ON CHARACTERIZING THE GAMMA AND THE Beta $q$-DISTRIBUTIONS

IMEN BOUTOURIA, IMED BOUZIDA, AND AFI MASMoudi

Abstract. In this paper, our central focus is upon gamma and beta $q$-distributions from a probabilistic viewpoint. The gamma and the beta $q$-distributions are characterized by investing the nature of the joint $q$-probability density function through the $q$-independence property and the $q$-Laplace transform.

1. Introduction


The quantum calculus has a lot of applications in different mathematical areas such as number theory, difference equation (see [7]), orthogonal polynomials, probability theory, . . . .

In mathematical physics and probability, the $q$-distribution is more general than classical distribution. It was introduced by Diaz [12, 13] in the continuous case and by Charalambos [9] in the discrete case. The construction of a $q$-distribution is the construction of a $q$-analogue of ordinary distribution. Mathai in [23] introduced the $q$-analogue of the gamma distribution with respect to Lebesgue measure. In this paper, gamma $q$-distribution is studied with respect to Jackson $q$-integral. If $q$ goes to 1, we obtain the ordinary calculus. This condition is the necessary condition in the theory of $q$-calculus.

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Charalambos was the pioneer to coin the notion of q-distribution in the discrete case [9]. As for the continuous case, Díaz et al. identified the Gaussian q-distribution [13].

A function \( p_q(x) \) is a q-probability density provided that it satisfies \( p_q(x) \geq 0, \forall x \in \mathbb{R} \), and \( \int_{\mathbb{R}} p_q(x) dq x = 1 \). The q-cumulative distribution function of a real-valued random variable \( X \), is the q-probability that \( X \) takes a value less than or equal to \( x \). It gives the area under the probability q-density function from \(-\infty\) to \( x \). It is defined by

\[
F_q(x) = \mathbb{P}_q(X \leq x) = \int_{-\infty}^{x} p_q(s)d_q s, \ x \in \mathbb{R}.
\]

Díaz et al. in [12] defined the gamma q-distribution in terms of

\[
\gamma_{q,a}(x) = \frac{\Gamma_q(a)x^{a-1}E_q^{-q(x)}}{\Gamma_q(a)}.
\]

In 1955 Lukacs [22] proved that \( X/Y \) and \( X + Y \) are independent if and only if \( X \) and \( Y \) are gamma distributed with the same scale parameter. Using the moment, in 1978 Findeisen [15] characterized the gamma distribution. Also, in 1999 Hwang and Hu [18] proved a characterization of the gamma distribution by the independence of the sample mean and the sample coefficient of variation. In 1967 I. Kotlarski [21] characterized the gamma distribution by the nature of joint distribution of the two quotients \( X_1/X_3, X_2/X_3 \) for three identically gamma distributed random variables.

Our work stands for an extension to the results given by I. Kotlarski [21].

Let \( X_1, X_2, X_3 \) be three positives independent real random variables and let \( Y_1 = \frac{X_1}{X_3} \) and \( Y_2 = \frac{X_2}{X_3} \).

The necessary and sufficient condition for \( X_k \) to be q-gamma distributed with parameters \( p_k \) \( (k = 1, 2, 3) \) is that the q-Laplace transform of the couple \((\log_q Y_1, \log_q Y_2)\) is given by

\[
\frac{\Gamma_q(p_1 - \theta_1)}{\Gamma_q(p_1)} \times \frac{\Gamma_q(p_2 - \theta_2)}{\Gamma_q(p_2)} \times \frac{\Gamma_q(p_3 + \theta_1 \oplus_q \theta_2)}{\Gamma_q(p_3)}.
\]

The beta q-distribution is also characterized in the same way.

This paper is structured as follows: In Section 2, some preliminary concepts related to q-derivative, q-integral, q-operator addition and some essential results are presented to build our work. In Section 3, we defined the joint q-density function by using the q-Fubini’s theorem. Besides, we introduced the notion of independence and marginal q-distribution. In Section 4, the gamma q-distribution is characterized by the nature of the joint q-distribution of the two quotients \( \frac{X_1}{X_3}, \frac{X_2}{X_3} \) for three identically q-gamma distributed random variables. Also, the beta q-distribution was characterized by the same way.
2. Preliminaries

In this section, some useful basic definitions [10, 11, 19] are introduced. We shall start with the \( q \)-derivative and the Jackson \( q \)-integral. Fixing a real number \( 0 < q < 1 \), the \( q \)-derivative of a function \( f : \mathbb{R} \to \mathbb{R} \) at \( x \in \mathbb{R} \setminus \{0\} \) is given by:

\[
D_q f(x) = \frac{f(qx) - f(x)}{(q - 1)x}.
\]

It is also known as the Jackson derivative.

It is manifestly linear,

\[
D_q (f(x) + g(x)) = D_q f(x) + D_q g(x).
\]

It has a product rule analogous to the ordinary ones, with two equivalent forms

\[
D_q (f(x)g(x)) = g(x)D_q f(x) + f(qx)D_q g(x) = g(qx)D_q f(x) + f(x)D_q g(x).
\]

Similarly, it satisfies a quotient rule,

\[
D_q \left( \frac{f(x)}{g(x)} \right) = \frac{g(x)D_q f(x) - f(x)D_q g(x)}{g(x)g(qx)} , \quad g \neq 0.
\]

For an integer \( n \geq 1 \), we have

\[
D_q x^n = [n]_q x^{n-1},
\]

where

\[
[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1}.
\]

We also denote, for all \( n \in \mathbb{N} \),

\[
[n]_q! = \begin{cases} 
1 & \text{if } n = 0, \\
[n]_q[n-1]_q \cdots [1]_q & \text{otherwise.}
\end{cases}
\]

For \( x \in \mathbb{R} \),

\[
[x]_q = \frac{1 - q^x}{1 - q}.
\]

If \( x \) goes to \( \infty \), we obtain \([\infty]_q = \frac{1}{1-q}\) which is called a \( q \)-analogue of \( \infty \).

Note that \([\infty]_q \) approaches 1 when \( q \) goes to 0 and goes to \( +\infty \) when \( q \) approaches 1.

We recall some usual notations used in the \( q \)-theory.

\[
(a + b)_q^n = \prod_{i=0}^{n-1} (a + q^i b), \quad \forall \ n \in \mathbb{N},
\]

\[
(1 + a)_q^\infty = \prod_{i=0}^{\infty} (1 + q^i a),
\]

\[
(1 + a)_q^t = \frac{(1 + a)_q^\infty}{(1 + q^t a)_q^\infty}, \quad \forall \ t \in \mathbb{R}.
\]

A right inverse of the \( q \)-derivative is obtained via the Jackson integral.
For $a, b \in \mathbb{R}$ the Jackson integral or $q$-integral of $f : \mathbb{R} \to \mathbb{R}$ on $[a, b]$ is defined by

$$\int_a^b f(x)d_qx = (1 - q) \sum_{n=0}^{\infty} q^n (bf(q^n b) - af(q^n a)).$$

It is clear if one lets $q$ approaches 1, then the $q$-derivative approaches the Newton derivative and the Jackson integral approaches the Riemann integral.

The $q$-analogue of the integration theorem by a variable change is given by

$$\int_{u(a)}^{u(b)} f(u)d_qu = \int_a^b f(u(x))d_{q^{1/\beta}}u(x), \quad \text{where} \quad u(x) = \alpha x^\beta. \quad (1)$$

The $q$-analogue of the rule of integration by parts is

$$\int_a^b g(x)D_qf(x)d_qx = [f(x)g(x)]_a^b - \int_a^b f(qx)D_qg(x)d_qx. \quad (2)$$

For any function $f(x)$ continuous at $x = 0$, we have

$$\int_0^a D_qf(x)d_qx = f(a) - f(0) \quad \text{and} \quad D_q \int_0^x f(t)d_qt = f(x). \quad (3)$$

Jackson in [19] proposed the $q$-analogue of the exponential function $e^x$ given by

$$e^x_q = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!},$$

It is clear that $e^0_q = 1$ and $D_q e^x_q = e^x_q$.

The $q$-analogue of the identity $e^x e^{-x} = 1$ is $e^x_q E^{-x}_q = 1$, where the function $E^x_q$ defined by $e^{x/\beta}_q$ is given also by

$$e^{x/\beta}_q = E^x_q = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{x^n}{[n]_q!}.$$

The $q$-logarithm function $\log_q(x)$ is the inverse of the $q$-exponential function $e^x_q$, and the function $\log_{q^{-1}}(x)$ is the inverse function of $E^x_q$.

In 1994 Chung et al. [10] proposed the $q$-addition operator and discussed its properties. The $q$-addition operator is defined by

$$\left\{ \begin{array}{ll}
(a \oplus_q b)^n = \sum_{k=0}^{n} qC^n_k a^k b^{n-k}, & \forall \ n \in \mathbb{N}, \quad (a \neq b,)
\\
(a \oplus_q a)^n = (a + a)^n = 2^n a^n,
\end{array} \right.$$

where

$$qC^n_k = \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

From the above definition, we have the following property:

$$k(a \oplus_q b) = ka \oplus_q kb, \quad \forall \ k \in \mathbb{R}.$$
It is easy to see that this operator is commutative, i.e., \( a \oplus_q b = b \oplus_q a \). In addition, if we take \( b = a \), then we get \( a \oplus_q a = a + a = 2a \). Finally if we take \( b = 0 \), we obtain \( a \oplus_q 0 = 0 \oplus_q a = a \).

The \( q \)-subtraction is defined as \( a \ominus_q b = a \oplus_q (-b) \), \( a \neq b \) and if we take \( b = a \), then we have \( a \ominus_q a = a - a = 0 \).

This operator permits to express the properties of \( q \)-logarithm and \( q \)-exponential functions in a more compact form:

(i) \( e_q^a e_q^b = e_q^{a+b} \),

(ii) \( e_q^a = (e_q^a)^n \), \( \forall n \in \mathbb{N} \),

(iii) \( \log_q(ab) = \log_q(a) \oplus_q \log_q(b) \),

(iv) \( \log_q(a^n) = n \log_q(a) \), \( \forall n \in \mathbb{N} \).

The only function \( f \) verified that \( f(a \oplus_q y) = f(x) f(y) \) is the \( q \)-exponential function \([7]\). The function \( \log_q(x) \) is the inverse function to \( E_q^x \), and it satisfies the same mentioned above properties as \( \log_q(x) \).

2.1. The gamma and beta \( q \)-distributions

Jackson in [19] showed that the \( q \)-beta function has the \( q \)-integral representation, which is a \( q \)-analogue of Euler’s formula:

\[
\beta_q(t, s) = \int_0^1 x^{t-1}(1 - qx)^{s-1} d_q x, \quad \forall \ t, s > 0. 
\]

The \( q \)-gamma function expressed as \( \Gamma_q \) is defined in [19] by

\[
\Gamma_q(t) = \int_0^1 x^{t-1} E_q^{-qx} d_q x, \quad \forall \ t > 0. 
\]

Jackson [19] proved the following properties of the \( q \)-gamma function:

(6) \( \Gamma_q(1) = 1, \)

(7) \( \Gamma_q(t + 1) = [t]_q \Gamma_q(t), \quad \forall \ t > 0, \)

(8) \( \Gamma_q(n + 1) = [n]_q!, \quad \forall n \in \mathbb{N}. \)

The relationship between the \( q \)-gamma and the \( q \)-beta functions is given by

\[
\beta_q(t, s) = \frac{\Gamma_q(t) \Gamma_q(s)}{\Gamma_q(t + s)}, \quad \forall \ t, s > 0. 
\]

Díaz et al. in [12] defined the gamma \( q \)-distribution in \([0, \frac{1}{1-q}]\) by

\[
\gamma_{q,a}(x) = \frac{x^{a-1} E_q^{-qx}}{\Gamma_q(a)} \mathbf{1}_{[0, \frac{1}{1-q}]}(x), 
\]

and the beta \( q \)-distribution in \([0, 1]\) by

\[
\beta_q(a, b)(x) = \frac{x^{a-1}(1 - qx)^{b-1}}{\beta_q(a, b)} \mathbf{1}_{[0, 1]}(x). 
\]
Díaz et al. in [13] defined the $q$-moment of a random variable $X$ with $q$-density function $f$ on $[a, b]$, by

$$qM(n) = \int_a^b x^n f(x)\,dq\,x.$$ 

In particular, the $q$-expectation of $X$ is given by

$$\mathbb{E}_q(X) = \int_a^b x f(x)\,dq\,x = qM_1.$$ 

3. Joint $q$-density function and independence marginal $q$-distribution

3.1. Joint $q$-density functions

In the univariate continuous case, Díaz et al. [13] identified the Gaussian $q$-distribution. Now, we introduce the bivariate $q$-density function of two random variables by using Fubini’s Theorem. Al-Ashwal in [6] proved the Leibniz’s Rule and Fubini’s Theorem in quantum calculus.

**Theorem 3.1.** Let $f$ be a function defined on the closed rectangle $R = [0, a] \times [0, b]$. Assuming that $f(t, s)$ is continuous at $t = 0$ uniformly and continuous at $s = 0$ uniformly. Then, the double $q$-integrals

$$\int_0^b \int_0^a f(t, s)\,dq\,s\,tdq\,s$$

and

$$\int_0^a \int_0^b f(t, s)\,dq\,s\,tdq\,s$$

exist and they are equal, that is

$$\int_0^b \int_0^a f(t, s)\,dq\,s\,tdq\,s = \int_0^a \int_0^b f(t, s)\,dq\,s\,tdq\,s$$

with

$$\int_0^a \int_0^b f(t, s)\,dq\,s\,tdq\,s = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} q^{j+k}ab(1-q)^2 f(q^ja, q^kb).$$

According to Theorem 3.1, we introduce the notion of joint $q$-density function.

**Definition 1.** Let $(X, Y)$ be a couple of positive random variables with continuous joint $q$-density function $f_{X,Y}(x, y)$. Then the $q$-cumulative function is given by

$$P_q(X \leq s, Y \leq t) = \int_{s\leq x, y\leq t} f_{X,Y}(x, y)\,dq\,x\,dq\,y.$$ 

Using Fubini’s Theorem, we can also write the $q$-integral as

$$P_q(X \leq s, Y \leq t) = \int_s^t \left( \int_0^s f_{X,Y}(x, y)\,dq\,y \right)\,dq\,x$$

$$= \int_s^t \left( \int_0^t f_{X,Y}(x, y)\,dq\,x \right)\,dq\,y.$$
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Note that as in the classical case, the $q$-joint density function $f_{X,Y}(x,y)$ verifies

\[
\begin{aligned}
&f_{X,Y}(x,y) \geq 0, \\
&\int \int f_{X,Y}(x,y) dq_x dq_y = 1.
\end{aligned}
\]

3.2. Independence and marginal $q$-distributions

Now, we are able to provide some properties of $q$-independence and marginal $q$-distributions.

**Definition 2.** Let $(X,Y)$ be a couple of random variables with joint $q$-density function $f_{X,Y}(x,y)$. Then the marginal $q$-densities of $X$ and $Y$ are given respectively by

\[
\begin{aligned}
f_X(x) &= \int_{\mathbb{R}} f_{X,Y}(x,y) dq_y, \\
f_Y(y) &= \int_{\mathbb{R}} f_{X,Y}(x,y) dq_x.
\end{aligned}
\]

**Definition 3.** Let $(X,Y)$ be a couple of random variables with joint $q$-density function $f_{X,Y}(x,y)$. The random variables $X$ and $Y$ are called independent if

\[
f_{X,Y}(x,y) = f_X(x)f_Y(y), \quad \forall x, y.
\]

**Definition 4.** The $q$-expectation of $XY$ (if it exists) is given by

\[
\mathbb{E}_q(XY) = \int \int xy f_{X,Y}(x,y) dq_x dq_y.
\]

Note that if $X$ and $Y$ are independent, then

\[
\mathbb{E}_q(XY) = \mathbb{E}_q(X)\mathbb{E}_q(Y).
\]

In fact, since $X$ and $Y$ are independent, then $f_{X,Y}(x,y) = f_X(x)f_Y(y)$. Hence,

\[
\begin{aligned}
\mathbb{E}_q(XY) &= \int \int xy f_X(x)f_Y(y) dq_x dq_y \\
&= \int x f_X(x) dq_x \int y f_Y(y) dq_y \\
&= \mathbb{E}_q(X)\mathbb{E}_q(Y).
\end{aligned}
\]

Studies on $q$-Laplace transform go back to W. H. Abdi in [1]. Research in the area was pursued in many works by W. H. Abdi in [2–4] and more recently in [7]. The $q$-version of the Laplace transform consists in choosing a $q$-version of the exponential function $e^{-\theta x}$ and then replacing the integral by the corresponding $q$-integral.

**Definition 5.** Let $f$ be a $q$-density function with bounded support $[0,A]$ and let $X$ be a random variable with $q$-density function $f$, the $q$-Laplace transform of $X$ is defined by
\[ qL_X(\theta) = \int_0^A E_q^{-\theta x} f(x) dq x = E_q (E_q^{-\theta X}). \]

Now, we extend the notion of \( q \)-Laplace transform in bivariate case.

**Definition 6.** Let \((X, Y)\) be a couple of a positive random variables with joint \( q \)-density function \( f(x, y) \) defined on \([0, A] \times [0, B]\).

The \( q \)-Laplace transform of \( f \) is given by
\[
qL_{(X,Y)}(\theta_1, \theta_2) = \int_0^A \int_0^B E_q^{-\theta_1 (x \oplus_q \theta_2 y)} f(x, y) dq x dq y = E_q (E_q^{-(\theta_1 X \oplus_q \theta_2 Y)}).
\]

### 4. Characterizing the gamma and the beta \( q \)-distributions

Díaz et al. [12] identified the gamma and the beta \( q \)-distributions. The gamma \( q \)-density function [4] is defined by
\[
\gamma_{q,a}(x) = \frac{1}{\Gamma_q(a)} x^{a-1} E_q^{-qx} 1_{[0,1]}(x),
\]
with \( q \)-moment
\[
M_q(n) = \frac{\Gamma_q(a+n)}{\Gamma_q(a)}.
\]

Now, we will characterize the gamma and the beta \( q \)-distributions. This characterization is based on the following two technical lemmas:

**Lemma 4.1.** Let \( X_1, X_2, X_3 \) be three independent real random variables, with common bounded support \([0, A]\) and let
\[
Z_1 = X_1 \ominus_q X_3 \text{ and } Z_2 = X_2 \ominus_q X_3.
\]

Then, the \( q \)-distribution of \((Z_1, Z_2)\) determines the \( q \)-distributions of \( X_1, X_2, X_3 \).

**Proof.** Let \( qL_{(Z_1, Z_2)}(\theta_1, \theta_2) \) be the \( q \)-Laplace transform function of the pair \((Z_1, Z_2)\), and let \( qL_{X_k}(\theta) \) be the \( q \)-Laplace transform function of \( X_k \) \((k = 1, 2, 3)\).

Assuming that \( X_1, X_2, X_3 \) are independent, we obtain
\[
qL_{(Z_1, Z_2)}(\theta_1, \theta_2) = E_q (E_q^{-\theta_1 Z_1 \oplus_q \theta_2 Z_2}) = E_q (E_q^{-\theta_1 X_1 \ominus_q \theta_2 \ominus_q X_3}) = qL_{X_1}(\theta_1) qL_{X_2}(\theta_2) qL_{X_3}(\ominus_q \theta_2).
\]

Let \( U_1, U_2 \) and \( U_3 \) be three independent random variables. If we take \( V_1 = U_1 \ominus_q U_3 \) and \( V_2 = U_2 \ominus_q U_3 \), then
\[
qL_{(V_1, V_2)}(\theta_1, \theta_2) = qL_{U_1}(\theta_1) qL_{U_2}(\theta_2) qL_{U_3}(\ominus_q \theta_2).
\]
If \((Z_1, Z_2)\) and \((V_1, V_2)\) have the same distribution, then their \(q\)-Laplace transform coincides.

\[
qL_{X_1}(\theta_1)L_{X_2}(\theta_2)L_{X_3}(\theta_1 \odot_q \theta_2) = qL_{U_1}(\theta_1)L_{U_2}(\theta_2)L_{U_3}(\theta_1 \odot_q \theta_2).
\]

Let \(p_1, p_2, p_3\) be three continuous functions such that

\[
qL_{U_1}(\theta_1) = qL_{X_1}(\theta_1)p_1(\theta_1),
\]

\[
qL_{U_2}(\theta_2) = qL_{X_2}(\theta_2)p_2(\theta_2),
\]

and

\[
qL_{U_3}(\theta_1 \odot_q \theta_2) = qL_{X_3}(\theta_1 \odot_q \theta_2)p_3(\theta_1),
\]

Then, equation (12) becomes

\[
qL_{X_1}(\theta_1)qL_{X_2}(\theta_2)qL_{X_3}(\theta_1 \odot_q \theta_2) = qL_{X_1}(\theta_1)p_1(\theta_1)qL_{X_2}(\theta_2)p_2(\theta_2)qL_{X_3}(\theta_1 \odot_q \theta_2)p_3(\theta_1).
\]

Which gives

\[
p_1(\theta_1)p_2(\theta_2)p_3(\theta_1) = 1,
\]

with \(p_k(\theta)\) is any function. Note that

\[
qL_{X_k}(0) = qL_{X_k}(0)p_k(0), \quad k \in \{1, 2, 3\}.
\]

Hence, \(p_k(0) = 1\) for \(k \in \{1, 2, 3\}\). In order to solve (17) we take \(\theta_1 = \theta\) and \(\theta_2 = 0\). So, we obtain

\[
p_1(\theta)p_3(-\theta) = 1.
\]

Setting \(\theta_2 = \theta\) and \(\theta_1 = 0\), we obtain

\[
p_2(\theta)p_3(0 \odot_q \theta) = p_2(\theta)p_3(\theta) = 1.
\]

Therefore

\[
\begin{align*}
p_1(\theta) &= \frac{1}{p_3(-\theta)}, \\
p_2(\theta) &= \frac{1}{p_3(\theta)},
\end{align*}
\]

By inserting them in (17), we have

\[
p_3(\theta_1 \odot_q \theta_2) = p_3(\theta_1)p_3(\theta_2).
\]

The only function that checks (19) and (22) is the \(q\)-exponential function. Hence, \(p_k(\theta) = E^C_{\theta}q, \quad k \in \{1, 2, 3\}\), with \(C\) is a constant.

Based on the equation (13), we obtain

\[
qL_{U_1}(\theta) = E^C_{\theta}qL_{X_k}(\theta) = qL_{X_k}(\theta), \quad k \in \{1, 2, 3\}.
\]

Since \(X_k\) and \(U_k\) are concentrated on the bounded interval \([0, A]\), then it’s necessary that \(C = 0\).
Therefore $X_k$ and $U_k$ have the same $q$-density function. As matter of fact, we deduce that
\begin{equation}
qL_{U_k}(\theta) = qL_{X_k}(\theta), \ k \in \{1, 2, 3\}.
\end{equation}

**Lemma 4.2.** Let $X_1$, $X_2$, $X_3$ be three positive independent real random variables with common bounded support $[0, A]$. Let $(Y_1,Y_2)$ be a couple of random variables such that $Y_1 = X_1^\gamma_1$ and $Y_2 = X_2^\gamma_2$. Then, the $q$-Laplace transform of the couple $(\Log_q(Y_1), \Log_q(Y_2))$ determines the distribution of $X_1$, $X_2$, $X_3$.

**Proof.** The proof of Lemma 4.2 is based on the proof of Lemma 4.1 because $\Log_q(Y_1), \Log_q(Y_2)$ satisfies the condition of Lemma 4.1.

**Theorem 4.3.** Let $X_1$, $X_2$, $X_3$ be three positive independent real random variables with common bounded support $[0, A]$ and let $Y_1 = X_1^\gamma_1$ and $Y_2 = X_2^\gamma_2$.

The necessary and sufficient condition for $X_k$ to be $q$-gamma distributed, $\gamma_q(p_k)$, with parameters $p_k$ $(k = 1, 2, 3)$ is that the $q$-Laplace transform of the couple $(\Log_q(Y_1), \Log_q(Y_2))$ has the following expression
\[
\frac{\Gamma_q(p_1 - \theta_1)}{\Gamma_q(p_1)} \times \frac{\Gamma_q(p_2 - \theta_2)}{\Gamma_q(p_2)} \times \frac{\Gamma_q(p_3 + \theta_1 \oplus q \theta_2)}{\Gamma_q(p_3)}.
\]

**Proof.** \( \Rightarrow \) Let $X_k \sim \gamma_q(p_k)$ $(k = 1, 2, 3)$. The $q$-Laplace transform of $\Log_q X_k$ is:
\[
qL_{\Log_q X_1}(\theta_1) = \int_0^\infty x_1^{-\theta_1} x_1^{p_1-1} E^{-q x_1}_{q(p_1)} dx_1
\]
\[
= \frac{1}{\Gamma_q(p_1)} \int_0^\infty x_1^{-\theta_1 + p_1 - 1} E^{-q x_1}_{q(p_1)} dx_1
\]
\[
= \frac{\Gamma_q(p_1 - \theta_1)}{\Gamma_q(p_1)}.
\]

In the same way, we show that:
\[
qL_{\Log_q X_2}(\theta_2) = \frac{\Gamma_q(p_2 - \theta_2)}{\Gamma_q(p_2)},
\]
and
\[
qL_{\Log_q X_3}(\theta_3) = \frac{\Gamma_q(p_3 + \theta_1 \oplus q \theta_2)}{\Gamma_q(p_3)}.
\]
The $q$-Laplace transform of the couple $(\Log_q Y_1, \Log_q Y_2)$ is
\[
qL_{(\Log_q Y_1, \Log_q Y_2)}(\theta_1, \theta_2) = qL_{\Log_q X_1}(\theta_1)qL_{\Log_q X_2}(\theta_2)qL_{\Log_q X_3}(\theta_3)
\]
\[
= \frac{\Gamma_q(p_1 - \theta_1)}{\Gamma_q(p_1)} \times \frac{\Gamma_q(p_2 - \theta_2)}{\Gamma_q(p_2)} \times \frac{\Gamma_q(p_3 + \theta_1 \oplus q \theta_2)}{\Gamma_q(p_3)}.
\]
expressed as the same way as the gamma $q$-distribution $\gamma_q$. At this level, we are able to characterize the beta $Y$-distribution. Several characterizations of the beta distribution have been determined [5, 20].

Proof. “⇒” By using Lemma 4.2 and the $q$-Laplace transform of the pair $(\log_q Y_1, \log_q Y_2)$, we obtain the desired result.

Several characterizations of the beta distribution have been determined [5, 20]. At this level, we are able to characterize the beta $q$-distribution by the same way as the gamma $q$-distribution. The beta $q$-density function [12] is expressed as 

$$f(x) = \frac{x^{a-1}(1-qx)^{b-1}}{\beta_q(a, b)} 1_{[0,1]}(x).$$

**Theorem 4.4.** Let $X_1$, $X_2$, $X_3$ be three independent random variables with support $[0,1]$ and let $Y_1 = \frac{X_1}{X_3}$ and $Y_2 = \frac{X_2}{X_3}$.

The necessary and sufficient condition for $X_1$, $X_2$, $X_3$ to be $q$-beta distributed with parameters $(p_i, p_{i+1})$ for $i = 1, 3, 5$ is that the $q$-Laplace transform of the couple $(\log_q Y_1, \log_q Y_2)$ has the following expression

$$\frac{\beta_q(p_1 - \theta_1, p_2)}{\beta_q(p_1, p_2)} \times \frac{\beta_q(p_3 - \theta_2, p_4)}{\beta_q(p_3, p_4)} \times \frac{\beta_q(p_5 + \theta_1 \oplus_q \theta_2, p_6)}{\beta_q(p_5, p_6)}.$$

**Proof.** “⇒” Let $X_1 \sim \beta_q(p_1, p_2)$, the $q$-Laplace transform of $\log_q X_1$ is

$$qL_{\log_q X_1}(\theta_1) = \frac{1}{\beta_q(p_1, p_2)} \int_0^1 x^{p_1 - \theta_1} (1-qx)^{p_2-1} dq x = \frac{\beta_q(p_1 - \theta_1, p_2)}{\beta_q(p_1, p_2)}.$$

In the same way, we show that

$$qL_{\log_q X_2}(\theta_2) = \frac{\beta_q(p_3 - \theta_2, p_4)}{\beta_q(p_3, p_4)},$$

and

$$qL_{\log_q X_3}(-\theta_1 \oplus_q \theta_2) = \frac{\beta_q(p_5 + \theta_1 \oplus_q \theta_2, p_6)}{\beta_q(p_5, p_6)}.$$

Now, we compute the $q$-Laplace transform of the couple $(\log_q Y_1, \log_q Y_2)$ with $Y_1 = \frac{X_1}{X_3}$ and $Y_2 = \frac{X_2}{X_3}$.

$$\begin{align*}
qL_{(\log_q Y_1, \log_q Y_2)}(\theta_1, \theta_2) &= qL_{\log_q X_1}(\theta_1)qL_{\log_q X_2}(\theta_2)qL_{\log_q X_3}(-\theta_1 \oplus_q \theta_2) \\
&= \frac{\beta_q(p_1 - \theta_1, p_2)}{\beta_q(p_1, p_2)} \times \frac{\beta_q(p_3 - \theta_2, p_4)}{\beta_q(p_3, p_4)} \times \frac{\beta_q(p_5 + \theta_1 \oplus_q \theta_2, p_6)}{\beta_q(p_5, p_6)}.
\end{align*}$$

“⇐” At this stage, if we take the $q$-distribution of the couple $(Y_1, Y_2)$ with $q$-Laplace transform of the pair $(\log_q Y_1, \log_q Y_2)$ given by

$$\begin{align*}
qL_{(\log_q Y_1, \log_q Y_2)}(\theta_1, \theta_2) &= \frac{\beta_q(p_1 - \theta_1, p_2)}{\beta_q(p_1, p_2)} \times \frac{\beta_q(p_3 - \theta_2, p_4)}{\beta_q(p_3, p_4)} \times \frac{\beta_q(p_5 + \theta_1 \oplus_q \theta_2, p_6)}{\beta_q(p_5, p_6)},
\end{align*}$$

...
and using Lemma 4.2, then the proof is complete. □

5. Conclusion

In this work, we have characterized the gamma and the beta $q$-distributions families. This characterization is based on the $q$-Laplace transform and $q$-independence properties. Departing from this result, we estimate that the gamma $q$-distribution would be characterized with the bivariate beta $q$-distribution in subsequent research. Having explored the gamma $q$-distribution, our work is a step that may be taken further. In future works, we aspire to introduce the Dirichlet $q$-distribution with their two kinds as well as to explore its characterization.

References


**Imen Boutouria**  
Laboratory of Probability and Statistics  
Sfax University  
B.P. 1171, Tunisia  
*Email address*: imen.boutouria@gmail.com

**Imed Bouzida**  
Laboratory of Probability and Statistics  
Sfax University  
B.P. 1171, Tunisia  
*Email address*: imed.bouzida@gmail.com

**Afif Masmoudi**  
Laboratory of Probability and Statistics  
Sfax University  
B.P. 1171, Tunisia  
*Email address*: afif.masmoudi@fss.rnu.tn