (2.6), the Dirac field equation can be written by

$$\psi(x,t) = L(t)\psi_0 + \int_0^t L(t-s)F(s)ds,$$

where

$$\begin{aligned} F &= -A_0\psi + \alpha^j A_j\psi, \\ L(t)\psi_0 &= e^{ic^2\langle\nabla/c\rangle t}\Pi_+^c(\nabla)\psi_0 + e^{-ic^2\langle\nabla/c\rangle t}\Pi_-^c(\nabla)\psi_0 \end{aligned}$$

with $\Pi_\pm^c(\nabla)$ introduced in (2.3). Since $\Pi_\pm^c(\nabla)$ has the smooth and bounded multiplier, it holds that, in particular,

$$\|\Pi_\pm^c(\nabla)f\|_{L^2(\mathbb{R}^2)} \le C\|f\|_{L^2(\mathbb{R}^2)} \tag{3.2}$$

for a uniform constant C which dose not depend on c. The H^s estimate of ψ follows immediately.

Proposition 1. *Let ψ be the solution to the Dirac equation* (2.2) *and let $s \in \mathbb{R}$. Then for any $T > 0$ there exists a uniform constant $C > 0$ such that*

$$\sup_{0\le t\le T} \|\psi(\cdot,t)\|_{H^s(\mathbb{R}^2)} \le C \left(\|\psi_0\|_{H^s(\mathbb{R}^2)} + \int_0^T \|F(\cdot,\tau)\|_{H^s(\mathbb{R}^2)} d\tau \right). \tag{3.3}$$

Proof. The L^2 estimate is straightforward by (3.2). Taking Λ^s to the both sides of (2.2), the $\dot{H}^s$ estimate of ψ follows from the duhamel formula for $\Lambda^s\psi$ and the L^2 estimate. □

Let us define

$$\|\psi\|_{X^s_T} = \sup_{t\in[0,T]} \|\psi\|_{H^s}.$$

Now the uniform existence of the local solution of (3.1) is obtained from the following proposition.

Proposition 2. *Let ψ and ψ' be two solutions of the Chern-Simons-Dirac equations* (3.1) *with initial data ψ_0 and ψ_0' in $C([0,T);H^s(\mathbb{R}^2))$ for $1/2 < s < 1$ and $T > 0$. Then there exist constants C independent of c such that the following estimates holds.*

(1) $\int_0^T \|A_\mu\psi\|_{H^s} \le CT\|\psi\|^3_{X^s_T}$.

(2) $\|\psi-\psi'\|_{X^s_T} \le CT\|\psi\|_{X^s_T}\|\psi-\psi'\|_{X^s_T} + CT\|\psi-\psi'\|_{X^s_T}\|\psi\|_{X^s_T}$.

By the fixed point theorem and the continuity argument, Proposition 2 implies there exist $T > 0$ and a uniform constant C such that there exists a unique local smooth solution of (3.1) in $C([0,T];H^s)$ for $\frac{1}{2} < s < 1$ satisfying

$$\|\psi\|_{X^s_T} \le C\|\psi_0\|_{H^s}. \tag{3.4}$$

Note that T and C will be chosen independent of c in the process of the proof.

Proposition 2 is proved in [9] (Propositions 4.2, 4.3) when $c = 1$ by using the charge estimate (3.3) and the bound of $\|A\|_{W^{k,p}}$ in terms of $\|\psi\|_{H^s}$. For the latter, [9] relies only on the classical functional inequalities and the Sobolev embeddings. Note that the constant in (3.3) is independent of c and further, there is no c appearance in A_μ equations in (3.1), hence the same proof goes through for Proposition 2. In the below we introduce the analysis in [9] for completeness of the paper.

The A_μ equations in (1.6) can be written symbolically as

$$\Delta A = \nabla J. \tag{3.5}$$

Using integral representation we may express

$$A_0 = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(-x_2+y_2, x_1-y_1)}{|x-y|^2} J(y)\,dy, \tag{3.6}$$

where $J = (J_1, J_2)$. Also (3.5) implies the integral representations we may express

$$A_j = \frac{\epsilon_{ij}}{2\pi}\int_{\mathbb{R}^2} \frac{x_i-y_i}{|x-y|^2} J_0(y)\,dy. \tag{3.7}$$

For $\|A_\mu\|_{L^p}$, we use the Hardy-Littlewood-Sobolev's inequality for the fractional integral operator $I_r(r>1)$,

$$I_r f(x) = \int_{\mathbb{R}^2} \frac{f(y)}{|x-y|^{\frac{2}{r}}} dy \quad \text{for } r>1.$$

Lemma 5. *Suppose $r>1$, $1<p<q<\infty$ and $1/r = 1-(1/p-1/q)$. Then there exists a constant C depending only on p and q such that*

$$\|I_r f\|_{L^q} \le C\|f\|_{L^p}.$$

Applying Hardy-Littlewood-Sobolev's inequality to (3.6)–(3.7), we have

$$\|A_\mu\|_{L^q} \le C\|J\|_{L^p} \quad \text{for } 1/q = 1/p - 1/2, \quad 1<p<2.$$

For $\|\partial_j A_\mu\|_{L^p}$ we can apply the Calderon-Zygmund inequality to obtain

$$\|\partial_j A_\mu\|_{L^p} \le C\|J\|_{L^p} \quad \text{for } 1<p<\infty.$$

Roughly we can assume $|J| = |\psi|^2$. By the Sobolev embedding

$$H^s \hookrightarrow L^r \quad \text{for } 2 \le r \le \frac{2}{1-s} \qquad \text{and} \qquad H^1 \hookrightarrow L^r \quad \text{for } 2 \le r < \infty,$$

so

$$\|J\|_{L^p} \le C\|\psi\|_{H^s}^2 \quad \text{for } 1 \le p \le \frac{1}{1-s}.$$

Also we have

$$\begin{aligned} \|A_\mu\|_{L^{\frac{2p}{2-p}}} &\le C\|J\|_{L^p} \le C\|\psi\|_{H^s}^2 \quad 1<p<2 \quad (1/2 \le s), \\ \|\nabla A_\mu\|_{L^p} &\le C\|J\|_{L^p} \le C\|\psi\|_{H^s}^2 \quad 1<p \le \frac{1}{1-s} \quad (0<s<1). \end{aligned} \tag{3.8}$$

In particular we have

$$\begin{aligned} \|A_\mu\|_{W^{1,\frac{1}{1-s}}} &\le C\|\psi\|_{H^s}^2, \\ \|A_\mu\|_{W^{s,\frac{2}{s}}} &\le C\|\psi\|_{H^s}^2 \end{aligned} \tag{3.9}$$

when $\frac{1}{2} < s < 1$, which follow by

$$\begin{aligned} &\|A_\mu\|_{L^{\frac{1}{1-s}}} \le \|J\|_{L^{\frac{2}{3-2s}}} \le C\|\psi\|_{H^s}^2, \\ &\|A_\mu\|_{\dot W^{1,\frac{1}{1-s}}} \le C\|J\|_{L^{\frac{1}{1-s}}}^2 \le C\|\psi\|_{L^{\frac{2}{1-s}}}^2 \le C\|\psi\|_{H^s}^2, \\ &\|A_\mu\|_{L^{\frac{2}{s}}} \le C\|J\|_{L^{\frac{2}{1+s}}} \le C\|\psi\|_{H^s}^2, \\ &\|A_\mu\|_{\dot W^{2,\frac{2}{s}}} = \|\Lambda^s A_\mu\|_{L^{\frac{2}{s}}} \le C\|J\|_{L^2} \le \|\psi\|_{H^s}^2. \end{aligned} \tag{3.10}$$

Next we introduce the well known product lemma which generalizes the Leibniz rule in the Sobolev space $W^{s,p}$ for $s>0$.

Proposition 3. *Let $f \in L^{p_1} \cap W^{s,p_3}$, $g \in L^{p_4} \cap W^{s,p_2}$, where $s>0$, $1<p<\infty$, p_1, $p_4 \in (1,+\infty]$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$. Then*

$$\|fg\|_{W^{s,p}(\mathbb{R}^2)} \le \|f\|_{L^{p_1}(\mathbb{R}^2)}\|g\|_{w^{s,p_2}(\mathbb{R}^2)} + \|f\|_{W^{s,p_3}(\mathbb{R}^2)}\|g\|_{L^{p_4}(\mathbb{R}^2)}.$$

Proof of Proposition 2. Applying Proposition 3 we have

$$\int_0^T \|A_\mu\psi\|_{H^s} \lesssim \int_0^T \|A_\mu\|_{L^\infty}\|\psi\|_{H^s} + \int_0^T \|A_\mu\|_{W^{s,\frac{2}{s}}}\|\psi\|_{L^{\frac{2}{1-s}}}.$$

By the embedding $W^{1,\frac{1}{1-s}}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ and (3.9) the both two terms are bounded by $CT\|\psi\|^3_{X^s_T}$ for some uniform constant C. For the second item we apply Proposition 1 to have

$$\|\psi-\psi'\|_{X^s_T} \le C\|\psi_0-\psi_0'\|_{H^s} + \int_0^T \|A_\mu(\psi-\psi')\|_{H^s} + \int_0^T \|(A'_\mu - A_\mu)(\psi-\psi')\|_{H^s},$$

where A_μ and A'_μ are the associated gauge field to ψ and ψ' respectively. Similarly in the first item the two terms are bounded by

$$CT\|\psi\|_{X^s_T}\|\psi-\psi'\|_{X^s_T} + CT\|\psi-\psi'\|_{X^s_T}\|\psi\|_{X^s_T}. \qquad \square$$

4. Convergence

In this section we prove Theorem 2. Let us remind that the Klein-Gordon formulation of (CSD) obtained in (1.5) and (1.6),

$$\begin{aligned}
&\frac{1}{c^2}\partial_t\partial_t\psi - \partial_j\partial_j\psi + c^2\psi \\
&= \frac{2i}{c}A^\alpha\partial_\alpha\psi + i\frac{1}{c^2}\partial_t A_0\psi + \frac{1}{c^2}A_\alpha A^\alpha\psi - \frac{i}{2c}\epsilon_{\alpha\beta\mu}J^\mu\gamma^\alpha\gamma^\beta\psi. \\
&\Delta A_j = \epsilon_{ji}\partial_i J_0, \quad i,j=1,2. \\
&\Delta A_0 = 2\epsilon_{ik}\partial_i J^k, \quad i,k=1,2.
\end{aligned} \tag{4.1}$$

As in (2.9) the first equation is equivalently written by

$$\partial_t\left(e^{\pm ic^2t}\psi_\pm\right) = \mp i(M-c^2)e^{\pm ic^2t}\psi_\pm \pm \frac{i}{2M}e^{\pm ic^2t}G, \tag{4.2}$$

where

$$\begin{aligned}
&\psi_\pm = P_\pm\psi := \frac{1}{2}\left(\psi \pm iM^{-1}\psi_t\right), \\
&G = 2icA^\alpha\partial_\alpha\psi + i\partial_t A_0\psi + A_\alpha A^\alpha\psi - \frac{ci}{2}\epsilon_{\alpha\beta\mu}J^\mu\gamma^\alpha\gamma^\beta\psi \\
&\quad = 2iA_0\partial_t\psi + 2ciA_j\partial_j\psi + i\partial_t A_0\psi + A_\alpha A^\alpha\psi - \frac{ci}{2}\epsilon_{\alpha\beta\mu}J^\mu\gamma^\alpha\gamma^\beta\psi.
\end{aligned} \tag{4.3}$$

Following [1], we use the transform

$$\tilde\psi_\pm = \frac{1}{2}\left(\psi \pm iM^{-1}(\psi_t - iA_0\psi)\right) \tag{4.4}$$

instead of (4.3), which is mainly due to technical conveniences. Plugging in (4.2)

$$\tilde\psi_\pm = \psi_\pm \pm \frac{1}{2}M^{-1}(A_0\psi),$$

we have

$$
\begin{aligned}
\partial_t\tilde{\psi}_\pm &= \partial_t\psi_\pm \pm \frac{1}{2}M^{-1}\partial_t(A_0\psi)\\
&= \mp iM\psi_\pm \pm \frac{i}{2}M^{-1}G \pm \frac{1}{2}M^{-1}\partial_t(A_0\psi)\\
&= \mp iM\tilde{\psi}_\pm + \frac{i}{2}A_0\psi \pm \frac{i}{2}M^{-1}G \pm \frac{1}{2}M^{-1}\partial_t(A_0\psi).
\end{aligned}
$$

In the last two terms $\pm\frac{1}{2}M^{-1}(\partial_t A_0\psi)$ is cancelled out so that it is written by

$$
\begin{aligned}
&\pm \frac{i}{2}M^{-1}G \pm \frac{1}{2}M^{-1}\partial_t(A_0\psi)\\
= \ &\pm \frac{1}{2}M^{-1}(-A_0\partial_t\psi - 2cA_j\partial_j\psi + iA_\alpha A^\alpha\psi + \frac{c}{2}\epsilon_{\alpha\beta\mu}J^\mu\gamma^\alpha\gamma^\beta\psi).
\end{aligned}
$$

Converting

$$
\psi = \tilde{\psi}_+ + \tilde{\psi}_-, \quad \partial_t\psi = -iM(\tilde{\psi}_+ - \tilde{\psi}_-) + iA_0\psi
$$

we have

$$
\begin{aligned}
A_0\partial_t\psi &= -iA_0M(\tilde{\psi}_+ - \tilde{\psi}_-)\\
&= -iMA_0(\tilde{\psi}_+ - \tilde{\psi}_-) - i[A_0, M](\tilde{\psi}_+ - \tilde{\psi}_-) + iA_0^2\psi.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\partial_t\tilde{\psi}_\pm \pm iM\tilde{\psi}_\pm = \ &\underbrace{\frac{i}{2}A_0(\tilde{\psi}_+ + \tilde{\psi}_-) \pm \frac{i}{2}A_0(\tilde{\psi}_+ - \tilde{\psi}_-)}_{=iA_0\tilde{\psi}_\pm}\\
&\pm \frac{1}{2}M^{-1}\Big(i[A_0, M](\tilde{\psi}_+ - \tilde{\psi}_-) \mp iA_0^2\psi + 2ciA_j\partial_j\psi\\
&\quad + A_\alpha A^\alpha\psi - \frac{ci}{2}\epsilon_{\alpha\beta\mu}J^\mu\gamma^\alpha\gamma^\beta\psi\Big).
\end{aligned}
$$

Following the previous observation (2.11), we introduce $\phi_\pm$ to subtract the rest energy

$$
\phi_\pm = e^{\pm ic^2t}\tilde{\psi}_\pm. \tag{4.5}
$$

Finally we arrive at the system

$$
i\partial_t\phi_\pm \mp (M - c^2)\phi_\pm = iA_0\phi_\pm \pm e^{\pm ic^2t}\mathcal{R}, \tag{4.6}
$$

where

$$
\begin{aligned}
\mathcal{R} &= \frac{1}{2}M^{-1}(i[A_0, M - c^2](\tilde{\psi}_+ - \tilde{\psi}_-) \mp iA_0^2\psi + 2ciA_j\partial_j\psi\\
&\quad + A_\alpha A^\alpha\psi - \frac{ci}{2}\epsilon_{\alpha\beta\mu}J^\mu\gamma^\alpha\gamma^\beta\psi), \qquad (4.7)\\
\Delta A_0 &= 2\partial_1(\psi^\dagger\alpha^2\psi) - 2\partial_2(\psi^\dagger\alpha^1\psi),\\
\Delta A_1 &= -\partial_2(\psi^\dagger\psi),
\end{aligned}
$$

$$\Delta A_2 = \partial_1(\psi^\dagger\psi).$$

Let us recall the uniform estimate (3.4)

$$\|\psi\|_{X_T^s} < C\|\psi_0\|_{H^s}$$

as well as (3.8) and (3.9), which in turn gives the uniform estimate

$$\|\tilde{\psi}_\pm\|_{X_T^s} < C\|\psi_0\|_{H^s} \tag{4.8}$$

by the following way. The bound

$$\|M^{-1}(A_\mu\psi)\|_{H^s} \le Cc^{-2}\|\psi_0\|_{H^s}^3 \tag{4.9}$$

is straightforward from Lemma 3 and the aforementioned uniform estimates. For the term $M^{-1}\psi_t$ we have by (2.2)

$$\|M^{-1}\psi_t\|_{H^s} \le \|c^2M^{-1}(Q^c(\nabla)\psi)\|_{H^s} + \|M^{-1}(A_\mu\psi)\|_{H^s}.$$

By (2.4) and $\Pi_\pm^c(\nabla)$ being smooth and bounded, we estimate

$$\|c^2M^{-1}(Q^c(\nabla)\psi)\|_{H^s} \le \|\Pi_+^c(\nabla)\psi\|_{H^s} + \|\Pi_-^c(\nabla)\psi\|_{H^s} \le C\|\psi\|_{H^s}.$$

In what follows we shall prove the convergence (1.9) and (1.10). First, in the level of initial data the separation to the upper and lower spinor of the limits is explained as follows. If ψ is the solution of the (CSD), two projections $\Pi_\pm^c\psi$ and $\psi_\pm = P_\pm\psi$ in (2.3) and (4.3) are related by

$$\psi_\pm = P_\pm\psi = \Pi_\pm^c\psi \pm \frac{1}{2}M^{-1}(\alpha^jA_j\psi - A_0\psi)$$

or

$$\tilde{\psi}_\pm = \Pi_\pm^c\psi \pm \frac{1}{2}M^{-1}(\alpha^jA_j\psi)$$

in H^s. So it holds that

$$\lim_{c\to\infty}\tilde{\psi}_{0\pm}^c = \lim_{c\to\infty}\Pi_\pm^c\psi_0^c \tag{4.10}$$

in H^s by (4.9) . Now due to (2.7) and the initial assumption on ψ_0^c, we have

(4.11)
$$\lim_{c\to\infty}\Pi_+^c\psi_0^c = \lim_{c\to\infty}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\psi_0^c = \begin{pmatrix}v_{0+}\\ 0\end{pmatrix},\ \lim_{c\to\infty}\Pi_-^c\psi_0^c = \lim_{c\to\infty}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\psi_0^c = \begin{pmatrix}0\\ 0\end{pmatrix}$$

in H^s by Lebesgue Dominated Convergence Theorem (LDCT).

Let us introduce the linear group

$$U_\pm^c(t) = e^{\pm it(M(|\nabla|)-c^2)}, \qquad S_\pm(t) = e^{\pm\frac{it}{2}\Delta}.$$

We shall use that $U_\pm^c(t)$ is unitary in H^s and it converges pointwise

$$U_\pm^c(t)f(x) \longrightarrow S_\pm(t)f(x)$$

for a sufficiently good function f when $c \to \infty$.

Let us show the case (i), that is ψ_0^c converges to $\begin{pmatrix} v_0^+ \\ 0 \end{pmatrix}$ for some $v_0^+ \in H^s$. The case (ii) can be proved similarly. Define

$$f(I) = \left\| \phi_+ - \begin{pmatrix} v_+ \\ 0 \end{pmatrix} \right\|_{L_t^\infty H^\sigma(I\times\mathbb{R}^2)} + \|\phi_-\|_{L_t^\infty H_x^\sigma(I\times\mathbb{R}^2)},$$

where σ is $1/2 < \sigma < s$ and $I = [0, T]$ is the uniform existence interval obtained in Theorem 1. Our goal is to show $f(I)$ converges to zero as c is going to infinity for some $T > 0$. We write

$$\begin{aligned} \phi_\pm^c(t) &= U_\pm^c(t)\phi_{0\pm}^c + \int_0^t U^c(t-s)(iA_0\phi_\pm^c \pm e^{\pm ic^2 s}\mathcal{R})ds, \\ v_+(t) &= S(t)v_{0+}, \end{aligned}$$

where $\phi_\pm, v_\pm, \phi_0^\pm, v_{0\pm}$ are given by (4.6) and (1.8). Then

$$\begin{aligned} \phi_+^c(t) - \begin{pmatrix} v_+ \\ 0 \end{pmatrix}(t) &= U^c(t)\left[\phi_{0+}^{\ c} - \begin{pmatrix} v_{0+} \\ 0 \end{pmatrix}\right] + [U^c(t) - S(t)]\begin{pmatrix} v_{0+} \\ 0 \end{pmatrix} \\ &\quad + \int_0^t U^c(t-s)\left[iA_0\phi_+^c + e^{ic^2 s}\mathcal{R}\right]ds \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{4.12}$$

In what follows we use the notation $a_c = o(1)$ if $\lim_{c\to\infty} a_c = 0$ holds.

Note that $\phi_{0\pm}^c = \tilde{\psi}_{0\pm}^c$. By (4.10) it holds that

$$\|I_1\|_{L_t^\infty H^s(I\times\mathbb{R}^2)} \le C\left\|\phi_{0+}^c - \begin{pmatrix} v_{0+} \\ 0 \end{pmatrix}\right\| \le C\left\|\Pi_+^c\psi_0^c - \begin{pmatrix} v_{0+} \\ 0 \end{pmatrix}\right\|_{H^s}$$

as c becomes large. Then

$$\|I_1\|_{L_t^\infty H^s(I\times\mathbb{R}^2)} = o(1)$$

follows from (4.11). Also we have

$$\|I_2\|_{L_t^\infty H^s(I\times\mathbb{R}^2)} = o(1)$$

by LDCT and the initial assumption on v_{0+}.

For I_3 we claim that

$$\begin{aligned} &\left\|\int_0^t U^c(t-s)\left[iA_0\phi_+\right]ds\right\|_{L_t^\infty H^s(I\times\mathbb{R}^2)} \\ &\le CT\left(\left\|\phi_+ - \begin{pmatrix} v_+ \\ 0 \end{pmatrix}\right\|_{L_t^\infty H^s(I\times\mathbb{R}^2)} + \|\phi_-\|_{L_t^\infty H^s(I\times\mathbb{R}^2)}\right) \end{aligned} \tag{4.13}$$

and

$$\left\|\int_0^t U^c(t-s)e^{ic^2 s}\mathcal{R}\,ds\right\|_{H^\sigma} = o(1) \text{ for } 1/2 < \sigma < s. \tag{4.14}$$

As in the proof of Proposition 2 we have

$$\begin{aligned}\int_0^T \|A_0\phi_+\|_{H^s}\ dt &\leq C\int_0^T \|\phi_+\|_{H^s}\|\ A_0\|_{L^\infty}\ dt + \int_0^T \|\phi_+\|_{W^{s,\frac{2}{s}}}\|A_0\|_{L^{\frac{2}{1-s}}}\ dt\\ &\leq C\|\phi_+\|_{L_t^\infty H^s}\int_0^T \|A_0\|_{W^{1,\frac{2}{1-s}}} + \|A_0\|_{L^{\frac{2}{1-s}}}\,dt.\end{aligned}$$

Let $V = \begin{pmatrix} e^{-ic^2t}v_+ \\ 0\end{pmatrix}$ then due to skew symmetry of α^1 and α^2, it holds that

$$V^\dagger\alpha^1 V = V^\dagger\alpha^2 V = 0.$$

Hence A_0 equation in (4.7) reads that

$$\Delta A_0 = 2\partial_1(\psi^\dagger\alpha^2\psi) - 2\partial_2(\psi^\dagger\alpha^1\psi) - \left[2\partial_1(V^\dagger\alpha^2V) - 2\partial_2(V^\dagger\alpha^1V)\right]. \tag{4.15}$$

Since

$$\begin{aligned}&\psi^\dagger\alpha^2\psi - V^\dagger\alpha^2 V\\ &= (\psi - V)^\dagger\alpha^2\psi + V^\dagger\alpha^2(\psi - V)\\ &= \left(\tilde{\psi}_+ + \tilde{\psi}_- - \begin{pmatrix} e^{-ic^2t}v_+ \\ 0\end{pmatrix}\right)^\dagger \alpha^2\psi + V^\dagger\alpha^2\left(\tilde{\psi}_+ + \tilde{\psi}_- - \begin{pmatrix} e^{-ic^2t}v_+ \\ 0\end{pmatrix}\right)\\ &= \left(e^{-ic^2t}\left(\phi_+ - \begin{pmatrix} v_+ \\ 0\end{pmatrix}\right) + e^{ic^2t}\phi_-\right)^\dagger\alpha^2\psi\\ &\quad + V^\dagger\alpha^2\left(e^{-ic^2t}\left(\phi_+ - \begin{pmatrix} v_+ \\ 0\end{pmatrix}\right) + e^{ic^2t}\phi_-\right),\end{aligned}$$

we write (4.15) symbolically as

$$\Delta A_0 = \partial_1\left[\left(|\phi_+ - \begin{pmatrix} v_+ \\ 0\end{pmatrix}| + |\phi_-|\right)\psi\right] + \partial_2\left[\begin{pmatrix} v_+ \\ 0\end{pmatrix}\left(|\phi_+ - \begin{pmatrix} v_+ \\ 0\end{pmatrix}| + |\phi_-|\right)\right].$$

Then by the same estimates yielding to (3.9), we have

$$\begin{aligned}&\|A_0\|_{W^{1,\frac{2}{1-s}}} + \|A_0\|_{L^{\frac{2}{1-s}}}\\ &\leq C(\|\psi\|_{L_t^\infty H^s(I\times\mathbb{R}^2)} + \|v_+\|_{L_t^\infty H^s(I\times\mathbb{R}^2)})\\ &\quad\left(\left\|\phi_+ - \begin{pmatrix} v_+ \\ 0\end{pmatrix}\right\|_{L_t^\infty H^s(I\times\mathbb{R}^2)} + \|\phi_-\|_{L_t^\infty H^s(I\times\mathbb{R}^2)}\right)\\ &\leq C\left(\left\|\phi_+ - \begin{pmatrix} v_+ \\ 0\end{pmatrix}\right\|_{L_t^\infty H^s(I\times\mathbb{R}^2)} + \|\phi_-\|_{L_t^\infty H^s(I\times\mathbb{R}^2)}\right).\end{aligned}$$

The first claim (4.13) is proved. The restriction $\sigma < s$ is only necessary for estimating $M^{-1}(ciA_j\partial_j\psi)$ for the second claim. By Lemma 3 we have

$$\|M^{-1}(ciA_j\partial_j\psi)\|_{\dot{H}^\sigma} \lesssim c^{-\delta}\|A_j\partial_j\psi\|_{\dot{H}^{\sigma-1+\delta}}$$

for a sufficiently small δ. A product estimate in homegeneous Sobolev spaces is given by the following.

Lemma 6. *Assume $s_1+s_2+s_3=\frac{d}{2}$ with $s_j+s_k>0$ for $j\neq k$. Then it holds that*

$$\|fg\|_{\dot{H}^{-s_1}(\mathbb{R}^d)}\lesssim\|f\|_{\dot{H}^{s_2}(\mathbb{R}^d)}\|g\|_{\dot{H}^{s_3}(\mathbb{R}^d)}.$$

We write $\sigma-1+\delta=s-1-\epsilon=-(1-s)_-$ by letting $\epsilon=s-\sigma-\delta$. By plugging in Lemma 6, $s_1=1-s+\epsilon$, $s_2=1-\epsilon$, $s_3=s-1$ we have

$$\|A\nabla\psi\|_{\dot{H}^{s-1-\epsilon}}\lesssim\|A\|_{\dot{H}^{1-\epsilon}}\|\nabla\psi\|_{\dot{H}^{s-1}}\lesssim\|\psi\|^3_{H^s}$$

for a small ϵ. The second inequality holds by the third estimate in (3.8).

It remains to estimate $\|M^{-1}(ciA_j\partial_j\psi)\|_{L^2}$. To do this, we need some elementary paracalculus technique. For notations used in the paragraph, see Chapter 2 in [5] for instance. Let us introduce the Littlewood-Paley decomposition

$$f=f_{-1}+\sum_{q=0}^{\infty}f_q,$$

$$\hat{f}_{-1}(\xi)=\chi(\xi)\hat{f}(\xi),\quad \hat{f}_q=\beta(\xi/2^q)\hat{f}(\xi),$$

where χ and β are nonnegative smooth radial functions with compact supports in $\{0\leq|\xi|<\frac{4}{3}\}$ and $\{\frac{3}{4}\leq|\xi|\leq\frac{8}{3}\}$ respectively such that

$$\chi(\xi)+\sum_{q\geq 0}\beta(\xi/2^q)=0.$$

We decompose the product $A\nabla\psi$ by

$$A\nabla\psi=\sum_{p,q\geq-1}A_p(\nabla\psi)_q=P_1+P_2+P_3,$$

where

$$P_1=\sum_{q+2\leq p}A_p(\nabla\psi)_q,\quad P_2=\sum_{p+2\leq q}A_p(\nabla\psi)_q,\quad P_3=\sum_{|p-q|\leq 1}A_p(\nabla\psi)_q.$$

The summand in P_1 is supported in $\{\frac{3\cdot 2^p}{4}\leq|\xi|\leq\frac{8\cdot 2^p}{3}\}$, in P_2 supported in $\{\frac{3}{4}2^q\leq|\xi|\leq\frac{8}{3}2^q\}$, and in P_3 supported in $\{|\xi|\leq\frac{16}{3}2^p\}$. We have

$$\begin{aligned}
&\|M^{-1}(ciA_j\partial_j\psi)\|_{L^2}\\
&\leq\|(c^2+|\nabla|^2)^{-1/2}(A\nabla\psi)\|_{L^2}\\
&\leq\sum_{q+2\leq p}\left\|\frac{2^p}{\sqrt{c^2+2^{2p}}}A_p\psi_q\right\|_{L^2}+\sum_{p+2\leq q}\left\|\frac{2^q}{\sqrt{c^2+2^{2q}}}A_p\psi_q\right\|_{L^2}\\
&\quad+\sum_{|p-q|\leq 1}\left\|\frac{2^q}{\sqrt{c^2+2^{2q}}}A_p\psi_q\right\|_{L^2}\\
&:=\mathcal{P}_1+\mathcal{P}_2+\mathcal{P}_3.
\end{aligned}$$

We estimate that for $p,q\geq-1$

$$\mathcal{P}_1\lesssim c^{-\epsilon}\sum_{q+2\leq p}2^{-\epsilon p}\|2^{2\epsilon p}A_p\|_{L^r}\|\psi_q\|_{L^{\tilde{r}}}$$

$$
\begin{aligned}
&\lesssim c^{-\epsilon}\left(\sum_{q+2\le p} 2^{-\epsilon p}\right)\|A\|_{W^{2\epsilon,r}}\|\psi\|_{L^{\tilde r}} \lesssim c^{-\epsilon}\|A\|_{W^{2\epsilon,r}}\|\psi\|_{L^{\tilde r}},\\
\mathcal{P}_2 &\lesssim c^{-\epsilon}\sum_{p+2\le q} 2^{\epsilon q}\|A_p\|_{L^{\tilde r}}\|\psi_q\|_{L^r} \lesssim c^{-\epsilon}\sum_{p+2\le q}\|A_p\|_{L^{\tilde r}}2^{\epsilon q}\|\psi_q\|_{H^{s'}}\\
&\lesssim c^{-\epsilon}\sum_{p+2\le q}\|A_p\|_{L^{\tilde r}}2^{-(s-s'-\epsilon)q}\|\psi_q\|_{H^s}\\
&\lesssim c^{-\epsilon}\left(\sum_{p+2\le q} 2^{-(s-s'-\epsilon)q}\right)\|A\|_{L^{\tilde r}}\|\psi\|_{H^s} \lesssim c^{-\epsilon}\|A\|_{L^{\tilde r}}\|\psi\|_{H^s},\\
\mathcal{P}_3 &\lesssim c^{-\epsilon}\sum_{p\sim q}2^{\epsilon q}\|A_p\|_{L^{\tilde r}}\|\psi_q\|_{L^r} \lesssim c^{-\epsilon}\left(\sum_p \|A_p\|_{L^{\tilde r}}^2\right)^{\frac12}\left(\sum_q 2^{2\epsilon q}\|\psi_q\|_{L^r}^2\right)^{\frac12}\\
&\lesssim c^{-\epsilon}\left(\|A_{-1}\|_{L^{\tilde r}} + \sum_{p\ge 0}2^{\delta p}\|A_p\|_{L^2}^2\right)^{\frac12}\left(\sum_q 2^{2\epsilon q}\|\psi_q\|_{H^{s'}}^2\right)^{\frac12}\\
&\lesssim c^{-\epsilon}(\|A\|_{L^{\tilde r}} + \|A\|_{\dot W^{1,2}})\left(\sum_q 2^{2\epsilon q}\|\psi_q\|_{H^{s'}}^2\right)^{\frac12} \lesssim c^{-\epsilon}\|\psi\|_{H^s}^3.
\end{aligned}
$$

The pair $r, \tilde r$ satisfies $1/r + 1/\tilde r = 1/2$. In $\mathcal{P}_2$ and $\mathcal{P}_3$ we choose r, s' such that $2 < r \le \frac{2}{1-s'}$ and $s' < s - \epsilon$ then use the Sobolev embedding. In $\mathcal{P}_3$ we use the Cauchy-Schwarz inequality and choose $\tilde r = \frac{2}{1-\delta}$ for $0 < \delta < 1$ then use (3.8). Now the estimates (3.8) and the Sobolev embedding lead that $\|M^{-1}(ciA_j\partial_j\psi)\|_{L^2} \lesssim c^{-\epsilon}$. Hence we have proved the second claim (4.14)

The same estimates yields that

(4.16)

$$
\|\phi_-\|_{L_t^\infty H^\sigma} \le o(1) + CT\left(\left\|\phi_+ - \begin{pmatrix} v_+ \\ 0\end{pmatrix}\right\|_{L_t^\infty H^s(I\times\mathbb{R}^2)} + \|\phi_-\|_{L_t^\infty H^s(I\times\mathbb{R}^2)}\right).
$$

Summed up, the above estimates (4.12)- (4.16) imply that

$$
f(I) \lesssim o(1) + Tf(I),
$$

which concludes $f(I) \to 0$ as $c \to \infty$ for some $T > 0$. Theorem 2 is proved.

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