REDUCED-ORDER BASED DISTRIBUTED FEEDBACK
CONTROL OF THE BENJAMIN-BONA-MAHONY-BURGERS
EQUATION†

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ABSTRACT. In this paper, we discuss a reduced-order modeling for the
Benjamin-Bona-Mahony-Burgers (BBMB) equation and its application to
a distributed feedback control problem through the centroidal Voronoi
tessellation (CVT). Spatial discretization to the BBMB equation is based
on the finite element method (FEM) using B-spline functions. To deter-
mine the basis elements for the approximating subspaces, we elucidate
the CVT approaches to reduced-order bases with snapshots. For the pur-
pose of comparison, a brief review of the proper orthogonal decomposition
(POD) is provided and some numerical experiments implemented includ-
ing full-order approximation, CVT based model, and POD based model.
In the end, we apply CVT reduced-order modeling technique to a feedback
control problem for the BBMB equation.

1. Introduction

The mathematical model of propagation of small amplitude long waves in
nonlinear dispersive media is described by the following Benjamin-Bona-Mahony-
Burgers equation [1]:

\[
\begin{cases}
  y_t - y_{xxt} - \alpha y_{xx} + \beta y_x + yy_x = f & \text{in } D \times [0, T], \\
  y(0, t) = y(L, t) = 0 & \text{on } [0, T], \\
  y(x, 0) = y_0(x) & \text{in } D,
\end{cases}
\]

(1)

where \( D = [0, L] \), \( \alpha > 0 \) and \( \beta \) are constants and \( f \) is a given forcing term. In
the physical case, the dispersive effect of (1) is the same as the Benjamin-Bona-
Mahony (BBM) equation, while the dissipative effect is the same as the Burgers
equation. The BBMB equation is an alternative model for the Korteweg-de
Vries-Burgers (KdVB) equation [2]. Stabilization of boundary feedback control

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for the equations BBM, KdVB and Burgers equation has been investigated in [3]-[6]. Zhang [7] and Laurent [8] demonstrated that the feedback controller is locally (globally) controllable and stabilizable at the case of the KdV equation.

In a computational simulation for (nonlinear) complex systems, standard discretization schemes (finite element, finite difference, etc.) may require a great degrees of freedom for the accurate simulation of fluid flows and using these approaches are expensive with respect to both storage and computing time. Therefore, the application of reduced-order models to physical systems and feedback control problems has received an increasing amount of attention in the recent. Roughly speaking, the reduced-order approach is based on projecting the dynamical system onto subspaces consisting of basis elements that contain characteristics of the expected solution. This is in contrast to the traditional numerical methods such as finite element techniques, where the elements of the subspaces are uncorrelated to the physical properties of the system that they approximate.

One of the most popular reduced-order modeling technique is the proper orthogonal decomposition (POD) method. POD begins with a set of snapshots that are generated by evaluating the computational solution of transient problems at several instants of time or by evaluating the computational solution for several values of the parameters appearing in the problem description or by a combination of the two generally. Thereafter, the well-known singular value decomposition (SVD) is applied to the snapshot matrices and project a solution on a proper orthogonal space. POD based model reduction has been applied with some success to several problems [10]-[27]. In case of the BBMB equation and a related feedback control problem, that kind of model reduction is studied in [28]-[29]. Meanwhile, the centroidal Voronoi tessellations (CVTs) have been successfully used in several applications to reduced-order modelings. In CVT reduced-order modeling, we start with a snapshot set just as is done in a POD-based setting. We then construct a special Voronoi clustering of the snapshot set for which the means of the clusters are also the generators of the corresponding Voronoi clusters. The generators of the Voronoi clustering make up the CVT reduce-dorder basis. We then use the CVT-based basis in the same way as one uses a POD-based basis to determine a very low-dimensional approximation to the solution of the complex dynamical system. CVT based model reduction to differential equations has been researched in [30]-[33].

The efficiency of CVT and POD reduced bases depend on their dimension, i.e., if a reduced basis is low-dimensional and can still approximate the state well, then accurate approximations of the solution of the dynamical system can be determined with lower cost. However, it decides the ability of a reduced basis to approximate the state of a system how much information is contained in the snapshot set used to generate the basis. Certainly, a reduced basis cannot contain more information than that contained in the snapshot set. Thus, crucial to the success of reduced-order modeling approaches to model reduction is the generation of good snapshot sets.
A conclusion obtained from [9] is that the solution of the generalized regularized long wave Burgers (GRLWB) equation decays like the solution of the corresponding linear equation when this solution has a small value. The system (1) is an important case of the GRLWB equation. Therefore, we can design the controller and control the system (1) using the linear quadratic regulator method. In this paper, we use the quadratic B-spline finite element method to convert the BBMB equation to a finite dimensional nonlinear ordinary differential equations, then, design the full-order control law, and last, using the reduced-order basis obtained by the CVT method applied to design low order controller and then reduce order of the control modeling.

The rest of paper goes as follows. In Section 2, we describe the B-spline finite element approximation of a solution of the BBMB equation. In Section 3, we show how CVT bases are defined and constructed and how they are used to determine very low-dimensional approximations. For a comparison reason, we also briefly review POD-based reduced-order bases and its reduced-order approximations. In Section 3, we compute an example to show how snapshots sets can be generated and to compare and contrast the performance of CVT and POD based reduced-order modeling. In Section 4, we apply a CVT based reduced-model to a distributed feedback control problem for the BBMB equation, which is our main purpose. Some numerical results for a distributed feedback control problem are also given.

2. Finite element approximation by the B-spline

Standard Lagrangian finite element basis functions offer only simple \( C^0 \)-continuity and therefore they cannot be used for the spatial discretization of the higher-order differential equations (e.g., third-order differential equation or forth-order differential equation), but the B-spline basis function can at least achieve \( C^1 \)-continuous globally, and such basis function is often used to solve the higher order differential equations.

Let us consider the BBMB equation with boundary conditions and the initial condition. We use a variational formulation to set a finite element method to approximate (1). Let \( H^1_0(D) = \{ y \in H^1(D) : y|_D = 0 \} \) and \( H^1(D) = \{ v \in L^2(D) : \frac{\partial v}{\partial x} \in L^2(D) \} \). Then a variational formulation is following: find \( y \in L^2(0,T;H^1_0(D)) \) such that

\[
\begin{align*}
\int_D y_t v dx + \int_D y_{xt} v' dx + \alpha \int_D y_x v' dx + \beta \int_D y_x v dx + \int_D y y_x v dx = \int_D f v dx \quad \text{for all } v \in H^1_0(D),
\end{align*}
\]

\( y(0, x) = y_0(x) \quad \text{in } D, \tag{2} \)

Now, we choose conforming finite element subspace for (2). Let \( V^h \) be a finite-dimensional subspace of and then define \( V^h_0 = V^h \cap H^1_0(D) \) Then the semi-discretization in space domain leads to the following problem: find \( y^h(t, \cdot) \in V^h_0 \),
where $y_0(x) \in V_0^h$ is an approximation, e.g., a projection, of $y_0(x)$.

We divide the interval $D = [0, L]$ into $N$ elements of equal length $h$ with the knots $x_i$, which are $0 = x_0 < x_1 < \cdots < x_N = L$. The set of splines \{B_{-1}, B_0, \cdots, B_N\} forms a basis for functions defined on $D$. Quadratic B-splines $B_i(x)$ with the required properties are defined by [34]

$$B_i(x) = \begin{cases} 
\frac{1}{h^2} \begin{pmatrix} (x_{i+2} - x_i)^2 - 3(x_{i+1} - x_i)^2 + 3(x_i - x)^2, & [x_{i-1}, x_i], \\
(x_{i+2} - x)^2 - 3(x_{i+1} - x)^2, & [x_i, x_{i+1}], \\
(x_{i+2} - x)^2, & [x_{i+1}, x_{i+2}], \\
0, & \text{otherwise,}
\end{pmatrix} 
\end{cases}$$

where $h = x_{i+1} - x_i, i = -1, 0, \cdots, N$.

The quadratic spline and its first derivative vanish outside the interval $[x_{i-1}, x_{i+2}]$. The spline function values and its first derivative at the knots are given by

$$\begin{cases} 
B_i(x_{i-1}) = B_i(x_{i+2}) = 0, 
B_i(x_i) = B_i(x_{i+1}) = 1;
\end{cases}$$

$$\begin{cases} 
B'_i(x_{i-1}) = B'_i(x_{i+2}) = 0, 
B'_i(x_i) = B'_i(x_{i+1}) = 1.
\end{cases}$$

Thus an approximate solution can be written in terms of the quadratic spline functions as

$$y^h(x, t) = \sum_{i=-1}^{N} a_i(t)B_i(x),$$

where $a_i(t)$ are yet undetermined coefficients.

Each spline covers three intervals so that three splines $B_{i-1}(x), B_i(x), B_{i+1}(x)$ cover each finite element $[x_i, x_{i+1}]$. All other splines are zero in this region. Using Eq.(5) and spline function properties (4), the nodal values of function $y^h(x, t)$ and its derivative at the knot $x_i$ and fixed time $\tilde{t}$ can be expressed in terms of the coefficients $a_i(\tilde{t})$ as

$$y^h(x_i, \tilde{t}) = a_{i-1}(\tilde{t}) + a_i(\tilde{t}), \quad \frac{\partial y^h(x, \tilde{t})}{\partial x}\bigg|_{x=x_i} = \frac{2}{h}(a_i(\tilde{t}) - a_{i-1}(\tilde{t})).$$

From (6) and homogeneous boundary conditions we get $a_{-1}(t) = -a_0(t)$ and $a_N(t) = -a_{N-1}(t)$. Hence we have

$$y^h(x, t) = \sum_{i=0}^{N-1} a_i(t)\phi_i(x),$$
where \( \phi_0(x) = (B_0(x) - B_{-1}(x)) \), \( \phi_i(x) = B_i(x) (i = 1, 2, \ldots, N-2) \), \( \phi_{N-1}(x) = B_{N-1}(x) - B_N(x) \). Hence \( N \) unknowns \( a_i(t) (i = 0, 1, \ldots, N - 1) \) for every moment of \( t \) can be determined.

According to Galerkin method the weighted function \( \psi^h(x) \) in (3) is chosen as \( \psi^h_i(x) = \phi_i(x) (i = 0, 1, \ldots, N - 1) \). Substituting (7) into (3) we obtain

\[
\begin{array}{l}
\sum_{i=0}^{N-1} \left( \int_D \phi_i \phi_j dx \right) \frac{da_i(t)}{dt} + \sum_{i=0}^{N-1} \left( \int_D \phi'_i \phi'_j dx \right) \frac{da_i(t)}{dt} + \alpha \sum_{i=0}^{N-1} \left( \int_D \phi'_i \phi'_j dx \right) a_i(t) + \beta \sum_{i=0}^{N-1} \left( \int_D \phi'_i \phi'_j dx \right) a_i(t) \\
+ \sum_{i=0}^{N-1} \sum_{k=0}^{N-1} \left( \int_D \phi_i \phi'_k \phi_j dx \right) a_i(t) a_k(t) = \int_D f \phi_j dx,
\end{array}
\]

(8)

Assume \( m_{ij} = (\phi_i, \phi_j) \), \( s_{ij} = (\phi'_i, \phi'_j) \), \( d_{ij} = (\phi'_i, \phi_j) \), \( n_{ijk} = (\phi_i, \phi'_k, \phi_j) \), \( f_j = (f, \phi_j) \), \( y_0^j = (y_0, \phi_j) \), and mass matrix \( M = (m_{ij}) \), stiff matrix \( S = (s_{ij}) \), \( D = (d_{ij}) \), \( N = (n_{ijk}) \), \( f = (f_0, f_1, \ldots, f_{N-1})^T \), \( y_0 = (y_0^0, y_0^1, \ldots, y_0^{N-1}) \), \( \bar{a}_0 = (a_0(t), a_1(t), \ldots, a_{N-1}(t))^T \), then system (8) can be written in the matrix form

\[
\begin{align*}
(M + S) \frac{d\bar{a}}{dt} + (\alpha S + \beta D) \bar{a} + (\bar{a})^T N\bar{a} &= f, \\
M\bar{a}_0 &= \bar{y}_0.
\end{align*}
\]

(9)

System (9) is a nonlinear ordinary differential equations which consists of \( N \) equations and \( N \) unknowns. Because \( M \) and \( M + S \) are invertible matrix, the system (9) can be written as standard first order nonlinear ordinary differential equations with initial condition, namely,

\[
\frac{d\bar{a}}{dt} = (M + S)^{-1} (f - ((\alpha S + \beta D) \bar{a} + (\bar{a})^T N\bar{a})), \quad \bar{a}_0 = \bar{y}_0,
\]

(10)

where, for simplicity take \( \bar{y}_0 = M^{-1} \bar{y}_0 \). Because the right terms in (10) are continuously differentiable, and the system (10) exists one and only one solution and have a zero equilibrium solution when forcing term \( f(x, t) \) tends to zero while time approach infinity. Therefore, letting the equilibrium solution as the starting point, we obtain the numerical solution of the system (1) by using Newton method. These approximate solutions are used to generate snapshots.

The \( M \) snapshot vectors

\[
\bar{a}_m = [a_0(t_m) \ a_1(t_m) \ \cdots \ a_{N-1}(t_m)]^T, \quad m = 1, \ldots, M
\]

are determined by evaluating the approximate solution of system (10) at \( M \) equally spaced time values \( t_m \), ranging from \( t_1 = 0 \) to \( t_M = T \).
Remark 1. For convenience, by nodal value we mean the solution of the differential equation at the knot, and coefficient refers to any coefficient appearing in Eqs. (5) and (6).

Remark 2. The property (6) will be used in the control solutions, so the nodal values of the full-order and reduced-order control solutions at a knot \( x_i \) equal the sum of the coefficients at the knots \( x_{i-1} \) and \( x_i \) (at other than the boundary points).

3. Reduced-order modeling for the BBMB equations

3.1. CVT reduced-order bases

Given a discrete set of snapshot vectors \( S = \{ \mathbf{a}_m \}_{m=1}^M \) belonging to \( \mathbb{R}^N \), the set \( \{ V_k \}_{k=1}^K \) is called a tessellation of \( \mathbb{R} \) if \( V_k \cap V_j = \phi \) for \( k \neq j \) and \( \bigcup_{k=1}^K V_k = S \). Given a set of points \( \{ z^*_k \}_{k=1}^K \) belonging to \( \mathbb{R}^N \), the Voronoi region \( V_k \) corresponding to the point \( z^*_k \) is defined by

\[
V_k = \{ \mathbf{a} \in \mathbb{R} : |\mathbf{a} - z^*_k| < |\mathbf{a} - z^*_j|, \text{ for } j = 1, \ldots, K, k \neq j \}
\]

The points are called generators. The set \( \{ V_k \}_{k=1}^K \) is a Voronoi tessellation or Voronoi diagram of \( S \), and each \( V_k \) is referred to as the Voronoi region corresponding to \( z^*_k \).

Given a density function \( \rho(\mathbf{a}) \geq 0 \) defined in \( S \), the mass centroid \( z^* \) of a subset \( V \subseteq S \) is defined by

\[
\sum_{\mathbf{a} \in V} \rho(\mathbf{a})|\mathbf{a} - z^*|^2 = \inf_{\mathbf{a} \in V^*} \sum_{\mathbf{a} \in V} \rho(\mathbf{a})|\mathbf{a} - z|^2,
\]

where the set \( V^* \) can be taken to be \( V \) or an even larger set. In case \( V^* = \mathbb{R}^N \), \( z^* \) is defined as the ordinary meaning of the center of mass, which is

\[
z^* = \frac{\sum_{\mathbf{a} \in V} \rho(\mathbf{a})\mathbf{a}}{\sum_{\mathbf{a} \in V} \rho(\mathbf{a})}.
\]

In this situation, \( z^* \neq z \) in general.

Let \( \{ z_k \}_{k=1}^K \) be the set of generators of a Voronoi tessellation \( \{ V_k \}_{k=1}^K \) and \( \{ z^*_k \}_{k=1}^K \) be the set of mass centroids of the Voronoi regions \( \{ V_k \}_{k=1}^K \). If \( z_k \) coincides with \( z^*_k \), i.e.,

\[
z_k = z^*_k \quad \text{for } k = 1, \ldots, K,
\]

then we refer to the Voronoi tessellation as being a centroidal Voronoi tessellation (CVT). Note that, in general, CVTs of a given set are not uniquely defined; see [35].

CVT have an optimization characterization. Let \( \{ z^*_k \}_{k=1}^K \) denote an arbitrary set of \( K \) vectors in \( \mathbb{R}^N \) and let \( \{ V_k \}_{k=1}^K \) denote a tessellation of the snapshot set \( \tilde{Y} \) into \( K \) disjoint subsets. Let

\[
\mathcal{F}(z_1, \ldots, z_K; V_1, \ldots, V_K) = \sum_{k=1}^K \sum_{\mathbf{a}_m \in V_k} \rho(\mathbf{a}_m)|\mathbf{a}_m - z^*_k|^2.
\]
Then we refer $F$ to as the **energy**. It was proved that a necessary condition for $K$ to be minimized holds when $\{V_k\}_{k=1}^K$ is a CVT of $S$; see [35]. Since, in practice, CVTs can only be approximately constructed, the energy is often used to monitor the quality of the results.

There are several algorithms known for constructing centroidal Voronoi tessellations of a given set. **Lloyd’s method** is a deterministic algorithm which is the obvious iteration between computing Voronoi diagrams and mass centroids, i.e., a given set of generators is replaced in an iterative process by the mass centroids of the Voronoi regions corresponding to those generators. **MacQueen’s method** is a probabilistic algorithm. There are other probabilistic methods that may be viewed as generalization of both the MacQueen and Lloyd methods and that are plausible to efficient parallelization. To see the detailed algorithms of constructing CVTs, refer to [35, 36].

In the reduced-order modeling context, given a set of snapshots $S$ belonging to $\mathbb{R}^N$, the CVT reduced-basis of dimension $K < N < M$ is the set of generators $\{\vec{z}_k\}_{k=1}^K$, also belonging to $\mathbb{R}^N$, of a CVT of the snapshot set. For reduced-order modeling applications, the snapshot vectors are usually coefficient vectors in the expansion of the finite element approximation of the solution of the partial differential equation at different moments in time. Thus, to each snapshot vector, there corresponds a finite element function.

**Remark 3.** In the computation here, the snapshots differ from the numerical solutions, as is usually noted in the reduced-order PDE model, i.e. rather than “modal values”, the snapshots are coefficients mentioned in Remark 1.

### 3.2. POD reduced-order bases

In this section, for the purpose of comparison with CVT method, we briefly describe reduced-order bases using POD method. Given a discrete set of snapshot vectors $\vec{A} = \{\vec{a}_m\}_{m=1}^M$ belonging to $\mathbb{R}^N$, where $M < N$, we form the $N \times M$ snapshot matrix $\vec{A}$ whose columns are the snapshot vectors $\vec{y}_m$:

$$\vec{A} = (\vec{a}_1 \quad \vec{a}_2 \quad \cdots \quad \vec{a}_M).$$

Let

$$\vec{U}^T \vec{A} \vec{V} = \left( \begin{array}{c} \sum \quad 0 \\ 0 \quad 0 \end{array} \right),$$

where $\vec{U}$ and $\vec{V}$ are $N \times N$ and $M \times M$ orthogonal matrices, respectively, and $\sum = \text{diag}(\sigma_1, \ldots, \sigma_{\tilde{M}})$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\tilde{M}}$ and $\tilde{M} \leq M$ be the singular value decomposition of $\vec{A}$. Here, $\tilde{M}$ is the rank of $\vec{A}$, i.e., the dimension of the snapshot set $\tilde{Y}$, which would be less than $M$ whenever the snapshot set is linearly dependent. It is well known [39] that if

$$\vec{U} = (\vec{u}_1 \quad \vec{u}_2 \quad \cdots \quad \vec{u}_N) \quad \text{and} \quad \vec{V} = (\vec{v}_1 \quad \vec{v}_2 \quad \cdots \quad \vec{v}_M),$$

then

$$\vec{A} \vec{u}_i = \sigma_i \vec{u}_i \quad \text{and} \quad \vec{A}^T \vec{v}_i = \sigma_i \vec{v}_i \quad \text{for} \quad i = 1, \ldots, \tilde{M}.$$
so that also
\[
\forall i, j, k \in \mathbb{N}, \quad \sigma_i^2 = \sigma_j^2 = \sigma_k^2 = 0,
\]
so that \(\sigma_i^2, i = 1, \ldots, M\), are the nonzero eigenvalues of \(AA^T\) (and also of \(A^TA\)) arranged in nondecreasing order. Note that the matrix \(C = A^TA\) is simply the correlation matrix for the set of snapshot vectors \(\{\tilde{\alpha}_m\}_{m=1}^M\), i.e., we have that \(C_{mn} = \tilde{\alpha}_m^T\tilde{\alpha}_n\).

In the reduced-order modeling context, given a set of snapshots \(\{\tilde{\alpha}_m\}_{m=1}^M\) belonging to \(\mathfrak{R}^N\), the POD reduced-basis of dimension \(K \leq M < N\) is the set \(\{\tilde{\alpha}_k\}_{k=1}^K\) of vectors also belonging to \(\mathfrak{R}^N\) consisting of the first \(K\) left singular vectors of the snapshot matrix \(\tilde{\alpha}\). Thus, one can determine the POD basis by computing the (partial) singular value decomposition of the \(N \times M\) matrix \(\tilde{\alpha}\). Alternately, one can compute the (partial) eigensystem \(\{\sigma_i^2, \tilde{\alpha}_i\}_{i=1}^K\) of the \(M \times M\) correlation matrix \(\tilde{C} = \tilde{\alpha}^TA\tilde{\alpha}\) and then set \(\tilde{\alpha}_k = \tilde{\alpha}_0, k = 1, \ldots, K\).

The \(K\)-dimensional POD basis has the obvious property of orthonormality. It also has several other important properties which we now mention. Let \(\{\tilde{q}_k\}_{k=1}^K\) be an arbitrary set of \(K\) orthonormal vectors in \(\mathfrak{R}^M\) and let \(\text{proj}_S(\tilde{\alpha})\) denote the projection of a vector \(\tilde{\alpha} \in \mathfrak{R}^N\) onto the set \(S = \text{span}(\{\tilde{q}_k\}_{k=1}^K)\). Further, let
\[
\mathcal{E}(\tilde{q}_1, \ldots, \tilde{q}_K) = \sum_{n=1}^N |\tilde{\alpha}_m - \text{proj}_S(\tilde{\alpha}_m)|^2,
\]
i.e., \(\mathcal{E}\) is the sum of the squares of the error between each snapshot vector \(\tilde{\alpha}_m\) and its projection \(\text{proj}_S(\tilde{\alpha}_m)\) onto the \(S\). Then, it can be shown that the POD basis \(\{\tilde{\alpha}_k\}_{k=1}^K\) minimize \(\mathcal{E}(\tilde{q}_1, \ldots, \tilde{q}_K)\) subject to any given \(K\)-dimensional orthonormal basis \(\{\tilde{q}_k\}_{k=1}^K\).

### 3.3. The CVT and POD based reduced-order models

Let \(\{\tilde{w}_k\}_{k=1}^K\) be a \(K\)-dimensional reduced-order basis of CVT or POD basis, i.e., corresponding to the snapshot set \(\{\tilde{\alpha}_m\}_{m=1}^M\), let \(\{\tilde{w}_k\}_{k=1}^K\) be generators of CVT or left singular vectors of POD. Recall \(V_0^h = V_h \cap H_0^1(0,1)\) and let \(V_0^K\) be a \(K\)-dimensional subspace of \(V_0^h\) spanned by reduced-order basis. For each \(\tilde{w}_k, k = 1, \ldots, K\), there is a finite element function such that
\[
\psi_k(x) = \sum_{j=1}^J w_{j,k} \phi_j(x) \in V_0^h
\]
where \(w_{j,k}\) denotes the \(j\)th component of \(\{\tilde{w}_k\}\). Let \(V_0^K = \text{span}(\{\tilde{w}_k\}_{k=1}^K) \subset V_0^h\). We then determine \(y^K(t, \cdot) \in V^K\) from the discrete problem:
\[
\begin{align*}
\int_D y_t^K v^k dx + \int_D y_{xt}^K (v^k)' dx + \alpha \int_D y_x^K (v^k)' dx + \beta \int_D y_{xx}^K (v^k) dx \\
+ \int_D y_x^K y_x^K v^k dx = \int_D f v^k dx \quad \text{for all } v^k \in V_0^K(D),
\end{align*}
(11)
\]
\[
y^K(0, x) = y_0^K(x) \quad \text{in } D,
\]
The reduced-order basis functions satisfy homogeneous boundary conditions. Since the reduced basis approximation of the form

$$y^K(t, \cdot) = \sum_{k=1}^{K} a_k(t) \psi_k(x), \in V_0^K.$$  

we may write

$$
\begin{aligned}
\sum_{k=0}^{K} \left( \int_D \psi_k \psi_j dx \right) \frac{da_k(t)}{dt} &+ \sum_{k=0}^{K} \left( \int_D \psi_k' \psi_j' dx \right) \frac{da_k(t)}{dt} \\
&+ \alpha \sum_{k=0}^{K} \left( \int_D \psi_k \psi_j' dx \right) a_k(t) + \beta \sum_{k=0}^{K} \left( \int_D \psi_k' \psi_j dx \right) a_k(t) \\
&+ \sum_{k=0}^{K} \sum_{l=0}^{K} \left( \int_D \psi_k \psi_l \psi_j dx \right) a_k(t) a_l(t) = \int_D f \psi_j dx, \\
\sum_{k=0}^{K} \left( \int_D \psi_k \psi_j dx \right) a_k(0) &= \int_D y_0(x) \psi_j dx, \quad j = 0, 1, \cdots, K.
\end{aligned}
$$

Equivalently, we have the system of nonlinear ordinary differential equations that determine the coefficient functions \(\{a_k(t)\}_{k=1}^{K}\)

$$
\begin{aligned}
(\bar{M} + \bar{S}) \frac{d\bar{a}}{dt} + (\alpha \bar{S} + \beta \bar{D}) \bar{a} + (\bar{a})^T \bar{N} \bar{a} &= \bar{f}, \\
\bar{M} \bar{a}_0 &= \bar{y}_0.
\end{aligned}
$$

where \(\bar{m}_{ij} = (\psi_i, \psi_j), \bar{s}_{ij} = (\psi_i', \psi_j'), \bar{a}_{ij} = (\psi_i', \psi_j), \bar{n}_{ijk} = (\psi_i \psi_k, \psi_j), f_j = (f, \psi_j), y_0^j = (y_0, \psi_j),\) and mass matrix \(\bar{M} = (m_{ij}),\) stiff matrix \(\bar{S} = (s_{ij}),\) \(\bar{D} = (d_{ij}),\) \(\bar{N} = (n_{ijk}),\) \(\bar{f} = (f_0, f_1, \cdots, f_{N-1})^T, \bar{y}_0 = (y_0^0, y_0^1, \cdots, y_0^{N-1}),\) \(\bar{a}_0 = (a_0(0), a_1(0), \cdots, a_{N-1}(0))^T, \bar{a}(t) = (a_0(t), a_1(t), \cdots, a_{N-1}(t))^T,\) We can observe that all of these matrices and tensors have full components; however, since \(K\) is usually chosen small, this does not cause any computational inefficiencies. Another important note is that matrices \(\bar{M}, \bar{S}, \bar{D}\) and tensor \(\bar{N}\) depends on the reduced-basis function \(\{\psi_k\}_{k=1}^{K}\), so that they may be all pre-computed by the matrices \(\bar{M}, \bar{S}, \bar{D}\), and the nonlinear tensor \(\bar{N} = (n_{ijk})\) from Section 2.

3.4. Computational experiments

Consider parameters \(\alpha = 0.01, \beta = 1,\) and an initial condition \(y_0(x) = \sin(\pi x)\) to (1) until a final time \(T = 10.\) Furthermore, we set a spatial interval to \([0, 1]\) and a full-order dimension to \(N = 128.\)

Spatial step size and time step size are \(h = 1/N\) and \(\Delta t = 1/100\) for discretizations, consequently 1001 snapshot vectors in \(\mathbb{R}^{128}\) are generated. The full-order solution is depicted in Figure 1. Note that for a small coefficient \(\alpha\) and an appropriate constant \(\beta,\) the finite element approximate solution of the BBMB equation fluctuates in a certain range.
We intend to compare two reduced-order methods (CVT and POD) of computation. We also verify that the CVT scheme is returning to the FEM with the B-spline answer as the number of basis functions increases. The CVT method is compared to the POD method for $K = 8$ dimensional basis. First of all, we apply the algorithm of the Lloyd method in subsection 3.1 to determine the generators of a CVT of the snapshot set; a set of generators is to be used as a reduced-order basis. The CVT generators are, here, computed from the normalized snapshot. The eight-dimensional CVT basis functions are displayed in Figure 2, and then for the same snapshot, the eight-dimensional POD basis functions are also displayed in Figure 3. It is important to note that, in contrast to the POD basis set, the CVT basis set of larger size is not built by augmenting the CVT basis set of smaller size, hence most of the elements of the larger set seem significantly different from any of those of the smaller set.

In some computations relating CVT and POD, we have a priori knowledge of the FEM solution. In this case, we measure a numerical error of the two approximating schemes, calculating the relative $L^2(D)$-error at time $t$ given by

$$E(t) = \left[ \frac{\int_D (y^N - y^K)^2 dx}{\int_D (y^N)^2 dx} \right]^{1/2},$$

where $y^N(t, x)$ denote a full-order solution and $y^K(t, x)$ denote a reduced-order solution. The actual errors - the simple difference between $y^K(t, x)$ and $y^N(t, x)$
are also shown graphically by computing $y^R(t, x)$ and $y^N(t, x)$ at $N + 2$ evenly-spaced points in each subinterval. Figure 5 displays for the 8 dimension cases and the CVT and POD based reduced-order models of the BBMB equation with corresponding actual error is plotted at these points. The plots of the relative $L^2(D)$-errors versus time (i.e. $E(t)$) are displayed in Figures 5. The reduced-order model are computed over varying dimensions $K = 4, 6,$ and $8$ for both CVT and POD cases. Except the four-dimensional cases, the CVT and POD based reduced-order models show almost no difference in the relative errors $E(t)$s. Even in Figures 4, one can observe that, for both the CVT and
Figure 4. Reduced-order solutions (first column) and actual errors (second column) of CVT (first row) and POD (second row) method for 8 dimension.

POD cases, $E(t)$s decrease as the size of the CVT basis set increase, moreover all the six-dimension models and all the eight-dimension models demonstrate the same accuracies in the graphs. In conclusion, we say that CVT and POD have a very similar accuracy.

4. Feedback control design

Now we describe our control problem: Find an optimal control $u^*(t)$ which minimizes the cost functional

$$J(u) = \int_{0}^{\infty} \left( ||y(t, \cdot)||_{L^2(D)}^2 + |u(t)|^2 \right) dt$$
subject to the constraint equations
\[
\begin{align*}
    y_t - y_{xxt} &= \alpha y_{xx} - \beta y_x - yy_x + f(x, t) \quad \text{in } D \times [0, T], \\
    y(0, t) &= y(L, t) = 0 \quad \text{on } [0, T], \\
    y(x, 0) &= y_0(x) \quad \text{in } D.
\end{align*}
\]
(14)

We will replace the forcing term by the special form \( b(x)u(t) \) in the system (14), then \( u(t) \) is the control input and \( b(x) \) is a given function used to distribute the control over the domain.

4.1. Linear quadratic regulator design.

Assuming that the nonlinear term in the BBMB equation is small, a sub-optimal feedback control \( u^* \) can be obtained by using the well-known linear quadratic regulator theory [40]-[42]. A full state feedback control is that to find an optimal control \( u^* \in L^2([0, T), L^2(D)) \) by minimizing the cost functional
\[
J(u) = \int_0^\infty (Q_y(t, \cdot), y(t, \cdot))_{L^2(D)} + (R_u(t), u(t)))dt
\]
subject to the constraint equations
\[ \dot{y}(t) = Ay(t) + Bu(t), \quad y(0) = y_0, \quad \text{for} \quad t > 0 \]
where \( Q : L^2(D) \to L^2(D) \) is a nonnegative definite self-adjoint weighting operator for state and \( R : L^2(D) \to L^2(D) \) is a positive definite weighting operator for the control. The optimal control \( u^*(t) \) can be found as
\[ u^*(t) = -\frac{1}{2} R^{-1} B^T \Pi y(t) = -K y(t), \]
where \( K \) is called the feedback operator and \( \Pi \) is symmetric positive definite solution of the algebraic Riccati equation
\[ \Pi A + A^T \Pi - \Pi R^{-1} B^T \Pi + Q = 0. \tag{15} \]
Here, the method of design controller is the same as the ordinary LQR scheme, but the properties (6) is considered in the whole process of the controller design.

### 4.2. Linear feedback controllers with state estimate feedback.

A simple, classical feedback control design, linear quadratic regulator (LQR), assumes the full state is “feed back” into the system by the control. However, knowledge of the full state is not possible for many complicated physical systems. As a realistic alternative, a compensator design provides a state estimate based on state measurements to be used in the feedback control law.

We do not assume that we have knowledge of the full state. Instead, we assume a state measurement of the form
\[ z(t) = Cy(t), \tag{16} \]
where \( C \in \mathcal{L}(L^2(D), \mathbb{R}^m) \). We can apply the theory and results to show that a stabilizing compensator based controller can be applied to the system [43].

The observer design is mainly needed in order to provide the feedback control law with estimated state variables. Therefore, the control law and observer are combined together into a complete system. The combined system is called compensator. This technique needs the availability of a limited measurement of the state as a condition. we assume that we have a system in the abstract form
\[ \dot{y}(t) = Ay(t) + G(y(t)) + Bu(t), \quad y(0) = y_0, \tag{17} \]
where \( y(t) \) is in a state space \( L^2(D) \) and \( u(t) \) is in a control space \( U \).

According to the given state measurement (16), a state estimate \( \hat{y}(t) \), is computed by solving the observer equation
\[ \hat{y}(t) = A\hat{y}(t) + G(\hat{y}(t)) + Bu(t) + L[z(t) - C\hat{y}(t)], \quad \hat{y}(0) = \hat{y}_0. \tag{18} \]
The feedback control law is given by
\[ u(t) = -K\hat{y}(t), \tag{19} \]
where \( K \) is called the feedback operator. Functional gain operator \( K \) and estimator gain operator \( L \) are determined linear quadratic regulator (LQR) and
Kalman estimator (LQE), respectively, in usual manner. According to the result of the above, we already know

$$K = R^{-1}B^T \Pi.$$  \hspace{1cm} (20)

Next, $P$ is found as the non-negative definite solution of

$$AP + PA^T - PC^T CP + \bar{Q} = 0,$$

where $\bar{Q}$ is a non-negative definite weighting operator. If the solution $P$ exists, we can define

$$L = PC^T.$$  \hspace{1cm} (21)

Form (16)-(21), we obtain the closed loop compensator as

$$\begin{aligned}
\dot{y}(t) &= Ay(t) - BK\tilde{y}(t) + G(y(t)), \\
\dot{\tilde{y}}(t) &= LCy(t) + (A - LC - BK)\tilde{y}(t) + G(\tilde{y}(t)), \\
y(0) &= y_0, \\
\tilde{y}(0) &= \tilde{y}_0.
\end{aligned}$$  \hspace{1cm} (22)

### 4.3. Reduced-order compensators

Implementation of the controller for a PDE system requires a numerical discretization. For example, use of a finite element method provides finite dimensional approximations of (16)-(17) of order $N$ (where order refers to the freedom of finite element), given by

$$\begin{aligned}
\dot{y}^N(t) &= A^N y^N(t) + G^N(y^N(t)) + B^N u^N(t), \\
y^N(0) &= y^N_0, \\
z^N(t) &= C^N y^N(t),
\end{aligned}$$

where, we take $A^N = (M + S)^{-1}(\alpha S + \beta D)$, and the $B^N$ is constructed by integration of product of two functions $b(x)$ and test function $\phi(x)$, and nonlinear term is $G^N(y) = (M + S)^{-1}y^T Ny$. In a full order compensator design, the order $N$ approximations are used to compute $K^N$ and $L^N$. Then finite dimensional approximations of the compensator equation (18) and control law (19) are given by

$$\begin{aligned}
\dot{\tilde{y}}^N(t) &= A^N \tilde{y}^N(t) + G^N(\tilde{y}^N(t)) + B^N u^N(t) \\
&\quad + L^N[z^N(t) - C^N \tilde{y}^N(t)], \\
\tilde{y}^N(0) &= \tilde{y}^N_0, \\
u^N(t) &= -K^N \tilde{y}^N(t),
\end{aligned}$$
respectively. The approximation to the closed-loop compensator system (which will henceforth be referred to as full order) is given by

\[
\begin{align*}
\dot{y}^N(t) &= A^N y^N(t) - B^N K^N \tilde{y}^N(t) + G^N (y^N(t)), \\
\dot{\tilde{y}}^N(t) &= L^N C^N y^N(t) + A^N \tilde{y}^N(t) - L^N C^N \tilde{y}^N(t) \\
y^N(0) &= y_0^N, \quad \tilde{y}^N(0) = \tilde{y}_0^N.
\end{align*}
\] (23)

Real-time control using the full order compensator may be impossible for many physical problems in that they may require large discretized systems for adequate approximation. Therefore, a reduced order compensator is required. A “reduce-then-design” approach has a potential drawback that important physics or information contained in the model can be lost before obtaining the controller; see [44]. Hence, in this paper, we adopt a “design-then-reduce” approach. In other words, a controller is designed based on the high order model, and then reduced.

\[
\begin{align*}
\dot{\tilde{y}}^K(t) &= A^K \tilde{y}^K(t) + G^K (\tilde{y}^K(t)) + B^K u^K(t) \\
&\quad + L^K [z^K(t) - C^K \tilde{y}^K(t)], \\
\tilde{y}^K(0) &= \tilde{y}_0^K, \\
u^K(t) &= -K^K \tilde{y}^K(t), \\
\dot{y}^K(t) &= A^K y^K(t) + G^K (y^K(t)) + B^K u^K(t), \\
y^K(0) &= y_0^K.
\end{align*}
\] (24)

The suggested control law (25) is substituted into equations (24) and (26) producing

\[
\begin{align*}
\dot{y}^K(t) &= A^K y^K(t) - B^K K^K \tilde{y}^K(t) + G^K (y^K(t)), \\
\dot{\tilde{y}}^K(t) &= L^K C^K y^K(t) + A^K \tilde{y}^K(t) - L^K C^K \tilde{y}^K(t) \\
&\quad - B^K K^K \tilde{y}^K(t) + G^K (\tilde{y}^K(t)), \\
y^K(0) &= y_0^K, \quad \tilde{y}^K(0) = \tilde{y}_0^K.
\end{align*}
\] (27)

In this work, reduced bases are formed using the CVT process as described in Section 3. The reduced bases are used to compute the compensator equation, feedback control law and model problem in (24)-(25). Then the reduced systems given by (27) are compared with the full order compensator system in (23). Backward Euler method is applied to solve numerical solution of the systems (23) and (27).

4.4. Computational experiments

For numerical computations, two example as follows are considered.

Example 1. We choose \( \alpha = 0.0001, \beta = 10, \) an initial condition \( y_0(x) = \sin(\pi x), \) and a final time \( T = 10. \)
The spatial domain is taken to be \([0, 1]\). Full-order dimension is \(N = 64\), reduced-order dimension is \(K = 8\). Spatial step size and time step size is \(h = 1/N\) and \(1/100\) respectively.

**Example 2.** We choose \(\alpha = 0.5, \beta = 1\), an initial condition

\[
   y_0(x) = \begin{cases} 
   1 - 3 \left| x - \frac{1}{2} \right| & \text{if } \frac{1}{6} \leq x \leq \frac{5}{6}, \\
   0 & \text{otherwise}
   \end{cases}
\]

and a final time \(T = 5\).

Notice that the second example has an initial condition ‘less’ smooth than the initial condition of the first example. In the second example, we also choose the spatial domain \([0, 1]\), full-order dimension \(N = 64\), reduced-order dimension \(K = 8\), spatial step size \(h = 1/N\), and time step size \(1/100\). The control input operator is \(B = \int_0^L b(x)\phi_i(x)dx\), where \(b(x) = x\) and \(\phi_i(x)\) is a test function \((i = 0, 1, \cdots, N - 1)\). A state weighting operator (used in Riccati equation calculations) \(Q\) is taken to be \(M + S\). We set a control weighting operator \(R = 10^{-6}\) and a weighting operator \(\bar{Q}\) is also chosen as the \(M + S\). Finally, we create the measurement operator \(C\) with \(Cy(t, x) = 8 \int_{3/4}^{5/6} y(t, x)dx\) for the state estimate feedback controller.

Figure 6 presents uncontrolled solutions of BBMB equation for given examples. Figure 7 and Figure 8 depict full-order controlled solutions and reduced-order controlled solutions respectively. In the whole control processes, we can find that the full-order controls and reduced-order controls are very stable for both the two examples. Figure 9 shows that the \(L^2(D)\)-norms for the solutions to the uncontrolled and controlled problems. The full-order control and the reduced-order control have almost the same effects in the two examples.
The BBMB equation have appeared frequently as models of physical phenomena [45]-[47]. In the BBMB equation, an appropriate parameter $\beta$ can control the dissipative phenomena, but it is difficult to achieve our desired situation. In this work, we used the CVT based reduced-order model and got a desired solution successfully. The numerical results tell us that in case of reduced-order modeling, the CVT method is as efficient and feasible as POD method.
Figure 9. $L^2(D)$-norms for the solutions of the uncontrolled and controlled problem vs. time for Example 1 (left) and Example 2 (right). For reduced-order controlled solution, 8 basis functions are used.

References


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