CENTRAL INDEX BASED SOME COMPARATIVE GROWTH ANALYSIS OF COMPOSITE ENTIRE FUNCTIONS FROM THE VIEW POINT OF $L^*$-ORDER

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ABSTRACT. In this paper, we discuss central index oriented and slowly changing function based some growth properties of composite entire functions.

1. Introduction, Definitions and Notations

Let $f$ be an entire function defined in the open complex plane $\mathbb{C}$. For entire $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$, the maximum modulus symbolized as $M_f(r)$, the maximum term denoted as $\mu_f(r)$ and the central index indicated as $\nu_f(r)$ are respectively defined as $\max_{|z|=r} |f(z)|$, $\max \{|a_n|r^n\}$ and $\max \{m, \mu_f(r) = |a_m|r^m\}$. Therefore, central index $\nu_f(r)$ of an entire function $f$ is the greatest exponent $m$ such that $|a_m|r^m = \mu_f(r)$. Obviously $M_f(r)$, $\mu_f(r)$ and $\nu_f(r)$ are real and increasing function of $r$. For another entire function $g$, $M_g(r)$ and $\mu_g(r)$ are also defined and the ratios $\frac{M_f(r)}{M_g(r)}$ when $r \to \infty$ as well as $\frac{\nu_f(r)}{\nu_g(r)}$ as $r \to \infty$ are called the comparative growth of $f$ with respect to $g$ in terms of their maximum moduli and the maximum term respectively. The prime object of the study of the growth investigation of entire functions has usually been done through their maximum moduli and maximum term. Though $\nu_f(r)$ is much weaker than $M_f(r)$ and $\mu_g(r)$ in some sense, from another angle of view $\frac{\nu_f(r)}{\nu_g(r)}$ as $r \to \infty$ is also called the growth of $f$ with respect to $g$ where $\nu_g(r)$ denotes the central index of entire $g$. Considering this, here we compare the central index of composition of two entire functions with their corresponding left and right factors under the treatment of the theories of slowly changing functions which in fact means

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that \( L(\ar) \sim L(r) \) as \( r \to \infty \) for every positive constant \( a \) i.e., \( \lim_{r \to \infty} \frac{L(\ar)}{L(r)} = 1 \) where \( L \equiv L(r) \). Actually in this paper we attempt to prove some results related to the growth rates of composite entire functions on the basis of central index using the idea of \( L^*-\text{order} \) (respectively, \( L^*-\text{lower order} \)) of an entire function where \( L^* \) is nothing but a weaker assumption of \( L \). Our notations are standard within the theory of Nevanlinna’s value distribution of entire functions and therefore we do not explain those in detail as those are available in [8]. To start our paper we just recall the following definitions which will be needed in the sequel:

**Definition 1.** The order \( \rho_f \) and lower order \( \lambda_f \) of an entire function \( f \) are define as

\[
\rho_f = \lim_{r \to +\infty} \sup_{r \to +\infty} \frac{\log \log M_f(r)}{\log r} = \lim_{r \to +\infty} \inf_{r \to +\infty} \frac{\log^2 M_f(r)}{\log r}.
\]

Therefore it seems reasonable to state suitably an alternative definition of order and lower order of entire function in terms of its central index. He and Xiao [3] introduced such a definition in the following way:

\[
\rho_f = \lim_{r \to +\infty} \sup_{r \to +\infty} \frac{\log \nu_f(r)}{\log r}.
\]

Let \( L \equiv L(r) \) be a positive continuous function increasing slowly i.e., \( L(\ar) \sim L(r) \) as \( r \to \infty \) for every positive constant \( a \). Considering \( L(r) = \log r \) and \( a = 10^{20} \), one can easily show that \( \lim_{r \to \infty} \frac{L(\ar)}{L(r)} = 1 \). Somasundaram and Thamizharasi [6] introduced the notions of \( L\text{-order} \) (respectively \( L\text{-lower order} \)) of entire functions. The more generalized concept for \( L\text{-order} \) and \( L\text{-lower order} \) for entire functions is \( L^*\text{-order} \) and \( L^*\text{-lower order} \) whose definition are as follows:

**Definition 2** ([6]). The \( L^*-\text{order} \( \rho_f^L \) and \( L^*-\text{lower order} \( \lambda_f^L \) of an entire function \( f \) are defined as

\[
\rho_f^L = \lim_{r \to +\infty} \sup_{r \to +\infty} \frac{\log \log M_f(r)}{\log [re^{L(r)}]} = \lim_{r \to +\infty} \inf_{r \to +\infty} \frac{\log^2 M_f(r)}{\log [re^{L(r)}]}.
\]

Taking \( f(z) = \exp z \) and \( L(r) = \log r \), one can easily verify that \( \rho_f = \lambda_f = 1 \) and \( \rho_f^L = \lambda_f^L = \frac{1}{2} \).

In terms of central index of entire functions, Definition 2 can be reformulated as:

**Definition 3.** The growth indicators \( \rho_f^L \) and \( \lambda_f^L \) of an entire function \( f \) are defined as:

\[
\rho_f^L = \lim_{r \to +\infty} \sup_{r \to +\infty} \frac{\log \nu_f(r)}{\log [re^{L(r)}]}.
\]
The concept of \((p, q)\)-\(\varphi\) order of entire function was introduced by Shen et al. [5] where \(p \geq q \geq 1\) and \(\varphi : [0, +\infty) \rightarrow (0, +\infty)\) be a non-decreasing unbounded function. Shen et al. [5] also established the equivalence of the definition of \((p, q)\)-\(\varphi\) order of entire function in terms of maximum modulus and central index under some certain condition. For details about it, one may see [5]. For particular if we consider \(p = 1, q = 1\) and \(\varphi (r) = re^{L(r)}\), then in view of Proposition 1.2 of [5], we can write that

\[
\rho_f^{L} = \lim_{r \to +\infty} \sup \log \log M_f(r) = \lim_{r \to +\infty} \sup \log \left[ \frac{\log \nu_f(r)}{\log \left( re^{L(r)} \right)} \right].
\]

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

**Lemma 1** ([3, Theorems 1.9 and 1.10] or [4, Satz 4.3 and 4.4]). Let \(f\) be any entire function, then

\[
\log \mu_f(r) = \log |a_0| + \int_0^r \frac{\nu_f(t)}{t} dt \text{ where } a_0 \neq 0,
\]

and for \(r < R\),

\[
M_f(r) < \mu_f(r) \left\{ \mu_f(R) + \frac{R}{R - r} \right\}.
\]

**Lemma 2** ([1]). Let \(f\) and \(g\) are any two entire functions with \(g(0) = 0\). Also let \(\beta\) satisfy \(0 < \beta < 1\) and \(c(\beta) = \frac{(1 - \beta) \beta}{\beta} = \frac{\beta}{1 + \beta}\). Then for all sufficiently large values of \(r\),

\[
M_f \left( c(\beta) M_g(\beta r) \right) \leq M_{fog}(r) \leq M_f \left( M_g(r) \right).
\]

In addition if \(\beta = \frac{1}{2}\), then for all sufficiently large values of \(r\),

\[
M_{fog}(r) \geq M_f \left( \frac{1}{8} M_g \left( \frac{r}{2} \right) \right).
\]

**Lemma 3.** Let \(f\) be an entire function with \(0 < \lambda_f \leq \rho_f < \infty\). Also let \(g\) be an entire function with non zero finite lower order. If \(0 < \alpha < \lambda_g\), then for all sufficiently large values of \(r\),

\[
\nu_{fog}(r) > \nu_f(\exp(r^\alpha))\).
\]

**Proof.** For any constant E, we get from the second part of Lemma 1, that

\[
\log M_f(r) < \nu_f(r) \log r + \log \nu_f(2r) + E \quad \text{cf. [2]}.
\]
Therefore from above we obtain that

\[ \log M_f(r) < \nu_f(2r) \log r + E \]

\( i.e., \) \( \log M_f(r) < \nu_f(2r) (1 + \log r) + E \)

\( i.e., \) \( \log M_f(r) < \nu_f(2r) \log (e \cdot r) + E \)

\( i.e., \) \( \log M_f \left( \frac{r}{2} \right) < \nu_f(r) \log \left( e \cdot \frac{r}{2} \right) + E \)

(1)

\( \log \nu_{fog}(r) > \log^{[2]} M_{fog} \left( \frac{r}{2} \right) - \log^{[2]} \left( e \cdot \frac{r}{2} \right) - \log \left( 1 + \frac{E}{\nu_{fog}(r) \log (e \cdot \frac{r}{2})} \right) \)

\( i.e., \) \( \lim_{r \to \infty} \frac{\log \nu_{fog}(r)}{\log \nu_f(\exp(r^\alpha))} \geq \lim_{r \to \infty} \frac{\log^{[2]} M_{fog} \left( \frac{r}{2} \right)}{\log \nu_f(\exp(r^\alpha))} - \frac{\log \left( 1 + \frac{E}{\nu_{fog}(r) \log (e \cdot \frac{r}{2})} \right)}{\log \nu_f(\exp(r^\alpha))} \)

(2)
\[
\liminf_{r \to \infty} \frac{\log^2 M_{\varphi \psi}(r)}{\log \nu_f(\exp(r^\alpha))} \geq \liminf_{r \to \infty} \frac{(\lambda_f - \varepsilon) \frac{1}{8} + (\lambda_f - \varepsilon) \frac{1}{2}(\lambda_g - \varepsilon)}{(\rho_f + \varepsilon) r^\alpha}
\]

As \(\alpha < \lambda_g\) we can choose \(\varepsilon > 0\) in such a way that

\[
\alpha < \lambda_g - \varepsilon.
\]

Thus from (5) and (6) we get that

\[
\lim_{r \to \infty} \frac{\log^2 M_{\varphi \psi}(r)}{\log \nu_f(\exp(r^\alpha))} = \infty.
\]

Therefore from (2) and (7) we obtain that

\[
i.e., \lim_{r \to \infty} \frac{\log \nu_{\varphi \psi}(r)}{\log \nu_f(\exp(r^\alpha))} = \infty.
\]

So from above we obtain for all sufficiently large values of \(r\) and \(K > 1\) that

\[
\log \nu_{\varphi \psi}(r) > K \log \nu_f(\exp(r^\alpha))
\]

\[i.e., \log \nu_{\varphi \psi}(r) > \log \nu_f(\exp(r^\alpha))^K\]

\[i.e., \nu_{\varphi \psi}(r) \geq \nu_f(\exp(r^\alpha)).\]

This proves the theorem.

In the line of Lemma 3, one can easily verify the following corollary and therefore its proof is omitted.

**Corollary 1.** Let \(f\) be an entire function with non zero lower order. Also let \(g\) be an entire function with \(0 < \lambda_g \leq \rho_g < \infty\). If \(0 < \alpha < \lambda_g\), then for all sufficiently large values of \(r\),

\[\nu_{\varphi \psi}(r) > \nu_g(\exp(r^\alpha)).\]

3. **Results**

In this section we present the main results of the paper.

**Theorem 4.** Let \(f\) be an entire function with non zero finite order and lower order and \(g\) be an entire function with non zero finite lower order. If \(0 < \lambda_f L_f^* \leq \rho_f L_f^* < \infty\), then for any \(A > 0\)

\[
\lim_{r \to \infty} \frac{\log^2 \nu_{\varphi \psi}(\exp(r^A))}{\log \nu_f(\exp(r^\alpha)) + K(r, A; L)} = \infty.
\]
where $0 < \alpha < \lambda_g$ and $K (r, A; L) = \begin{cases} 0 & \text{if } r^\alpha = o \{ L \left( \exp \left( \alpha r^A \right) \right) \} \\ L \left( \exp \left( \alpha r^A \right) \right) & \text{as } r \to \infty \\ \text{otherwise} \end{cases}$.

Proof. Let $0 < \alpha < \alpha' < \lambda_g$. Now from the definition of $L^*$-lower order we obtain in view of Lemma 3, for for all sufficiently large values of $r$ that

$$
\log \nu_{fog} \left( \exp (r^A) \right) \geq \log \nu_f \left( \exp (r^A) \right)^{\alpha'}
$$

i.e., $\log \nu_{fog} \left( \exp (r^A) \right)$

$$
\geq \left( \lambda^*_{f'} - \varepsilon \right) \cdot \log \left\{ \left( \exp (r^A) \right)^{\alpha'} \cdot L \left( \exp (r^A) \right)^{\alpha'} \right\}
$$

i.e., $\log \nu_{fog} \left( \exp (r^A) \right)$

$$
\geq \left( \lambda^*_{f'} - \varepsilon \right) \cdot \left\{ \left( \exp (r^A) \right)^{\alpha'} \left( 1 + \frac{L \left( \exp (r^A) \right)^{\alpha'}}{\left( \exp (r^A) \right)^{\alpha'}} \right) \right\}
$$

i.e., $\log \nu_{fog} \left( \exp (r^A) \right) \geq O (1) + \alpha' \log \exp (r^A)$

$$
+ \log \left\{ 1 + \frac{L \left( \exp (r^A) \right)^{\alpha'}}{\left( \exp (r^A) \right)^{\alpha'}} \right\}
$$

i.e., $\log \nu_{fog} \left( \exp (r^A) \right) \geq O (1) + \alpha' r^A$

$$
+ \log \left\{ 1 + \frac{L \left( \exp (r^A) \right)^{\alpha'}}{\exp \left( \alpha' r^A \right)} \right\}
$$

i.e., $\log \nu_{fog} \left( \exp (r^A) \right) \geq O (1) + \alpha' r^A$

$$
+ \log \left[ 1 + \frac{L \left( \exp \left( \alpha' r^A \right) \right)}{\exp \left( \alpha' r^A \right)} \right]
$$

i.e., $\log \nu_{fog} \left( \exp (r^A) \right) \geq O (1) + \alpha' r^A + L \left( \exp \left( \alpha r^A \right) \right)$

$$
- \log \left[ \exp \left\{ L \left( \exp \left( \alpha r^A \right) \right) \right\} \right]
$$

$$
+ \log \left[ 1 + \frac{L \left( \exp \left( \alpha' r^A \right) \right)}{\exp \left( \alpha' r^A \right)} \right]
$$
i.e., \[ \log^2 \nu_{fog}(\exp(r^A)) \geq O(1) + \alpha' r^A + L(\exp(\exp(\alpha r^A))) \]

\[ + \log \left[ \frac{\exp(\alpha r^A) + L(\exp(\exp(\alpha r^A)))}{\exp\{L(\exp(\exp(r^A)))\} \cdot \exp(\alpha r^A)} \right] \]

(8)

\[ \text{i.e., } \log^2 \nu_{fog}(\exp(r^A)) \geq O(1) + \alpha' r^{(A-\alpha)} r^\alpha + L(\exp(\exp(r^A))) \]

Again we have for all sufficiently large values of \( r \) that

\[ \log \nu_f(\exp(r^a)) \leq (\rho_f^{L^*} + \varepsilon) \log \left\{ \exp(r^a) e^{L(\exp(r^a))} \right\} \]

i.e., \[ \log \nu_f(\exp(r^a)) \leq (\rho_f^{L^*} + \varepsilon) \{ \log \exp(r^a) + L(\exp(r^a)) \} \]

i.e., \[ \log \nu_f(\exp(r^a)) \leq (\rho_f^{L^*} + \varepsilon) \{ r^a + L(\exp(r^a)) \} \]

(9)

\[ \text{i.e., } \frac{\log \nu_f(\exp(r^a)) - (\rho_f^{L^*} + \varepsilon) L(\exp(r^a))}{(\rho_f^{L^*} + \varepsilon)} \leq r^a. \]

Now from (8) and (9) it follows for all sufficiently large values of \( r \) that

(10)

\[ \log^2 \nu_{fog}(\exp(r^A)) \]

\[ \geq O(1) + \left( \frac{\alpha' r^{(A-\alpha)}}{\rho_f^{L^*} + \varepsilon} \right) \left[ \log \nu_f(\exp(r^a)) - \left( \rho_f^{L^*} + \varepsilon \right) L(\exp(r^a)) \right] \]

\[ + L(\exp(\exp(r^A))) \]

(11)

\[ \text{i.e., } \frac{\log^2 \nu_{fog}(\exp(r^A))}{\log \nu_f(\exp(r^a))} \geq \frac{L(\exp(\exp(r^A))) + O(1)}{\log \nu_f(\exp(r^a))} \]

\[ + \frac{\alpha' r^{(A-\alpha)}}{\rho_f^{L^*} + \varepsilon} \left\{ 1 - \frac{(\rho_f^{L^*} + \varepsilon) L(\exp(r^a))}{\log \nu_f(\exp(r^a))} \right\}. \]

Again from (10) we get for all sufficiently large values of \( r \) that

\[ \frac{\log^2 \nu_{fog}(\exp(r^A))}{\log \nu_f(\exp(r^a)) + L(\exp(\exp(r^A)))} \geq \frac{O(1) - \alpha' r^{(A-\alpha)} L(\exp(r^a))}{L(\exp(\exp(r^A))) + L(\exp(\exp(r^A)))} \]

\[ + \frac{\alpha' r^{(A-\alpha)}}{\rho_f^{L^*} + \varepsilon} \log \nu_f(\exp(r^a)) \]

\[ + L(\exp(\exp(r^A))) \]
\[
\frac{\log^2 \nu_{fog} \left( \exp \left( r^A \right) \right)}{\log \nu_f \left( \exp \left( r^\alpha \right) \right) + L \left( \exp \left( \alpha r^A \right) \right)} \geq \frac{O(1) - \alpha' r^{(A-\alpha)} L(\exp(r^\alpha))}{L(\exp(\exp(\alpha r^A))) + 1} \\
+ \left( \frac{\nu_f^{(A-\alpha)}}{\rho_f^{1+\varepsilon}} \right) \log \nu_f \left( \exp \left( r^\alpha \right) \right) + \frac{1}{1 + \frac{L(\exp(\exp(\alpha r^A)))}{\log \nu_f (\exp(r^\alpha))}}.
\]

Case I. If \( r^\alpha = o \left\{ L \left( \exp \left( \alpha r^A \right) \right) \right\} \) then it follows from (11) that
\[
\lim_{r \to 1} \log^2 \nu_{fog} \left( \exp \left( r^A \right) \right) = \infty.
\]

Case II. \( r^\alpha \neq o \left\{ L \left( \exp \left( \alpha r^A \right) \right) \right\} \) then two sub cases may arise.

Sub case (a). If \( L \left( \exp \left( \alpha r^A \right) \right) = o \left\{ \log \nu_f \left( \exp \left( r^\alpha \right) \right) \right\} \), then we get from (12) that
\[
\lim_{r \to 1} \log^2 \nu_{fog} \left( \exp \left( r^A \right) \right) = \infty.
\]

Sub case (b). If \( L \left( \exp \left( \alpha r^A \right) \right) \sim \log \nu_f \left( \exp \left( r^\alpha \right) \right) \) then
\[
\lim_{r \to 1} \frac{L \left( \exp \left( \alpha r^A \right) \right)}{\log \nu_f \left( \exp \left( r^\alpha \right) \right)} = 1
\]
and we obtain from (12) that
\[
\lim_{r \to 1} \log^2 \nu_{fog} \left( \exp \left( r^A \right) \right) = \infty.
\]

Combining Case I and Case II we may obtain that
\[
\lim_{r \to 1} \log^2 \nu_{fog} \left( \exp \left( r^A \right) \right) = \infty,
\]
where
\[
K(r, A; L) = \begin{cases} 
0 & \text{if } r^\mu = o \left\{ L \left( \exp \left( \alpha r^A \right) \right) \right\} \\
L \left( \exp \left( \alpha r^A \right) \right) & \text{as } r \to \infty \\
& \text{otherwise}
\end{cases}
\]

This proves the theorem.

\[\square\]

**Theorem 5.** Let \( f \) be an entire function with non zero finite order and lower order and \( g \) be an entire function with non zero finite lower order. If \( \lambda_f^+ > 0 \) and \( \rho_g^+ \leq \infty \) then for any \( A > 0 \)
\[
\lim_{r \to 1} \log^2 \nu_{fog} \left( \exp \left( r^A \right) \right) = \infty,
\]
where $0 < \alpha < \lambda_g$ and $K (r, A; L) = \left\{ \begin{array}{ll}
0 & \text{if } r^\alpha = o \left\{ L \left( \exp \left( \exp \left( \alpha r^A \right) \right) \right) \right\} \\
L \left( \exp \left( \exp \left( \alpha r^A \right) \right) \right) & \text{otherwise}.
\end{array} \right.$

The proof is omitted because it can be carried out in the line of Theorem 4.

REFERENCES


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