THE RANGE OF $r$-MAXIMUM INDEX OF GRAPHS

JEONG-OK CHOI

Abstract. For a connected graph $G$, an $r$-maximum edge-coloring is an edge-coloring $f$ defined on $E(G)$ such that at every vertex $v$ with $d_G(v) \geq r$ exactly $r$ incident edges to $v$ receive the maximum color. The $r$-maximum index $\chi'_r(G)$ is the least number of required colors to have an $r$-maximum edge coloring of $G$. In this paper, we show how the $r$-maximum index is affected by adding an edge or a vertex. As a main result, we show that for each $r \geq 3$ the $r$-maximum index function over the graphs admitting an $r$-maximum edge-coloring is unbounded and the range is the set of natural numbers. In other words, for each $r \geq 3$ and $k \geq 1$ there is a family of graphs $G(r, k)$ with $\chi'_r(G(r, k)) = k$. Also, we construct a family of graphs not admitting an $r$-maximum edge-coloring with arbitrary maximum degrees: for any fixed $r \geq 3$, there is an infinite family of graphs $F_r = \{G_k : k \geq r + 1\}$, where for each $k \geq r + 1$ there is no $r$-maximum edge-coloring of $G_k$ and $\Delta(G_k) = k$.

1. Introduction

In this paper we consider only simple graphs. The edge-connectivity of a connected graph $G$ is the least integer $k$ such that there exist $k$ edges whose deletion increases the number of components of the remaining graph. We denote the edge-connectivity of $G$ by $\kappa'(G)$. In particular, we call an edge a cut-edge of a nontrivial connected graph $G$ if $G - e$ is not connected. A $k$-factor of a graph $G$ is a $k$-regular spanning subgraph of $G$. Given an edge-coloring $f$ of $G$, not necessarily proper, a maximum color of a vertex $v \in V(G)$ is $\max_{e \in E_v(G)} f(e)$, where $E_v(G)$ is the set of incident edges to $v$. A color $c$ appears $l$ times at $v$ under $f$ if exactly $l$ incident edges to $v$ receive the color $c$.

For hypergraphs a vertex coloring is called a unique-maximum coloring if the maximum color in each edge appears exactly once on the vertices of the edge. There are results on unique-maximum coloring of hypergraphs in [1]. For a graph this coloring is just a proper coloring of the vertices. For a planar graph $G$, we can define an associated hypergraph with $V(G)$ as the vertex set and the
faces as hyperedges. This unique-maximum coloring is called *unique-maximum coloring of a planar graph with respect to faces*. In other words, it is a vertex coloring such that for every face the maximum color occurs exactly once on the vertices of the face. The *unique-maximum index* of a planar graph $G$ is the minimum number of colors for $G$ to have a unique-maximum coloring. The parameter is studied in [2,3].

Without a restriction on planar graphs, a similar concept on edges can be applied: a *unique-maximum edge-coloring* of a graph is an edge-coloring such that at every vertex the maximum color appears exactly once [4]. In [4] S. Jendrol’ and Vrbjarová generalized the concept by fixing the number of maximum colors appearing at each vertex. For any natural number $r$, we say that a graph $G$ has an *$r$-maximum edge-coloring* if there is an edge-coloring $f$ defined on $E(G)$ such that at every vertex $v$ with $d_G(v) \geq r$ the maximum color appears $r$ times at $v$ under $f$. If $\Delta(G)$ is smaller than $r$, then $r$-maximum coloring problems become trivial by using one color for all the edges. In fact, every edge-coloring is an $r$-maximum edge-coloring. On the other hand, if $\Delta(G)$ is larger than $r$, properties on $r$-maximum coloring reflect various aspects of structure of graphs. For example, the existence of an $r$-factor of a graph implies the existence of an $r$-maximum coloring of the graph. Note that the converse is not true. (See also [6], [7].)

The *$r$-maximum index* of a graph $G$, denoted by $\chi'_r(G)$, is the smallest number of required colors to have an $r$-maximum edge coloring of $G$. A natural question on $r$-maximum index is what the possible numbers for $\chi'_r(G)$ will be. When $r \leq 2$, every graph admits an $r$-maximum edge coloring. Moreover, $\chi'_r(G) \leq 3$ for $r = 1, 2$.

**Theorem 1.1** ([4]). For $r = 1, 2$, $\chi'_r(G) \leq 3$ for all connected nontrivial graph $G$.

In addition, they characterized graphs according to the value of $\chi'_r(G) = i$, $i = 1, 2, 3$. On the other hand, not every graph has an $r$-maximum edge-coloring when $r \geq 3$. For $r \geq 3$ they expected $\chi'_r(G) \leq 3$ for every graph $G$ admitting an $r$-maximum edge-coloring.

As answering this question, we show that for each $r \geq 3$ the $r$-maximum index function over the graphs admitting an $r$-maximum edge-coloring is unbounded and the range is the set of natural numbers. In other words, for each $r \geq 3$ and $k \geq 1$ there is a family of graphs $G(r,k)$ with $\chi'_r(G(r,k)) = k$. Also, we construct a family of graphs not admitting an $r$-maximum edge-coloring with arbitrary maximum degrees: for any fixed $r \geq 3$, there is an infinite family of graphs $\mathcal{F}_r = \{G_k : k \geq r + 1\}$, where for each $k \geq r + 1$ there is no $r$-maximum edge-coloring of $G_k$ and $\Delta(G_k) = k$.

We make a remark that there is a similar name for the parameter called a *maximum edge coloring* or known as the Maximum Edge Coloring (MEC) problem. This maximum edge coloring is a generalization of a proper edge coloring. (For this topic, see [5].) The $r$-maximum edge-coloring is not related
to a proper edge-coloring, and it is completely different from the maximum edge coloring. In this paper, we only discuss on \( r \)-maximum edge-coloring.

In the next section, we first explore effects of an \( r \)-maximum edge-coloring and the \( r \)-maximum index as adding an edge in many different ways.

2. Preliminaries

The maximum edge-coloring of a graph is related to \( r \)-regular subgraphs. It is trivial to see the following results.

**Proposition 2.1.** If \( G \) has an \( r \)-factor, then \( \chi'(G) \leq 2 \).

**Proposition 2.2.** If \( G \) has an \( r \)-maximum edge-coloring \( f \), then all the incident edges to a vertex of degree \( r \) must get the same color under \( f \).

**Proposition 2.3.** Suppose that \( \delta(G) \geq r \) and \( G \) has an \( r \)-maximum edge-coloring \( f \). Let \( c \) be the maximum value of \( f \). Then the induced subgraph by the edges receiving \( c \) is \( r \)-regular.

The \( r \)-maximum index of a graph is not inherited from its supergraph. All four combinations are possible. For example, \( \chi'(K_r) = 1 \) but \( \chi'(K_{2r}) = 2 \). But there is a \( 2r \)-vertex graph whose \( r \)-maximum index does not exist [4]. We provide sufficient conditions to make an \( r \)-maximum index as an invariant under some operations on a graph.

**Proposition 2.4.** Given a graph \( H \) admitting an \( r \)-maximum edge-coloring with \( \chi'(H) > 1 \), let \( H' \) be a graph obtained by adding a new leaf to a vertex \( v \) with \( d_H(v) > r \). Then \( \chi'(H') \leq \chi'(H) \).

*Proof.* For any given \( r \)-maximum edge-coloring of \( H \), the number of colors that all incident edges to \( v \) received is at least two because \( d_H(v) \geq r \). If we assign a smaller color than the maximum color at \( v \) to the newly added edge, then it is still an \( r \)-maximum edge-coloring. Therefore, \( \chi'(H') \leq \chi'(H) \). \( \square \)

By arguing similarly we obtain the following fact.

**Proposition 2.5.** Given a graph \( H \) admitting an \( r \)-maximum edge-coloring with \( \chi'(H) > 1 \), let \( H' \) be a graph obtained by adding a new edge connecting two vertices \( u \) and \( v \) with \( d_H(u), d_H(v) > r \). Then \( \chi'(H') \leq \chi'(H) \).

*Proof.* Since both of \( d_H(u) \) and \( d_H(v) \) are strictly greater than \( r \), for each of \( u \) and \( v \) there is an incident edge that has a non-maximum color among the incident edges to the vertex. Let \( c_u \) and \( c_v \) be a non-maximum color at \( u \) and \( v \), respectively. Without loss of generality we assume that \( c_u \leq c_v \). We assign \( c_u \) for the color of the newly added edge. The edge-coloring of \( H' \) is now an \( r \)-maximum edge-coloring. Therefore, \( \chi'(H') \leq \chi'(H) \). \( \square \)

For example, note that the star \( K_{m,1} \) with \( m \geq r + 1 \) has an \( r \)-maximum edge-coloring. For a graph \( G \) admitting an \( r \)-maximum edge-coloring assume that \( G \) has a vertex \( v \) of degree at least \( r + 1 \). Let \( u \) be the vertex of degree
m in $K_{m,1}$. We denote $G_1 + G_2$ the disjoint union of two graphs $G_1$ and $G_2$.

If $G' = (G + K_{m,1}) \cup \{uw\}$, then $\chi'_r(G') \leq \max\{\chi'_r(G), 2\}$ by Proposition 2.5. We generalize this observation as follows.

**Corollary 2.6.** For $G_i$ with $i = 1, 2$ suppose that each graph $G_i$ admits an $r$-maximum edge-coloring. Also suppose that $d_{G_1}(u) > r$ and $d_{G_2}(v) > r$ for some vertices $u \in V(G_1)$ and $v \in V(G_2)$. Let $G' = (G_1 + G_2) \cup \{uw\}$. Then $\chi'_r(G') \leq \max\{\chi'_r(G_1), \chi'_r(G_2)\}$.

**Theorem 2.7.** If a connected $n$-vertex graph $G$ has only one vertex of degree larger than $r$ which is not a cut-vertex and all the rest of the vertices have degree $r$, then $G$ does not have an $r$-maximum edge-coloring.

**Proof.** Suppose to the contrary that $G$ has an $r$-maximum edge-coloring, say $f$. Let $u$ be the vertex of maximum degree. Then $\chi'_r(G) > 1$. Otherwise, the unique color appears more than $r$ times at $u$. Therefore, there are at least two edges $e_1$ and $e_2$ incident to $u$ such that $f(e_1) \neq f(e_2)$. Let $x$ be the end vertex of $e_1$ different from $u$ and $y$ be the end vertex of $e_2$ different from $u$. Since $u$ is not a cut-vertex, $G - u$ is connected. Therefore, there is an $x, y$-path in $G - u$, say this path $xv_1v_2 \cdots v_ty$ for some $t \geq 0$. Then by applying Proposition 2.2 to $f$ on $G$, the edge $xv_1$ must get the same color as $f(e_1)$, and we repeat this argument until we reach the edge $v_ty$. If $t = 0$, $xy$ must get the same color as $f(e_1)$ and $f(e_2)$. Now we have a contradiction: $f(e_1) = f(v_ty) = f(e_2)$.

There is a relation between the edge-connectivity and $r$ on the maximum edge-coloring. If $r \leq \kappa'(G)$, then $G$ admits an $r$-maximum edge-coloring if and only if $G$ has an $r$-factor since $r \leq \kappa'(G) \leq \delta(G)$.

### 3. Main results

In this section we show that for each $r \geq 3$, $\chi'_r(G)$ is arbitrary.

It is easy to see that we can characterize all graphs admitting an $r$-maximum edge-coloring according to the values 1, 2, and at least 3 for $r$-maximum index in the following way.

**Theorem 3.1.** Let $G$ be a connected graph. Then

1. $\chi'_r(G) = 1$ if and only if $\Delta(G) \leq r$.
2. $\chi'_r(G) = 2$ if and only if $\Delta(G) > r$ and $G$ has a spanning subgraph $H$ such that $d_H(v) = r$ if $d_G(v) \geq r$ and $d_H(u) = t$ for some $t < r$ if $d_G(v) < r$.
3. For $G$ admitting an $r$-maximum edge-coloring, $\chi'_r(G) \geq 3$ if and only if $G$ does not satisfies any of the conditions above.

Note that in particular, if $\delta(G) \geq r$, a spanning subgraph $H$ satisfying the condition in (2) in Theorem 3.1 is an $r$-factor of $G$.

It is trivial to see that $\chi'_r(G) = 1$ if $r \geq \Delta(G)$. However, $\chi'_r(G)$ may not exist if $r < \Delta(G)$. In fact, it is known that for every $r \geq 3$ there is a graph
For any fixed $r \geq 3$, there is an infinite family of graphs $F_r = \{G_k : k \geq r + 1\}$, where for each $k \geq r + 1$ there is no $r$-maximum edge-coloring of $G_k$ and $\Delta(G_k) = k$.

**Proof.** Let $H$ be an $r$-regular graph such that $\alpha'(H) \geq 2$ and $\kappa'(H) \geq 3$, where $\alpha'(H)$ is the size of a maximum matching in $H$. For any $r \geq 3$, such $H$ exists, for example $H = K_{r+1}$. We pick two independent edges $e = xy$ and $e' = uv$. Let $H_i$ be an $i$-th copy of $H$. In this case we rename the vertices and the edges by putting subscript $i$. For example, the corresponding vertex to $x$ in $H_i$ is denoted by $x_i$.

We construct $G_k$ as follows:

1. We take $k$ copies of $H$, and take disjoint union of them: we start with $H_1 + H_2 + \ldots + H_k$.
2. We delete the edges: $\{e_1, e_2, \ldots, e_k, e'_1, \ldots, e'_k\}$ from (1).
3. Add a new vertex $z$ to (2).
4. Add new edges: $\{y_1x_2, y_2x_3, \ldots, y_{k-1}x_k, y_kx_1\}$ to (2).
5. Add new edges: $\{u_iz, v_iz : i = 1, 2, \ldots, k\}$ to (2).

**(Case 1)** $r$ is odd.

Let $G = G_{\frac{r+1}{2}}$. Then $\Delta(G) = r + 1 = d_G(z)$. All the rest of the vertices have degree $r$.

**Claim 1.** $G$ does not admit an $r$-maximum edge-coloring.

**Proof.** Suppose that there is an $r$-maximum edge-coloring of $G$, say $f$. Let $c = f(u_1z)$. Note that all the other incident edges to $u_1$ appear in $H_1$ and must receive the color $c$ since the degree of $u_1$ in $G$ is $r$. Moreover, by Proposition 2.2, all the incident edges to each vertex except for $z$ must receive only one (maximum) color. For every $i = 1, 2, \ldots, \frac{r+1}{2}$, the graph $H_i - \{e_i, e'_i\}$ is connected since $\kappa'(H_i) \geq 3$. Therefore, for any vertex $w \in V(H_1) - \{u_1\}$, there is a $w, u_1$-path in $H_1 - \{e_1, e'_1\}$. Applying the same argument as in the proof of Theorem 2.7, we conclude that in $G$ the edges from $H_1 - \{e_1, e'_1\}$ must receive the color $c$ under $f$. In fact, the incident edges of $a$ and the incident edges of $b$ must get the same color under $f$ whenever there is an $a, b$-path in $G$ for vertices $a$ and $b$ with $d_G(a) = d_G(b) = r$. Hence, we get $f(u_i z) = f(v_j z)$ and $f(u_i z) = f(u_j z)$ for all $i, j$. Then $c$ appears $r + 1$ times at $z$ under $f$, which is contradiction. \hfill $\Box$

If we add arbitrarily many leaves making $z$ as their neighbor, then we can make the degree of $z$ become an arbitrary size at least $r + 2$. In this case, by repeating the proof of Claim 1, the color of the previously existing edges must get the same color anyway. This forces that all $r + 1$ incident edges to
z must get the maximum color. Therefore this new graph does not have an r-maximum edge-coloring.

(Case 2) \( r \) is even.

We let \( G = G_{2+1}^r \). Then \( \Delta(G) = r + 2 \). For larger maximum degree we add leaves to \( z \) as in \( r \) odd case. In this case we can make \( \Delta(G) \geq r + 2 \). Then the proof is exactly the same as above.

For \( \Delta(G) = r + 1 \) when \( r \) is even, we make the following construction: from the above graph \( G \) with \( \Delta(G) = r + 2 \), let \( G' \) be the graph adding a new vertex \( w \) and a new edge \( v_1w \) after deleting the edge \( v_1z \). Then \( \Delta(G') = r + 1 \). Applying the same argument from above a color assigned on \( u_1z \) forces the rest of the edges of \( G' \) to have the same color. Therefore, it is not possible to admit an \( r \)-maximum edge-coloring.

\[ \square \]

In a graph admitting an \( r \)-maximum edge-coloring, a structure that forces a certain color on certain edges may require many colors to be used. In [4] the authors proposed an open problem asking whether \( \chi'_r(G) \leq 3 \) for every \( r \geq 3 \) and for every graph \( G \) admitting an \( r \)-maximum edge-coloring. But unlike the case \( r = 2 \), from \( r \geq 3 \) we show that some graphs require many colors to allow an \( r \)-maximum coloring. Our construction requires \( r \geq 3 \) and with \( r = 2 \) the construction does not contain a structure forcing many colors anymore. We will construct a family of graphs using \( H^m_n \), which is described below. (Figure 1 presents \( H^3_3 \).)

**Construction of \( H^m_n \):**

- Let \( V = \{v_1, v_2, \ldots, v_n\} \).
- Let \( W = \{w_1^1, \ldots, w_1^m, w_2^1, \ldots, w_2^m, \ldots, w_n^1, \ldots, w_n^m\} \).
- Let \( X = \{x_1^1, \ldots, x_1^m, x_2^1, \ldots, x_2^m, \ldots, x_n^1, \ldots, x_n^m\} \).
- Let \( U = \{u_1^1, \ldots, u_1^m, u_2^1, \ldots, u_2^m, \ldots, u_n^1, \ldots, u_n^m\} \).
- \( V(H^m_n) = V \cup W \cup X \cup U \).

There are three types of edges.

- **Edges between \( V \) and \( W \):** Each \( v_i \) has exactly \( m \) neighbors in \( W: w_i^1, \ldots, w_i^m \).
- **Edges between \( W \) and \( X \):** Each \( x_i^j \) has exactly \( m - 1 \) neighbors in \( W \): starting from \( w_i^j \), take the first consecutive \( m - 1 \) vertices in order in \( W \) as neighbors of \( x_i^j \) reading subscripts and superscripts modulo \( m \).
- **Edges between \( X \) and \( U \):** Each \( x_i^j \) in \( X \) has only one neighbor \( u_i^j \) in \( U \).

We see that every vertex in \( U \) is a leaf. All other vertices have degree \( m \).

**Lemma 3.3.** For any \( m \geq 3 \) and \( n \geq 1 \), \( \chi'_m(H^m_n) = 1 \). In any \( m \)-maximum edge-coloring of \( H^m_n \), any color \( c \) assigned on the edge incident to a leaf forces \( c \) as the color for the rest of the edges.
Figure 1: $H_3^3$

Two vertices $w_1^1$ and $x_1^1$ are adjacent in $H_3^3$.
There are 9 leaves and 21 vertices of degree 3.

Proof. Assigning the same color to all edges satisfies the condition for an $m$-maximum edge-coloring. Now we will show that edge-coloring with one color is the only possible way.

Consider an $m$-maximum edge-coloring of $H_m^n$. Let $c$ be the color that was assigned on an edge $e = x_1^1u_1^1$. Since the degree of $x_1^1$ is $m$, all incident edges to this vertex must receive the same color. In other words, if a color appears at a vertex of degree $m$, then that color is the only color appearing at the vertex.

Note that for any two vertices $y,y'$ with degree $m$, if there is a $y,y'$-path via the vertices of degree $m$, then $y$ and $y'$ must get the same color for their incident edges. Note that $H_m^n$ is connected. In fact, we can see that for any $x_i^t$ and $w_j^{t'}$, there is an $x_i^t,w_j^{t'}$-path using vertices only in $W \cup X$. To see this it is sufficient to show that there is an $x_i^1,w_j^{t'}$-path using vertices only in $W \cup X$. It is because that $x_i^1$ and $w_j^{t'}$ are always adjacent for every $i$. Once the vertices in $W \cup X$ allow only one color $c$, then the vertices in $V \cup U$ are forced to allow color $c$ only. \hfill \Box

Given an integer $L \geq 2$, let $H_m^n(k)$ be the $k$-th copy of $H_m^n$ for $k = 1, 2, \ldots, L$.
Let $G(m, L)$ be the graph constructed in the following way:

- Concatenate $H_m^n(k)$ and $H_m^n(k+1)$ for $k = 1, 2, \ldots, L-1$.
- Identify $u_1^i$ in $V(H_m^n(k))$ and $v_{m(i-1)+t}$ in $V(H_m^n(k+1))$ for $1 \leq i \leq m^k$ and $1 \leq t \leq m$.

Note that there are only three values for the degrees: 1, $m$, and $m+1$. The leaves of $G(m, L)$ are the leaves of $H_m^n(L)$.

Theorem 3.4. $\chi'_r(G(r, L)) = L$ for $r \geq 3$. In other words, for any fixed integer $r \geq 3$, the $r$-maximum index function (over the graphs admitting an $r$-maximum edge-coloring) is unbounded and the range is the set of natural numbers.
Proof. Let \( r \geq 3 \) and consider \( G(r, L) \). First of all, if we assign color \( k \) for the edges of \( E(H^{r}_{L}(k)) \) in \( G(r, L) \), then this edge-coloring becomes an \( r \)-maximum edge-coloring. Therefore, \( \chi'_{r}(G(r, L)) \leq L \).

Let \( f \) be an optimal \( r \)-maximum edge-coloring of \( G(r, L) \). Let \( e = f(e) \), where \( e \) is the edge incident to a leaf in \( H^{r}_{L}(L) \). By the proof of Lemma 3.3 all the edges of \( H^{r}_{L}(L) \) must receive color \( c \). Now in \( G(r, L) \) the color \( c \) already appeared \( r \) times at every vertex of degree \( m+1 \). Therefore, the uncolored incident edge at every such vertex must receive a smaller color, say \( c_1 \). Applying the same argument, all the edges of \( H^{r}_{L-1}(L-1) \) will get color \( c_1 \). We inductively apply this argument to conclude that all the edges of \( H^{r}_{L-i}(L-i) \) will get color \( c_i \), where \( c > c_1 > \cdots > c_{i-1} > c_i > \cdots > c_{L-1} \) and \( i = 0, 1, \ldots, L-1 \). Therefore, \( f \) requires at least \( L \) distinct values, and \( \chi'_{r}(G(r, L)) \geq L \). □

In [4] the family of trees and the family of complete graphs were characterized in terms of an \( r \)-maximum index. They are bounded above by \( 3 \) for every \( r \). Note that \( G(r, L) \) is bipartite for \( r \geq 3 \). Therefore, the \( r \)-maximum index function is not bounded over the family of bipartite graphs. It would be an interesting question whether there is any well-known family of graphs that has a bounded range for the \( r \)-maximum index function when \( r \geq 3 \).

Remark 3.5. Note that the above construction with \( G(2, L) \) does not give the same conclusion as in Theorem 3.4. The graph \( H^{2}_{L}(k) \) is disconnected for each \( k \), and \( G(2, L) \) is a tree. Therefore, in \( G(2, L) \), at any level \( k \) a color assigned on an edge incident to a leaf does not necessarily force the same color for the rest of the edges in the copy \( H^{2}_{L}(k) \).

References


Jeong-Ok Choi  
Division of Liberal Arts and Sciences  
Gwangju Institute of Science and Technology  
Gwangju 61005, Korea  
Email address: jchoi351@gist.ac.kr