Korean J. Math. **26** (2018), No. 3, pp. 459–465
https://doi.org/10.11568/kjm.2018.26.3.459

A NOTE ON NONLINEAR SKEW LIE TRIPLE DERIVATION BETWEEN PRIME $*$-ALGEBRAS

Ali Taghavi, Mojtaba Nouri, and Vahid Darvish

Abstract. Recently, Li et al proved that Φ which satisfies the following condition on factor von Neumann algebras

$$\Phi([[A,B]_*,C]_*) = [[\Phi(A),B]_*,C]_* + [[A,\Phi(B)]_*,C]_* + [[A,B]_*,\Phi(C)]_*$$

where $[A,B]_* = AB - BA^*$ for all $A,B \in \mathcal{A}$, is additive $*$-derivation. In this short note we show the additivity of Φ which satisfies the above condition on prime $*$-algebras.

1. Introduction

Let $\mathcal{R}$ be a $*$-ring. For $A,B \in \mathcal{R}$, denoted by $A \diamond B = AB + BA^*$ and $[A,B]_* = AB - BA^*$, which are $*$-Jordan product and $*$-Lie product, respectively. These products are found playing a more and more important role in some research topics, and its study has recently attracted many author's attention (for example, see [2, 4, 6, 7]).

Let define λ-Jordan $*$-product by $A \diamond_\lambda B = AB + \lambda BA^*$. We say the map Φ with property of $\Phi(A \diamond_\lambda B) = \Phi(A) \diamond_\lambda B + A \diamond_\lambda \Phi(B)$ is a λ-Jordan $*$-derivation map. It is clear that for $\lambda = -1$ and $\lambda = 1$, the λ-Jordan $*$-derivation map is a $*$-Lie derivation and $*$-Jordan derivation, respectively [1]. We should mention here whenever we say Φ preserves derivation, it means $\Phi(AB) = \Phi(A)B + A\Phi(B)$.

Received May 06, 2018. Revised July 15, 2018. Accepted July 17, 2018.
2010 Mathematics Subject Classification: 46J10, 47B48, 46L10.
Key words and phrases: Lie triple derivation, prime $*$-algebra, additive map.

Recently, Yu and Zhang in [9] proved that every non-linear $*$-Lie derivation from a factor von Neumann algebra into itself is an additive $*$-derivation. Also, Li, Lu and Fang in [3] have investigated a non-linear λ-Jordan $*$-derivation. They showed that if $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra without central abelian projections and λ is a non-zero scalar, then $\Phi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ is a non-linear λ-Jordan $*$-derivation if and only if Φ is an additive $*$-derivation.

In [8] we showed that $*$-Jordan derivation map (i.e., $\phi(A \diamond_1 B) = \phi(A) \diamond_1 B + A \diamond_1 \phi(B)$) on every factor von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is additive $*$-derivation.

The authors of [5] introduced the concept of Lie triple derivations. A map $\Phi : \mathcal{A} \to \mathcal{A}$ is a nonlinear skew Lie triple derivations if

(1.1)

$$\Phi([[A,B]_*,C]_*) = [[\Phi(A),B]_*,C]_* + [[A,\Phi(B)]_*,C]_* + [[A,B]_*,\Phi(C)]_*$$

for all $A, B, C \in \mathcal{A}$ such that $[A,B]_* = AB - BA^*$. They showed that if Φ preserves the above characterizations on factor von Neumann algebras then Φ is additive $*$-derivation.

In this paper, we prove that if $\mathcal{A}$ is a prime $*$-algebra then $\Phi : \mathcal{A} \to \mathcal{A}$ which holds in (1.1) is additive.

We say that $\mathcal{A}$ is prime, that is, for $A, B \in \mathcal{A}$ if $A\mathcal{A}B = \{0\}$ then $A = 0$ or $B = 0$.

2. Main Results

Our main theorem is as follows:

Theorem 2.1. *Let $\mathcal{A}$ be a prime $*$-algebra with a non-trivial projection. Then the map $\Phi : A \to A$ satisfies in the following condition*

(2.1)

$$\Phi([[A,B]_*,C]_*) = [[\Phi(A),B]_*,C]_* + [[A,\Phi(B)]_*,C]_* + [[A,B]_*,\Phi(C)]_*$$

where $[A,B]_* = AB - \lambda BA^*$ for all $A, B, C \in \mathcal{A}$ is additive.

Proof. Let P_1 be a nontrivial projection in $\mathcal{A}$ and $P_2 = I_{\mathcal{A}} - P_1$. Denote $\mathcal{A}_{ij} = P_i\mathcal{A}P_j,\ i,j = 1,2$, then $\mathcal{A} = \sum_{i,j=1}^{2} \mathcal{A}_{ij}$. For every $A \in \mathcal{A}$ we may write $A = A_{11} + A_{12} + A_{21} + A_{22}$. In all that follow, when we write A_{ij}, it indicates that $A_{ij} \in \mathcal{A}_{ij}$. For showing additivity of Φ on $\mathcal{A}$, we use above partition of $\mathcal{A}$ and give some claims that prove Φ is additive on each $\mathcal{A}_{ij},\ i,j = 1,2$.

We prove the above theorem by several claims.

CLAIM 1. *We show that* $\Phi(0) = 0$.

This claim is easy to prove.

CLAIM 2. *For each* $A_{11} \in \mathcal{A}_{11}$ *and* $A_{22} \in \mathcal{A}_{22}$ *we have*

$$\Phi(A_{11} + A_{22}) = \Phi(A_{11}) + \Phi(A_{22}).$$

We show that

$$T = \Phi(A_{11} + A_{22}) - \Phi(A_{11}) - \Phi(A_{22}) = 0.$$

For $i \in \mathbb{C}$, we can write that

$$\begin{aligned} &[[\Phi(iI), P_1]_*, A_{11} + A_{22}]_* + [[iI, \Phi(P_1)]_*, A_{11} + A_{22}]_* \\ &+[[iI, P_1]_*, \Phi(A_{11} + A_{22})]_* = \Phi([[iI, P_1]_*, A_{11} + A_{22}]_*) \\ &= \Phi([[iI, P_1]_*, A_{11}) + \Phi([[iI, P_1]_*, A_{22}]_*) \\ &= \Phi([[iI, P_1]_*, A_{11} + A_{22}]_*) + [[iI, \Phi(P_1)]_*, A_{11} + A_{22}]_* \\ &+[[iI, P_1]_*, \Phi(A_{11}) + \Phi(A_{22})]_*. \end{aligned}$$

It follows that

$$[[iI, P_1]_*, T]_* = 0.$$

So,

$$T_{11} = T_{12} = T_{21} = 0.$$

Similarly, by applying the same proof for P_2 we have $T_{22} = 0$.

CLAIM 3. *For each* $A_{11} \in \mathcal{A}_{11}$, $A_{12} \in \mathcal{A}_{12}$, $A_{21} \in \mathcal{A}_{21}$, $A_{22} \in \mathcal{A}_{22}$ *we have*

$$\Phi(A_{11} + A_{12} + A_{21} + A_{22}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22}).$$

We show that for T in $\mathcal{A}$ the following holds

(2.2)

$$T = \Phi(A_{11} + A_{12} + A_{21} + A_{22}) - \Phi(A_{11}) - \Phi(A_{12}) - \Phi(A_{21}) - \Phi(A_{22}) = 0.$$

We can write

$$\begin{aligned}
&[[\Phi(P_1),(A_{11}+A_{12}+A_{21}+A_{22})]_*,P_2]_*\\
&\qquad+[[P_1,\Phi(A_{11}+A_{12}+A_{21}+A_{22})]_*,P_2]_*\\
&\qquad+[[P_1,(A_{11}+A_{12}+A_{21}+A_{22})]_*,\Phi(P_2]_*\\
&=\Phi([[P_1,(A_{11}+A_{12}+A_{21}+A_{22})]_*,P_2]_*)\\
&=\Phi([[P_1,A_{11}]_*,P_2]_*)+\Phi([[P_1,A_{12}]_*,P_2]_*)\\
&\qquad+\Phi([[P_1,A_{21}]_*,P_2]_*)+\Phi([[P_1,A_{22}]_*,P_2]_*)\\
&=[[\Phi(P_1),(A_{11}+A_{12}+A_{21}+A_{22})]_*,P_2]_*\\
&\qquad+[[P_1,(\Phi(A_{11})+\Phi(A_{12})+\Phi(A_{21})+\Phi(A_{22}))]_*,P_2]_*\\
&\qquad+[[P_1,(A_{11}+A_{12}+A_{21}+A_{22})]_*,\Phi(P_2)]_*.
\end{aligned}$$

Therefore,

$$[[P_1,T]_*,P_1]_*=0.$$

So, $T_{12}=0$.
Similarly, one can show that

$$[[P_2,T]_*,P_1=0.$$

We obtain $T_{21}=0$.
From Claim 2 we have

$$\begin{aligned}
&[[\Phi(i(P_1-P_2),I]_*,(A_{11}+A_{12}+A_{21}+A_{22})]_*\\
&\qquad+[[i(P_1-P_2),\Phi(I)]_*,(A_{11}+A_{12}+A_{21}+A_{22})]_*\\
&\qquad+[[i(P_1-P_2),I]_*,\Phi(A_{11}+A_{12}+A_{21}+A_{22})]_*\\
&=\Phi([[i(P_1-P_2),I]_*,(A_{11}+A_{12}+A_{21}+A_{22})]_*\\
&=[\Phi(i(P_1-P_2),I]_*,(A_{11}+A_{22})]_*\\
&\qquad+\Phi([[i(P_1-P_2),I]_*,A_{12}]_*+\Phi([[i(P_1-P_2),I]_*,A_{21}]_*\\
&=\Phi(i(P_1-P_2),I]_*,A_{11}]_*+\Phi(i(P_1-P_2),I]_*,A_{22}]_*\\
&\qquad+\Phi(i(P_1-P_2,I]_*,A_{21}]_*+\Phi(i(P_1-P_2,I]_*,A_{21}]_*\\
&=\Phi(i(P_1-P_2),I]_*,(A_{11}+A_{12}+A_{21}+A_{22})]_*\\
&\qquad+[[i(P_1-P_2),\Phi(I)]_*,(A_{11}+A_{12}+A_{21}+A_{22})]_*\\
&\qquad+[[i(P_1-P_2),I]_*,(\Phi(A_{11})+\Phi(A_{12})+\Phi(A_{21})+\Phi(A_{22})]_*.
\end{aligned}$$

So, we obtain

$$[[i(P_1-P_2),I]_*,T]_*=0.$$

So, $T_{11}=T_{22}=0$.

Claim 4. *For each $A_{ij}, B_{ij} \in \mathcal{A}_i$ such that $i \neq j$, we have*

$$\Phi(A_{ij}+B_{ij}) = \Phi(A_{ij}) + \Phi(B_{ij}).$$

It is easy to show that

$$\left[\left[i\frac{I}{2}, (P_i + A_{ij})\right]_*, i(P_j + B_{ij})\right]_* = A_{ij} - B_{ij} - A_{ij}^* - B_{ij}A_{ij}^*.$$

From Claim 3 we have

$$\begin{aligned}
&-\Phi(A_{ij}+B_{ij}) - \Phi(A_{ij}^*) - \Phi(B_{ij}A_{ij}^*)\\
&= \Phi\left(\left[\left[i\frac{I}{2}, (P_i + A_{ij})\right]_*, i(P_j + B_{ij})\right]_*\right)\\
&\quad + [\Phi\left(i\frac{I}{2}\right), (P_i + A_{ij})]_*, i(P_j + B_{ij})]_*\\
&\quad + \left[\left[\left(i\frac{I}{2}\right), \Phi(P_i + A_{ij})\right]_*, i(P_j + B_{ij}\right]_*\\
&= \left[\left[\Phi\left(i\frac{I}{2}\right), (P_i + A_{ij})\right]_*, (P_j + B_{ij})\right]_*\\
&\quad + \left[\left[\left(i\frac{I}{2}\right), (\Phi(P_i) + \Phi(A_{ij})\right]_*, (P_j + B_{ij})\right]_*\\
&\quad + \left[\left[\left(i\frac{I}{2}\right), (P_i + A_{ij})\right]_*, (\Phi(iP_j) + \Phi(iB_{ij})\right]_*\\
&= \Phi\left(\left[\left[\left(i\frac{I}{2}\right), P_i\right]_*, iP_j\right]_*\right) + \Phi\left(\left[\left[\left(i\frac{I}{2}\right), P_i\right]_*, iB_{ij}\right]_*\right)\\
&\quad + \Phi\left(\left[\left[\left(i\frac{I}{2}\right), A_{ij}\right]_*, iP_j\right]_*\right) + \Phi\left(\left[\left[\left(i\frac{I}{2}\right), A_{ij}\right]_*, iB_{ij}\right]_*\right)\\
&= -\Phi(B_{ij}) - \Phi(A_{ij}) - \Phi(A_{ij}^*) - \Phi(B_{ij}A_{ij}^*).
\end{aligned}$$

So,

$$\Phi(A_{ij}+B_{ij}) = \Phi(A_{ij}) + \Phi(B_{ij}).$$

Claim 5. *For each $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$ such that $1 \leq i \leq 2$, we have*

$$\Phi(A_{ii}+B_{ii}) = \Phi(A_{ii}) + \Phi(B_{ii}).$$

We show that

$$T = \Phi(A_{ii}+B_{ii}) - \Phi(A_{ii}) - \Phi(B_{ii}) = 0.$$

We can write

$$\begin{aligned}&[[\Phi(iP_j), I]_*, (A_{ii}+B_{ii})]_* + [[iP_j, \Phi(I)]_*, (A_{ii}+B_{ii})]_*\\ &\quad + [[iP_j, I]_*, \Phi(A_{ii}+B_{ii})]_*\\ &= \Phi([[iP_j, I]_*, A_{ii}+B_{ii}]_*) = \Phi([[iP_j, I]_*, A_{ii}]_*) + \Phi([[iP_j, I]_*, B_{ii}]_*)\\ &= [[\Phi(iP_j), I]_*, (A_{ii}+B_{ii})]_* + [[iP_j, \Phi(I)]_*, (A_{ii}+B_{ii})]_*\\ &\quad + [[iP_j, I]_*, (\Phi(A_{ij}) + \Phi(B_{ij})]_*.\end{aligned}$$

Hence

$$[[iP_j, I]_*, T]_* = 0.$$

So, $T_{ij} = T_{ji} = T_{jj} = 0$. From Claim 4 for each $C_{ij} \in \mathcal{A}_{ij}$ we have

$$\begin{aligned}&[[\Phi(iP_i), (A_{ii}+B_{ii})]_*, C_{ij}]_* + [[iP_i, \Phi(A_{ii}+B_{ii})]_*, C_{ij}]_*\\ &+[[iP_i, (A_{ii}+B_{ii})]_*, \Phi(C_{ij})]_*] = \Phi([[iP_i, (A_{ii}+B_{ii})]_*, C_{ij}]_*)\\ &= \Phi([[iP_i, A_{ii}]_*, C_{ij}]_*) + \Phi([[iP_i, B_{ii}]_*, C_{ij}]_*\\ &= [[\Phi(iP_i, (A_{ii}+B_{ii})]_*, C_{ij}]_* + [[iP_i, (A_{ii}+B_{ii})]_*, \Phi(C_{ij})]_*\\ &+[[iP_i, (\Phi(A_{ii}) + \Phi(B_{ij})]_*, C_{ij}]_*.\end{aligned}$$

From primeness of $\mathcal{A}$ we have $T_{ii} = 0$.

Hence, the additivity of Φ comes from the above claims.

References

[1] Z. Bai and S. Du, *The structure of non-linear Lie derivations on factor von Neumann algebras*, Linear Algebra Appl. **436** (2012), 2701–2708.

[2] J. Cui and C.K. Li, *Maps preserving product $XY - YX^*$ on factor von Neumann algebras*, Linear Algebra Appl. **431** (2009), 833–842.

[3] C. Li, F. Lu and X. Fang, *Nonlinear $\xi-$Jordan $*$-derivations on von Neumann algebras*, Linear and Multilinear Algebra. **62** (2014), 466–473.

[4] C. Li, F. Lu and X. Fang, *Nonlinear mappings preserving product $XY + YX^*$ on factor von Neumann algebras*, Linear Algebra Appl. **438** (2013), 2339–2345.

[5] C. LI, F Zhao and Q. Chen, *Nonlinear Skew Lie Triple Derivations between Factors*, Acta Mathematica Sinica **32** (2016), 821–830.

[6] L. Molnár, *A condition for a subspace of B(H) to be an ideal*, Linear Algebra Appl. **235** (1996), 229–234.

[7] A. Taghavi, V. Darvish and H. Rohi, *Additivity of maps preserving products $AP \pm PA^*$ on C^*-algebras*, Mathematica Slovaca **67** (2017), 213–220.

[8] A. Taghavi, H. Rohi and V. Darvish, *Non-linear $*$-Jordan derivations on von Neumann algebras*, Linear Multilinear Algebra **64** (2016), 426–439.

[9] W. Yu and J. Zhang, *Nonlinear $*$-Lie derivations on factor von Neumann algebras*, Linear Algebra Appl. **437** (2012), 1979–1991.

Ali Taghavi
Department of Mathematics, Faculty of Mathematical Sciences
University of Mazandaran P. O. Box 47416-1468 Babolsar, Iran.
E-mail: taghavi@umz.ac.ir

Mojtaba Nouri
Department of Mathematics, Faculty of Mathematical Sciences
University of Mazandaran P. O. Box 47416-1468 Babolsar, Iran.
E-mail: mojtaba.nori2010@gmail.com

Vahid Darvish
Department of Mathematics, Faculty of Mathematical Sciences
University of Mazandaran P. O. Box 47416-1468 Babolsar, Iran.
E-mail: vahid.darvish@mail.com