MINIMAL AND HARMONIC REEB VECTOR FIELDS ON TRANS-SASAKIAN 3-MANIFOLDS

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Abstract. In this paper, we obtain some necessary and sufficient conditions for the Reeb vector field of a trans-Sasakian 3-manifold to be minimal or harmonic. We construct some examples to illustrate main results. As applications of the above results, we obtain some new characteristic conditions under which a compact trans-Sasakian 3-manifold is homothetic to either a Sasakian or cosymplectic 3-manifold.

1. Introduction

In differential geometry of almost contact metric manifolds, trans-Sasakian manifolds are an important field of research because such manifolds include the well known $\alpha$-Sasakian manifolds (see [29]), $\beta$-Kenmotsu manifolds (see [29]) and cosymplectic manifolds (see [2]) as their special cases. In [25], the notion of trans-Sasakian manifolds $M$ was proposed for the first time which is an almost contact metric manifold such that $M \times \mathbb{R}$ belongs to the class $W_4$ of Hermitian manifolds (see [20]). Note that Hermitian manifolds of class $W_4$ are closely related to locally conformally Kähler manifolds.

The local structures of trans-Sasakian manifolds were classified by Marrero in [21], namely a connected trans-Sasakian manifold of dimension greater than three is of class either $C_5$ or $C_6$. In general, a trans-Sasakian manifold of type $(\alpha, \beta)$ is said to be proper if it is of class either $C_5$ or $C_6$, or equivalently, it is of type either $(\alpha, 0)$, or $(0, \beta)$ or $(0, 0)$. However, there exist many trans-Sasakian 3-manifolds which are not proper (see [1], [3], [23] and [24]). Therefore, to find on what condition a trans-Sasakian 3-manifold is proper is an interesting problem. S. Deshmukh et al. in [11], [12], [13], [14] and [15] gave various conditions under which a compact trans-Sasakian 3-manifold is homothetic to either a Sasakian 3-manifold or a cosymplectic 3-manifold. Trans-Sasakian 3-manifolds under some curvature restrictions were also studied in [6], [7], [8] and [9].

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It is well known that the Reeb vector field plays important role in geometry of trans-Sasakian 3-manifolds. In this paper, we start to investigate the minimality and harmonicity of the Reeb vector field of a trans-Sasakian 3-manifold $M$. In fact, we prove that the Reeb vector field of $M$ is minimal if and only if it is harmonic. After giving some equivalent conditions for the Reeb vector field of $M$ to be minimal or harmonic, we also construct several concrete examples to illustrate our main results. At last, as an application of the above results, we give some characteristic conditions for a compact trans-Sasakian 3-manifold being homothetic to either a Sasakian or a cosymplectic manifold. These results can be regarded as generalizations of those in [6], [7], [12] and [15].

2. Preliminaries

2.1. Minimal and harmonic vector fields

Let $(M, g)$ be a Riemannian manifold of dimension $m$ and $(T^1M, g_S)$ its unit tangent sphere bundle furnished with the standard Sasakian metric $g_S$. Let $V$ be a unit vector field of $M$. Then there exists a metric $g$ on $M$ induced from $g_S$ via $V$ which can be written as follows:

$$ (V^*g_S)(X,Y) = g(X,Y) + g(\nabla_X V, \nabla_Y V) $$

for any vector fields $X$ and $Y$ on $M$, where $\nabla$ denotes the Levi-Civita connection of the metric $g$. We define a $(1,1)$-type tensor field $L_V$ on $M$ by

$$ L_V = \text{id} + (\nabla V)^t \circ \nabla V, $$

where $\text{id}$ is the identity map and $(\nabla V)^t$ is the transpose of $\nabla V$. Now (2.1) can be written as $V^*g_S = g(L_V \cdot , \cdot )$. When $M$ is compact and orientable, the volume of $V$ is defined as the volume of the corresponding submanifold $(M, V^*g_S)$ of $(T^1M, g_S)$ and can be written as

$$ \text{Vol}(V) = \int_M f(V)dv_g, $$

where $f(V) = \sqrt{\det(L_V)}$. We define another $(1,1)$-type tensor field $K_V$ by

$$ K_V = f(V)(L_V)^{-1} \circ (\nabla V)^t. $$

According to Gil-Medrano and Llinares-Fuster [17], $V$ is a critical point for the volume function if and only if the following 1-form

$$ \omega_V(X) = \text{trace}\{Y \mapsto (\nabla_Y K_V)X\} $$

vanishes on the distribution $\mathcal{D}_V$ determined by all vector fields orthogonal to $V$. Following Gil-Medrano [16], such a critical point is said to be a minimal vector field even when $M$ is non-compact and non-orientable.
The energy of $V$ is defined as the energy of the map from $(M,g)$ into $(T^1M,gs)$ and can be written as

$$E(V) = \frac{m}{2} \Vol(M,g) + \frac{1}{2} \int_M \|\nabla V\|^2 dv_g.$$ 

Following Gil-Medrano [16], a unit vector field $V$ is a critical point for the energy function if and only if the following 1-form

(2.5) $\rho_V(X) = \text{trace} \{ Y \rightarrow (\nabla_Y (\nabla V)^t)X \}$

vanishes on the distribution $\mathcal{D}^V$ (see [31, 32]). A unit vector field $V$ satisfying this condition is said to be harmonic.

Moreover, the map $V : (M,g) \rightarrow (T^1M,gs)$ defines a harmonic map if and only if $V$ is a harmonic vector field and, in addition, the following 1-form

(2.6) $\bar{\rho}_V(X) = \text{trace} \{ Y \rightarrow R(\nabla_Y V, V)X \}$

vanishes for any vector field $X$ on $M$ (see [16]), where $R$ denotes the Riemannian curvature tensor defined by $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$.

### 2.2. Trans-Sasakian manifolds

According to D. E. Blair [2], an almost contact metric structure defined on a smooth differentiable manifold $M$ of dimension $2n+1$ is a $(\phi, \xi, \eta, g)$-structure satisfying

(2.7) $\phi^2 = -\text{id} + \eta \otimes \xi, \; \eta(\xi) = 1,$

$$\phi^* g = g - \eta \otimes \eta,$$

where $\phi$ is a $(1, 1)$-type tensor field, $\xi$ is a tangent vector field called the characteristic or the Reeb vector field and $\eta$ is a 1-form called the almost contact form.

A Riemannian manifold $M$ furnished with an almost contact metric structure is said to be an almost contact metric manifold, denoted by $(M, \phi, \xi, \eta, g)$.

Let $M$ be an almost contact metric manifold of dimension $2n+1$. On the product $M \times \mathbb{R}$ there exists an almost complex structure $J$ defined by

$$J \left( X, f \frac{d}{dt} \right) = \left( \phi X - f \xi, \eta(X) \frac{d}{dt} \right),$$

where $X$ denotes a vector field tangent to $M^{2n+1}$, $t$ is the coordinate of $\mathbb{R}$ and $f$ is a $\mathcal{C}^\infty$-function on $M^{2n+1} \times \mathbb{R}$.

An almost contact metric manifold is said to be normal if the above almost complex structure $J$ is integrable. An almost contact metric manifold is said to be a trans-Sasakian manifold (see [21]) if it is normal and $d\eta = \alpha \Phi$, $d\Phi = 2\beta \eta \wedge \Phi$, where $\alpha = \frac{1}{2} \text{tr}(\phi \nabla \xi)$, $\beta = \frac{1}{2} \text{div} \xi$ and $\Phi(\cdot, \cdot) = g(\cdot, \phi \cdot)$. It is known that an almost contact metric manifold $M$ is trans-Sasakian if and only if there exist two smooth functions $\alpha$ and $\beta$ satisfying

(2.8) $(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X)$

for any vector fields $X$ and $Y$. 
Usually, a trans-Sasakian manifold is denoted by \((M, \phi, \xi, \eta, \alpha, \beta)\) and is called a trans-Sasakian manifold of type \((\alpha, \beta)\). From the definition of trans-Sasakian manifolds, putting \(Y = \xi\) in (2.8) and using (2.7) we have

\[
\nabla_X \xi = -\alpha \phi X + \beta (X - \eta(X) \xi)
\]

for any vector field \(X\).

By Propositions 1, 2 and Corollary 1 of [22], we observe that a normal almost contact metric 3-manifold is always trans-Sasakian. Therefore, by the definition of trans-Sasakian manifolds, we state that an almost contact metric 3-manifold is trans-Sasakian if and only if it is normal.

Note that a trans-Sasakian 3-manifold is an \(\alpha\)-Sasakian manifold if \(\alpha \in \mathbb{R}^\ast\) and \(\beta = 0\) (see [29]), a \(\beta\)-Kenmotsu manifold if \(\beta \in \mathbb{R}^\ast\) and \(\alpha = 0\) (see [29]), or a cosymplectic manifold if \(\alpha = \beta = 0\) (see [2]).

In this paper, all manifolds are assumed to be connected.

3. Minimal and harmonic Reeb vector fields on trans-Sasakian 3-manifolds

A trans-Sasakian manifold of type \((\alpha, \beta)\) is of \(C_6\)-class if \(\beta = 0\) (see [5]). As seen in [21], \(\alpha\) on a trans-Sasakian manifold of \(C_6\)-class of dimension greater than three is a constant. Then the trans-Sasakian manifolds of \(C_6\)-class of dimension greater than 3 are just \(\alpha\)-Sasakian manifolds. However, \(\alpha\) on a trans-Sasakian 3-manifold of \(C_6\)-class is not necessarily a constant.

A trans-Sasakian manifold of type \((\alpha, \beta)\) is of \(C_5\)-class if \(\alpha = 0\) (see [5]). On such manifolds of dimension greater than three there holds naturally \(d\beta \wedge \eta = 0\) (see [24]). However, the above equation does not necessarily hold for dimension three. The set of all trans-Sasakian manifolds of \(C_5\)-class contains the set of all \(\beta\)-Kenmotsu manifolds as its proper subset. For trans-Sasakian manifolds of \(C_5\)-class with non-constant function \(\beta\) we refer the reader to [1], [3], [6] and [24]. Note that a trans-Sasakian manifold of \(C_5\)-class of dimension greater than three is also called a \(f\)-cosymplectic manifold (see [1]) or a \(f\)-Kenmotsu manifold (see [24]).

It has been proved that the Reeb vector field of a Sasakian manifold is harmonic (see [31]) and also is a harmonic map (see [19]). It has been proved in [18, Theorem 2.2] that every unit strongly normal geodesic vector field is minimal. Applying this one knows that the Reeb vector field of a cosymplectic manifold is always minimal. Also, the Reeb vector of a \(\beta\)-Kenmotsu or a cosymplectic manifold is also harmonic (see [26]). For the minimality of the Reeb vector fields of almost Kenmotsu 3-manifolds and almost cosymplectic 3-manifolds we refer the reader to [30] and [27] respectively. In view of the above statements, in this paper we concentrate only on the study of the minimality and harmonicity of the Reeb vector fields on trans-Sasakian manifolds of dimension three.

In what follows, let \(M\) be a trans-Sasakian 3-manifold of type \((\alpha, \beta)\). In this section, we aim to give several equivalent conditions for the Reeb vector field of
$M$ to be minimal or harmonic. As an application, we also present a sufficient and necessary condition for the Reeb vector field of $M$ defining a harmonic map.

The following lemma was proved in [9, Theorem 3.2] (see also [15]).

**Lemma 3.1** ([9]). On a trans-Sasakian 3-manifold of type $(\alpha, \beta)$ we have

$$\xi(\alpha) + 2\alpha\beta = 0. \tag{3.1}$$

In this paper, we denote by $\nabla f$ the gradient of a smooth function $f$ on $M$. Moreover, putting $n = 1$ in [9, Proposition 3.4] we obtain the following lemma.

**Lemma 3.2** ([9]). On a trans-Sasakian 3-manifold of type $(\alpha, \beta)$ we have

$$Q\xi = \phi(\nabla\alpha) - \nabla\beta + (2(\alpha^2 - \beta^2) - \xi(\beta))\xi, \tag{3.2}$$

where $Q$ denotes the Ricci operator associated with the Ricci tensor $S$ which is defined by $S(X, Y) = \text{trace}(X \to R(X, Y))$.

On an $n$-dimensional Riemannian manifold $(M, g)$, the rough Laplacian operator $\bar{\Delta}$ acting on any smooth vector field $X$ is defined by

$$\bar{\Delta}X = \sum_{i=1}^{n} (\nabla_{e_i}\nabla_{e_i}X - \nabla_{\nabla_{e_i}e_i}X),$$

where $\{e_i : i = 1, \ldots, n\}$ is a local orthonormal frame on the manifold. If there exist a vector field $V$ and a smooth function $f$ such that $\bar{\Delta}V = fV$, we say that $V$ is an eigenvector field of $\bar{\Delta}$ with eigenfunction $f$.

**Lemma 3.3** ([12]). On a trans-Sasakian 3-manifold of type $(\alpha, \beta)$ we have

$$\bar{\Delta}\xi = -\phi(\nabla\alpha) + \nabla\beta - (2(\alpha^2 + \beta^2) + \xi(\beta))\xi. \tag{3.3}$$

Now we are ready to prove the following.

**Theorem 3.1.** On a trans-Sasakian 3-manifold $M$ the following six conditions are equivalent to each other.

1. The Reeb vector field is minimal.
2. The Reeb vector field is harmonic.
3. There hold $e(\alpha) - \phi e(\beta) = 0$ and $\phi e(\alpha) + e(\beta) = 0$ for any vector field $e$ orthogonal to the Reeb vector field.
4. There holds $\nabla\phi + \phi(\nabla\beta) + 2\alpha\beta\xi = 0 \iff \phi(\nabla\alpha) - \nabla\beta + \xi(\beta)\xi = 0$.
5. The Reeb vector field is an eigenvector field of the Ricci operator.
6. The Reeb vector field is an eigenvector field of the rough Laplacian $\bar{\Delta}$.

**Proof.** For each point $p$ on $M$, we may choose a local orthonormal frame $\{\xi, e, \phi e\}$ on certain neighbourhood $U$ of $p$. Using (2.7)-(2.9), we see that the Levi-Civita connection $\nabla$ of $M$ can be written as the following (see [9]):

$$\nabla_{\xi}e = 0, \quad \nabla_{\xi}e = \lambda e, \quad \nabla_{\xi}\phi e = -\lambda e,$$

$$\nabla_{e}\xi = \beta e - \phi e, \quad \nabla_{e}e = -\beta e + \gamma e, \quad \nabla_{e}\phi e = \alpha\xi - \gamma e,$$

$$\nabla_{\phi e}\xi = \alpha e + \beta\phi e, \quad \nabla_{\phi e}e = -\alpha\xi - \delta e, \quad \nabla_{\phi e}e = -\beta\xi + \delta e.$$
where \( \lambda, \gamma \) and \( \delta \) are smooth functions on \( U \).

From the last line of (3.4), \( \nabla \xi \) and its transpose can be written as the following forms:

\[
\nabla \xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & \alpha \\ 0 & -\alpha & \beta \end{pmatrix} \quad \text{and} \quad (\nabla \xi)^t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & -\alpha \\ 0 & \alpha & \beta \end{pmatrix}
\]

with respect to \( \{\xi, e, \phi e\} \), respectively. From (2.2), \( L_\xi \) can be written as the following

\[
L_\xi = E + (\nabla \xi)^t \circ \nabla \xi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha^2 + \beta^2 + 1 & 0 \\ 0 & 0 & \alpha^2 + \beta^2 + 1 \end{pmatrix}
\]

with respect to \( \{\xi, e, \phi e\} \). Therefore, we have \( f(\xi) = \sqrt{\text{det}(L_\xi)} = \alpha^2 + \beta^2 + 1 \).

Consequently, by (2.3) we have

\[
K_\xi = f(\xi)(L_\xi)^{-1} \circ (\nabla \xi)^t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & -\alpha \\ 0 & \alpha & \beta \end{pmatrix}
\]

In view of (3.5) and (3.7), we see that the two operators \( (\nabla \xi)^t \) and \( K_\xi \) are the same. Then, the equivalence between (1) and (2) follows from (2.4) and (2.5).

Now we compute the covariant derivative of \( K_\xi \) with respect to \( \{\xi, e, \phi e\} \).

\[
(\nabla_\xi K_\xi)e = (\xi(\beta)e + \xi(\alpha)\phi e, \\
(\nabla_e K_\xi)e = (\alpha^2 - \beta^2)\xi + e(\beta)e + e(\alpha)\phi e, \\
(\nabla_{\phi e} K_\xi)e = -2\alpha\beta\xi + \phi e(\beta)e + \phi e(\alpha)\phi e,
\]

where we have used (3.4). From (2.4) we have

\[
\omega_\xi(e) = \text{trace}(Y \rightarrow (\nabla_Y K_\xi)e) = e(\beta) + \phi e(\alpha).
\]

Similarly, we continue to compute the derivative of \( K_\xi \) with respect to \( \{\xi, e, \phi e\} \).

\[
(\nabla_\xi K_\xi)e = -\xi(\alpha)e + \xi(\beta)\phi e, \\
(\nabla_e K_\xi)e = 2\alpha\beta\xi - e(\alpha)e + e(\beta)\phi e, \\
(\nabla_{\phi e} K_\xi)e = (\alpha^2 - \beta^2)\xi - \phi e(\alpha)e + \phi e(\beta)\phi e,
\]

where we have used (3.4). From (2.4) we also have

\[
\omega_\xi(\phi e) = \text{trace}(Y \rightarrow (\nabla_Y K_\xi)\phi e) = -e(\alpha) + \phi e(\beta).
\]

Thus, from (2.4) (resp. (2.5)), the equivalence between (1) and (3) (resp. (2) and (3)) follows from (3.9) and (3.11).

Let \( e \) be a vector field orthogonal to \( \xi \). Thus, \( e(\beta) + \phi e(\alpha) = 0 \) is equivalent to \( g(\nabla \alpha + \phi \nabla \beta, \phi e) = 0 \) and \( e(\alpha) - \phi e(\beta) = 0 \) is equivalent to \( g(\nabla \alpha + \phi \nabla \beta, e) = 0 \).

Thus, we see that the Reeb vector field \( \xi \) is minimal or harmonic if and only if \( \nabla \alpha + \phi \nabla \beta \) is collinear with \( \xi \) and this is equivalent to \( \nabla \alpha + \phi \nabla \beta = \eta(\nabla \alpha + \phi \nabla \beta)\xi = -2\alpha \beta \xi \), where we have used Lemma 3.1. By (2.7), the action of \( \phi \) on
the previous relation gives \( \phi(\nabla \alpha) - \nabla \beta + \xi(\beta) \xi = 0 \). Conversely, the action of \( \phi \) on the previous relation gives that \( \nabla \alpha + \phi(\nabla \beta) + 2\alpha \beta \xi = 0 \).

If the Reeb vector field \( \xi \) is an eigenvector field of the Ricci operator, from Lemma 3.2 we have

\[
Q\xi = \eta(Q\xi)\xi = 2(\alpha^2 - \beta^2 - \xi(\beta))\xi.
\]

Comparing the above relation with (3.2) we obtain that
\[
\nabla \alpha + \phi(\nabla \beta) + 2\alpha \beta \xi = 0.
\]

Conversely, it is easy to check that the minimality or harmonicity of \( \xi \), together with (3.2), implies (3.12).

If the Reeb vector field \( \xi \) is an eigenvector field of the rough Laplacian \( \overline{\Delta} \), from Lemma 3.3 we have

\[
\overline{\Delta} \xi = \eta(\Delta \xi)\xi = -2(\alpha^2 + \beta^2)\xi.
\]

Comparing the above relation with (3.3) we obtain
\[
\phi(\nabla \alpha) - \nabla \beta + \xi(\beta) \xi = 0.
\]

Conversely, it is easy to check that the minimality or harmonicity of \( \xi \), together with (3.3), implies (3.13). This completes the proof.

An almost contact metric manifold is said to be \( \eta \)-Einstein if the Ricci operator is given by
\[
Q = a\text{id} + b\eta \otimes \xi,
\]
where \( a, b \) are smooth functions.

It has been proved in [9, Theorem 4.1] that the Ricci operator \( Q \) of a trans-Sasakian 3-manifold is given by

\[
Q = \left( \frac{r}{2} + \xi(\beta) - \alpha^2 - \beta^2 \right) \text{id} - \left( \frac{r}{2} + \xi(\beta) - 3\alpha^2 + 3\beta^2 \right) \eta \otimes \xi
\]

\[
+ \eta \otimes (\phi(\nabla \alpha) - \nabla \beta) - g(\nabla \beta - \phi(\nabla \alpha), \cdot) \otimes \xi.
\]

**Corollary 3.1.** On a trans-Sasakian 3-manifold the following three statements are equivalent to each other.

1. The Reeb vector field is minimal or harmonic.
2. The manifold is \( \eta \)-Einstein.
3. The Ricci operator is given as the following:

\[
Q = \left( \frac{r}{2} + \xi(\beta) - \alpha^2 - \beta^2 \right) \text{id} - \left( \frac{r}{2} + 3\xi(\beta) - 3\alpha^2 + 3\beta^2 \right) \eta \otimes \xi.
\]

**Proof.** The proof follows directly from Theorem 3.1 and (3.14). □

As another application of Theorem 3.1, we have:

**Theorem 3.2.** The Reeb vector field of a trans-Sasakian 3-manifold defines a harmonic map if and only if \( \nabla \alpha + \phi(\nabla \beta) + 2\alpha \beta \xi = 0 \) and \( \beta(\alpha^2 - \beta^2 - \xi(\beta)) = 0 \).

**Proof.** Applying relation (2.9) and Lemma 3.1, we compute

\[
R(X, Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{X,Y} \xi
\]

as the following:

\[
R(X, Y)\xi = Y(\alpha)\phi X - X(\alpha)\phi Y + X(\beta)(Y - \eta(Y))\xi
\]

\[
- Y(\beta)(X - \eta(X))\xi + (\alpha^2 - \beta^2)(\eta(Y) X - \eta(X) Y)
\]

\[
+ 2\alpha \beta(\eta(Y) \phi X - \eta(X) \phi Y)
\]

(3.15)
for any vector fields $X, Y$. Using (3.15) and Lemma 3.1, we obtain $R(\phi X, \xi) = (\alpha^2 - \beta^2 - \xi(\beta))\phi X$ for any vector field $X$. Thus, applying again (2.9), from (2.6) and Lemma 3.1 we obtain

$$R(\phi X, \xi) = \alpha \phi X - \beta X, \xi$$

Thus, applying again (2.9), from (2.6) and Lemma 3.1 we obtain

$$\rho(\xi) = -\operatorname{trace}\{Y \rightarrow R(\phi Y, \xi)\} + \beta g(\xi, \xi) = 2\beta(\alpha^2 - \beta^2 - \xi(\beta)).$$

Similarly, applying (3.15) we obtain $R(X, e) = e(\alpha)\phi X - e(\beta)X - (X(\alpha) + 2\alpha\beta\eta(X))\phi e + (X(\beta) + (\beta^2 - \alpha^2)\eta(X))e + e(\beta)\eta(X)\xi$. Thus, applying again (2.9), from (2.6) and Lemma 3.1 we obtain

$$\rho(e) = -\operatorname{trace}\{Y \rightarrow R(\phi Y, \xi)e\} + \beta g(\xi, e)$$

$$= -\alpha \sum_{i=1}^{3} g(R(E_i, e)\xi, \phi E_i) + \beta g(\xi, e)$$

$$= -\alpha(\phi e(\beta) - e(\alpha)) - \beta(\phi e(\alpha) + e(\beta)),$$

where $\{E_1, E_2, E_3\}$ is a local orthonormal frame of the tangent space at a point of the manifold.

Similarly, using (3.15) we obtain $R(X, \phi e) = \phi e(\alpha)\phi X - \phi e(\beta)X + (X(\alpha) + 2\alpha\beta\eta(X))\phi e + (X(\beta) + (\beta^2 - \alpha^2)\eta(X))\phi e + \phi e(\beta)\eta(X)\xi$. Thus, applying again (2.9), from (2.6) and Lemma 3.1 we obtain

$$\rho(\phi e) = -\operatorname{trace}\{Y \rightarrow R(\phi Y, \xi)\phi e\} + \beta g(\xi, \phi e)$$

$$= -\alpha \sum_{i=1}^{3} g(R(E_i, \phi e)\xi, \phi E_i) + \beta g(\xi, \phi e)$$

$$= -\alpha(\phi e(\alpha) + e(\beta)) + \beta(\phi e(\beta) - e(\alpha)).$$

Thus, the proof follows from (2.6), (3.16)-(3.18) and Theorem 3.1. □

The following two corollaries follows directly from Theorems 3.1 and 3.2.

**Corollary 3.2.** The Reeb vector field of a 3-dimensional $\beta$-Kenmotsu manifold is harmonic but is never a harmonic map.

**Corollary 3.3.** The Reeb vector field of a 3-dimensional $\alpha$-Sasakian manifold or cosymplectic manifold is a harmonic map.

### 4. Examples

Except for the above three typical examples of trans-Sasakian manifolds, one would like to know the minimality and harmonicity of non-proper trans-Sasakian 3-manifolds. Next, we construct some concrete examples to illustrate our main results.

**Example 4.1.** Let $(x, y, z)$ be the standard Cartesian coordinates of $\mathbb{R}^3$. We consider a manifold $M$ defined by $M := \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$. Let

$$e_1 = z \left( \frac{\partial}{\partial x} + y \frac{\partial}{\partial z} \right), \quad e_2 = z \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}.$$
Let $g$ be the Riemannian metric defined on $M$ by $g(e_i, e_j) = \delta_{ij}$, $i, j \in \{1, 2, 3\}$, where $\delta_{ij}$ denotes the Kronecker symbol. On $M$ we can define an almost contact metric structure $(\phi, \xi, \eta, g)$ as the following:

$$
\xi = e_3, \eta(\cdot) = g(e_3, \cdot), \phi e_1 = e_2, \phi e_2 = -e_1, \phi \xi = 0.
$$

It has been proved in [6, p. 798] that $M$ is a trans-Sasakian 3-manifold of type $(-\frac{1}{2}z^2, -\frac{1}{2}z)$. From Theorem 3.1, it is easy to see that the Reeb vector field of $M$ is neither harmonic nor a harmonic map.

**Example 4.2.** Let $(x, y, z)$ be the canonical Cartesian coordinates in $\mathbb{R}^3$. We can define on $\mathbb{R}^3$ as almost contact metric structure $(\phi, \xi, \eta, g)$ as the following:

$$
\xi = \frac{\partial}{\partial z}, \eta = dz - ydx,
$$

$$
\phi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & y & 0 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} e^2 + y^2 & 0 & -y \\ 0 & e^2 & 0 \\ -y & 0 & 1 \end{pmatrix}.
$$

It has been proved in [3, p. 202] that $(\mathbb{R}^3, \phi, \xi, \eta, g)$ is a trans-Sasakian 3-manifold of type $(-\frac{1}{2}z^2, \frac{1}{2}z)$. Obviously, from Theorem 3.2, we see that the Reeb vector field of this structure is neither harmonic nor a harmonic map.

**Lemma 4.1** ([21]). Let $(M, \phi, \xi, \eta, g)$ be a Sasakian 3-manifold and $f$ be a non-constant positive function on $M$. Then, $(M, \phi, \xi, \eta, g')$ is a trans-Sasakian 3-manifold of type $(\frac{1}{f}, \frac{1}{27}(f))$, where the Riemannian metric $g'$ is defined by $g' = fg + (1 - f)\eta \otimes \eta$.

Applying the above lemma, now we construct a large class of trans-Sasakian 3-manifolds whose Reeb vector fields may be harmonic or harmonic maps. Firstly, let us recall the following well known examples of trans-Sasakian 3-manifolds.

**Example 4.3.** Let $(x, y, z)$ be the canonical Cartesian coordinates in $\mathbb{R}^3$. On $\mathbb{R}^3$ there exists a standard Sasakian structure (see Blair [2, p. 60]) defined as the following:

$$
\xi = 2 \frac{\partial}{\partial z}, \eta = \frac{1}{2}(dz - ydx),
$$

$$
\phi = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & y & 0 \end{pmatrix} \quad \text{and} \quad g = \frac{1}{4} \begin{pmatrix} 1 + y^2 & 0 & -y \\ 0 & 1 & 0 \\ -y & 0 & 1 \end{pmatrix}.
$$

The orthonormal $\phi$-basis is given by $\{\xi, e_1 := 2 \frac{\partial}{\partial y}, e_2 := \phi e_1 = 2(\frac{\partial}{\partial z} + y \frac{\partial}{\partial z})\}$. Let $f$ be a positive function on $\mathbb{R}^3$. From Lemma 4.1, $(\mathbb{R}^3, \phi, \xi, \eta, g')$ is a trans-Sasakian 3-manifold of type $(\frac{1}{f}, \frac{1}{27}(f))$, where $g' = fg + (1 - f)\eta \otimes \eta$. 
Proposition 4.1. The Reeb vector field of the trans-Sasakian 3-manifold defined in Example 4.3 is minimal or harmonic if and only if the following system of partial differential equations hold:

\[(4.1) \quad y \left( \frac{\partial f}{\partial z} \right)^2 - fy \frac{\partial^2 f}{\partial z^2} + \frac{\partial f}{\partial x} \frac{\partial f}{\partial z} - f \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial f}{\partial y} = 0, \]

\[(4.2) \quad f \frac{\partial^2 f}{\partial z \partial y} - \frac{\partial f}{\partial x} - y \frac{\partial f}{\partial z} - \frac{\partial f}{\partial y} \frac{\partial f}{\partial z} = 0. \]

Proof. As seen before, the trans-Sasakian 3-manifold defined in Example 4.3 is of type \((\frac{1}{f}, f\frac{\partial f}{\partial z}(f))\). Applying \(\alpha = \frac{1}{f}\) and \(\beta = \frac{1}{f^2} \frac{\partial f}{\partial z}\), and choosing a local orthogonal \(\phi\)-basis \(\{\xi, e : = \frac{\partial}{\partial y}, \phi e = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}\}\), then the remaining proof follows directly from Theorem 3.1. Notice that the above basis is not necessarily orthonormal for the metric \(g'\). \(\square\)

Proposition 4.2. The Reeb vector field of the trans-Sasakian 3-manifold defined in Example 4.3 is a harmonic map if and only if either \(4.1, 4.2\) and the following partial differential equation

\[(4.3) \quad 1 + \left( \frac{\partial f}{\partial z} \right)^2 - 2f \frac{\partial^2 f}{\partial z^2} = 0\]

hold or the manifold is \(\alpha\)-Sasakian or a cosymplectic manifold.

Proof. Suppose that the Reeb vector field \(\xi\) of the trans-Sasakian 3-manifold is a harmonic map. Notice that the second condition of Theorem 3.2 is equivalent to either \(\beta = 0\) or \(\alpha^2 - \beta^2 - \xi(\beta) = 0\) holds on certain open subset of the manifold. From the first condition of Theorem 3.2, \(\beta = 0\) implies that \(\alpha\) is a constant. In this context, the manifold is \(\alpha\)-Sasakian if \(\alpha \in \mathbb{R} - \{0\}\) or cosymplectic if \(\alpha = 0\). The remaining proof follows from the Theorem 3.2 and Proposition 4.1. \(\square\)

We now construct some concrete non-proper trans-Sasakian 3-manifolds with minimal or harmonic Reeb vector fields.

Example 4.4. Let \(M := \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}\) and \((x, y, z)\) be the standard Cartesian coordinates of \(\mathbb{R}^3\). On \(M\) we consider three vector fields defined as the following:

\[e_1 = \frac{x}{z} \frac{\partial}{\partial x}, \quad e_2 = \frac{y}{z} \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}.\]

Now we define a trans-Sasakian structure \((M, \phi, \xi, \eta, g)\) on \(M\) as the following:

\[\xi = e_3, \quad \eta = g(e_3, \cdot),\]

\[\phi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\]

with respect to the basis \(\{e_1, e_2, e_3\}\).
It has been proved in [10, p. 262] that \((M, \phi, \xi, \eta, g)\) is a trans-Sasakian 3-manifold of type \((-1, \frac{1}{z})\). Obviously, by Theorems 3.1 and 3.2, one observes that the Reeb vector field of \(M\) is minimal (or harmonic) but is never a harmonic map.

**Example 4.5.** Let \(M := \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}\) and \((x, y, z)\) be the standard Cartesian coordinates of \(\mathbb{R}^3\). On \(M\) we consider three vector fields defined as the following:

\[
e_1 = e^x \frac{\partial}{\partial x}, \quad e_2 = e^y \frac{\partial}{\partial y}, \quad e_3 = -\frac{\partial}{\partial z}.
\]

Now we define a trans-Sasakian structure \((M, \phi, \xi, \eta, g)\) on \(M\) as the following:

\[
\xi = e_3, \quad \eta = g(e_3, \cdot),
\]

\[
\phi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

with respect to the base \(\{e_1, e_2, e_3\}\).

It has been proved in [28, p. 161] that \((M, \phi, \xi, \eta, g)\) is a trans-Sasakian 3-manifold of type \((\frac{1}{z}, -1)\). Obviously, from Theorems 3.1 and 3.2, it is easy to see that the Reeb vector field of \(M\) is minimal (or harmonic) but is never a harmonic map.

Finally, before closing this section, we construct a large class of trans-Sasakian 3-manifolds whose Reeb vector fields are either harmonic or harmonic maps.

**Example 4.6.** Let \((x, y, z)\) be the standard Cartesian coordinates of \(\mathbb{R}^3\). On \(\mathbb{R}^3\) we consider a Riemannian metric \(g\) as the following:

\[
g = \frac{1}{e^{2f(z)}} \, dx \otimes dx + \frac{1}{e^{2f(z)}} \, dy \otimes dy + dz \otimes dz,
\]

where \(f(z)\) is a non-constant smooth function on \(\mathbb{R}^3\). From the above metric, an orthonormal \(\phi\)-basis \(\{e_1, e_2, e_3\}\) of \(\mathbb{R}^3\) is given by

\[
e_1 = e^{f(z)} \frac{\partial}{\partial x}, \quad e_2 = e^{f(z)} \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}.
\]

Now we define a trans-Sasakian structure \((\phi, \xi, \eta, g)\) on \(\mathbb{R}^3\) as the following:

\[
\xi = e_3, \quad \eta = g(e_3, \cdot), \quad \phi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

with respect to the \(\phi\)-basis \(\{e_1, e_2, e_3\}\). According to a direct calculation, we see that \((\mathbb{R}^3, \phi, \xi, \eta, g)\) is a trans-Sasakian 3-manifold of type \((0, -f'(z))\).

**Remark 4.1.** Example 4.6 corrects a little mistake in [4, Example 3.9]. Moreover, if we take \(f(z) = 2 \ln z\), Example 4.6 is just that of in [33, Section 3]. Also, if we take \(f(z) = z^2\), Example 4.6 is just that of in [1, Section 5.2].
Applying Theorems 3.1 and 3.2, we have the following two propositions.

**Proposition 4.3.** The Reeb vector field of the trans-Sasakian structure defined in Example 4.6 is always minimal or harmonic.

**Proposition 4.4.** The Reeb vector field of the trans-Sasakian structure defined in Example 4.6 defines a harmonic map if and only if

\[(f'(z))^2 - f''(z) = 0,\]

or equivalently,

\[f(z) = -\ln(c_1 + c_2 z),\]

where \(c_1\) and \(c_2\) are arbitrary constants.

## 5. Trans-Sasakian 3-manifolds homothetic to Sasakian or cosymplectic manifolds

In this section, we obtain some applications of main results shown in Section 3 and characterize some new conditions for a compact trans-Sasakian 3-manifold being proper.

First, we need the following lemma which is useful in proofs of main results.

**Lemma 5.1.** Let \(M\) be a compact trans-Sasakian 3-manifold of type \((\alpha, \beta)\) such that the Reeb vector field is minimal or harmonic. Then, \(\alpha\) is a constant.

**Proof.** Suppose that the Reeb vector field \(\xi\) is minimal or harmonic, by Theorem 3.1 we have

\[(5.1) \quad e(\alpha) - \phi e(\beta) = 0\]

and

\[(5.2) \quad e(\alpha) + e(\beta) = 0.\]

for any unit vector field \(e\) orthogonal to \(\xi\). Applying (3.4), (5.1) and Lemma 3.1, now we compute the usual Laplacian of \(\alpha\) as the following:

\[
\Delta \alpha = \nabla_e \nabla \alpha - (\nabla_e e)\alpha + \nabla \phi \nabla e \alpha - (\nabla \phi e)\alpha + \nabla \xi \nabla \alpha - (\nabla \xi \alpha)
\]

\[
= e(\phi(\beta)) - (\gamma \phi(\beta)\alpha - \phi(\beta)) - (\delta \phi(\beta\xi) + \xi(\phi(-2\alpha \beta))
\]

\[
= (\nabla \phi e - \nabla \phi e)(\beta + \gamma \phi(\alpha) - \delta e(\alpha) - 2\alpha \xi(\beta)
\]

\[
= 0.
\]

Since \(M\) is assumed to be compact, from (5.2) we see that \(\alpha\) is a constant. \(\square\)

From Theorem 3.1 and Lemma 5.1, we have:

**Corollary 5.1.** The Reeb vector field \(\xi\) of a compact trans-Sasakian 3-manifold is minimal or harmonic if and only if \(\beta\) is invariant along the distribution orthogonal to \(\xi\).

In order to give applications, we need the following lemma.

**Lemma 5.2.** Let \((M, g)\) be a Riemannian manifold. If \(M\) admits a Killing vector field \(\xi\) of constant length satisfying

\[k^2(\nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y} \xi) = g(Y, \xi) X - g(X, Y) \xi\]

for a non-zero constant \(k\) and any vector fields \(X, Y\), then \(M\) is homothetic to a Sasakian manifold.
Theorem 5.1. Let $M$ be a compact trans-Sasakian 3-manifold of type $(\alpha, \beta)$. Then the Reeb vector field defines a harmonic map if and only if $M$ is homothetic to either a Sasakian or a cosymplectic manifold.

Proof. Let the Reeb vector field $\xi$ of $M$ be a harmonic map. By Lemma 5.1 and Theorem 3.2 we have that $\alpha$ is a constant and $\beta(\alpha^2 - \beta^2 - \xi(\beta)) = 0$. If $\alpha \neq 0$, by Lemma 3.1 we have $\beta = 0$ and hence by Lemma 5.2 we see that $M$ is homothetic to a Sasakian manifold.

Next we consider the other case $\alpha = 0$. Suppose that $\beta = 0$, then in this context $M$ is homothetic to a cosymplectic manifold. Now suppose that $M$ is not a cosymplectic manifold, by Theorem 3.2 we have

\begin{equation}
(5.3) \quad \xi(\beta) = -\beta^2.
\end{equation}

From (2.9) we see that $\text{div}\xi = 2\beta$. Thus, using (5.3) we compute the divergence of $\beta \xi$ as the following:

\begin{equation}
(5.4) \quad \text{div}(\beta \xi) = \xi(\beta) + \beta \text{div} \xi = \beta^2.
\end{equation}

Applying the divergence theorem on compact manifold $M$, we observe that $\beta$ is zero. By the above analyses, we see that $\alpha = 0$ implies only that $M$ is homothetic to a cosymplectic manifold.

The converse follows from Corollary 3.3. This completes the proof. $\square$

Remark 5.1. Theorem 5.1 generalizes Theorem 4.1 of [15] because the vanishing of $Q \xi$ implies that $\xi$ defines a harmonic map. Moreover, Theorem 5.1 extends Theorem 3.1 of [7] because $\xi$-projective flatness of trans-Sasakian 3-manifolds implies that $\xi$ is minimal or harmonic. From Corollary 3.1, since the minimality or harmonicity of $\xi$ implies that the manifold is $\eta$-Einstein, then Theorem 5.1 extends also Theorem 6.1 of [6].

In fact, Theorem 5.1 can be improved as the following two forms.

Corollary 5.2. Let $M$ be a compact trans-Sasakian 3-manifold of type $(\alpha, \beta)$. The Reeb vector field is an eigenvector field of the Ricci operator satisfying $Q \xi = \lambda \xi$ and $\xi(\lambda) = 0$ for a smooth function $\lambda$ if and only if $M$ is homothetic to either a Sasakian or a cosymplectic manifold.

Proof. Suppose that $\xi$ is an eigenvector field of the Ricci operator satisfying $Q \xi = \lambda \xi$ and $\xi(\lambda) = 0$. From Theorem 3.1, $Q \xi = \lambda \xi$ is equivalent to that $\xi$ is minimal or harmonic and $Q \xi = 2(\alpha^2 - \beta^2 - \xi(\beta)) \xi$. On the other hand, by Theorem 5.1, we know $\alpha$ is a constant. Thus, $\xi(\lambda) = 0$ is equivalent to

\begin{equation}
(5.5) \quad 2\beta \xi(\beta) + \xi(\xi(\beta)) = 0.
\end{equation}

Since $\xi$ is harmonic and $\alpha$ is a constant, from Theorem 3.1 we have $\nabla \beta = \xi(\beta) \xi$. Using this and (5.5) we compute the usual Laplacian of $\beta$ as the following:

\begin{equation}
(5.6) \quad \Delta \beta = \text{div}(\nabla \beta) = \sum_{i=1}^{3} g(E_i(\xi(\beta)) \xi + \xi(\beta)(-\phi E_i + \beta E_i - \beta \eta(E_i) \xi), E_i)
= \xi(\xi(\beta)) + 2\beta \xi(\beta) = 0.
\end{equation}
Since $M$ is compact, then the harmonicity of $\beta$ means that it is a constant. By Lemma 3.1 we have $\alpha \beta = 0$ and hence $\beta = 0$. In fact, if $\beta \neq 0$ we have that $\text{div} \xi = 2\beta$ is a non-zero constant, this contradicts the compactness of $M$. The remaining proof follows from Lemma 5.2. The converse follows from Corollary 3.3. This completes the proof.

The above result generalizes Theorem 3.2 of [15].

**Corollary 5.3.** Let $M$ be a compact trans-Sasakian 3-manifold of type $(\alpha, \beta)$. The Reeb vector field is an eigenvector field of the rough Laplacian operator $\Delta$ satisfying $\Delta \xi = \lambda \xi$ and $\xi(\lambda) = 0$ for a smooth function $\lambda$ if and only if $M$ is homothetic to either a Sasakian or a cosymplectic manifold.

**Proof.** Suppose that $\xi$ is an eigenvector field of the rough Laplacian operator $\Delta$ satisfying $\Delta \xi = \lambda \xi$ and $\xi(\lambda) = 0$. From Theorem 3.1, $\Delta \xi = \lambda \xi$ is equivalent to that $\xi$ is minimal or harmonic and $\Delta \xi = -2(\alpha^2 + \beta^2)\xi$. Thus, by Theorem 5.1, we know $\alpha$ is a constant. Now, $\xi(\lambda) = 0$ is equivalent to

$$2\beta \xi(\beta) = 0.$$  

Using this and (5.7) we compute the divergence of $\beta^2 \xi$ as the following:

$$\text{div}(\beta^2 \xi) = 3 \beta^2 \xi(\beta) + \beta^3 \text{div} \xi = 2\beta^4,$$

where we have used $\text{div} \xi = 2\beta$. Since $M$ is compact, applying the divergence theorem on (5.8), we obtain $\beta = 0$. The remaining proof follows from Lemma 5.2. The converse follows from Corollary 3.3. This completes the proof. □

The above result generalizes Theorem 3.1 of [12].

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