Generalized Derivations on ∗-prime Rings

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Abstract. Let I be a ∗-ideal on a 2-torsion free ∗-prime ring and F : R → R a
generalized derivation with an associated derivation d : R → R. The aim of this paper is
to explore the condition under which generalized derivation F becomes a left centralizer
i.e., associated derivation d becomes a trivial map (i.e., zero map) on R.

1. Introduction

Let R be an associative ring. A mapping ∗ : R → R is said to be an involution on
R if
(i) ∗∗(x) = x,
(ii) ∗(x + y) = ∗(x) + ∗(y) and
(iii) ∗(xy) = ∗(y) ∗(x)
hold for all x, y ∈ R. A ring R equipped with an involution ‘∗’ is said to be ring with
involution. A ring R with involution ‘∗’ is said to be ∗-prime if aRb = aR ∗(b) = {0}
implies that either a = 0 or b = 0. Note that every prime ring with involution ‘∗’ is ∗-
prime but the converse is not true in general. It can be easily seen that a ∗-prime ring
is semiprime. To show this, let aRa = {0}. Therefore, we have ar ∗(a)sar ∗(a) = 0
for all r, s ∈ R. Also ar ∗(a)s ∗(ar ∗(a)) = ar ∗(a)sar ∗(r) ∗(a) = 0 for all r, s ∈ R.
Hence by ∗-primeness, we get ar ∗(a) = 0. Also, we have assumed that ara = 0.
Therefore, by ∗-primeness, we find that a = 0, as required. An additive mapping
d : R → R is called a derivation if d(xy) = d(x)y + xd(y) for all x, y ∈ R. An
additive mapping F : R → R is called a generalized derivation if there exists a

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derivation \( d : R \to R \) such that \( F(xy) = F(x)y + xd(y) \) for all \( x, y \in R \). An additive mapping \( F : R \to R \) is called a left centralizer if \( F(xy) = F(x)y \) for all \( x, y \in R \). The situation when generalized derivation becomes a left centralizer arises when associated derivation is trivial map i.e., zero map. In this paper, our aim is to discuss the situation when derivation associated with the generalized derivation on an appropriate subset of a \(*\)-prime ring \( R \) becomes zero i.e., generalized derivation becomes a left centralizer.

2. Identities Related to Generalized Derivations on \(*\)-ideals of \(*\)-prime Rings

Over the last three decades, several authors have explored various identities involving automorphisms or derivations on an appropriate subset of a prime or semiprime ring (see [1, 2, 3, 4, 5], where further references can be found). The purpose of this section is to prove some results which are of independent interest and related to generalized derivations on \(*\)-prime rings. We begin this section with the following well known result which are needed for developing the proof of the results presented in this section.

**Lemma 2.1.** Let \( R \) be a \(*\)-prime ring and \( I \) a nonzero \(*\)-ideal of \( R \). If \( aIb = aI(b) = \{0\} \) or \( aIb = *(a)Ib = \{0\} \) for all \( a, b \in I \), then either \( a = 0 \) or \( b = 0 \).

**Proof.** Given that \( aIb = aI*(b) = \{0\} \) for any \( a, b \in R \). Hence, \( axrb = axr*(b) = 0 \) for all \( a, x \in I \) and \( r \in R \). Since \( R \) is \(*\)-prime, we find that either \( ax = 0 \) for all \( x \in I \) or \( b = 0 \). But, since \( I \) is a \(*\)-ideal, \( ax = 0 \) implies that \( ax = ar*(x) = 0 \) for all \( x \in I \) and \( r \in R \). Now, \(*\)-primeness of \( R \) yields that either \( I = \{0\} \) (a contradiction) or \( a = 0 \). Similarly, we can prove that if \( aIb = *(a)Ib = \{0\} \) for all \( a, b \in I \), then either \( a = 0 \) or \( b = 0 \). \( \square \)

**Theorem 2.2.** Let \( R \) be a noncommutative \(*\)-prime ring and \( I \) a nonzero \(*\)-ideal of \( R \). Suppose \( F_1, F_2 : R \to R \) are two generalized derivations with associated derivations \( d_1, d_2 : R \to R \) respectively such that \( d_1F_1 \neq 0 \) on \( I \) and \( * \) commutes with \( d_2, F_1 \). If \( F_1(x) \circ F_2(y) = 0 \) for all \( x, y \in I \) or \( [F_1(x), F_2(y)] = 0 \) for all \( x, y \in I \), then \( d_2 = 0 \).

**Proof.** Replace \( y \) by \( yr \) in \( F_1(x) \circ F_2(y) = 0 \) and also use this condition, to get

\[
F_2(y)[r, F_1(x)] + y(F_1(x) \circ d_2(r)) + [F_1(x), y]d_2(r) = 0 \text{ for all } x, y \in I \text{ and } r \in R.
\]

Replacing \( r \) by \( F_1(x) \) in the above relation, we find that

\[
y(F_1(x) \circ d_2(F_1(x))) + [F_1(x), y]d_2(F_1(x)) = 0 \text{ for all } x, y \in I.
\]

Now, replacing \( y \) by \( yz \), we get

\[
[F_1(x), y]zd_2(F_1(x)) = 0 \text{ for all } x, y, z \in I.
\]

Also, replacing \( x \) by \( x + t \) in (2.2) and using (2.2), we get

\[
[F_1(x), y]zd_2(F_1(t)) + [F_1(t), y]zd_2(F_1(x)) = 0 \text{ for all } x, y, z, t \in I.
\]
Using (2.2) and (2.3), we find that
\[
[F_1(x), y]zd_2(F_1(t)) + R[F_1(x), y]zd_2(F_1(t)) = -[F_1(t), y]zd_2(F_1(x))R[F_1(x), y]zd_2(F_1(t)) = \{0\} \text{ for all } x, y, z, t \in I.
\]

Since \(\ast\)-primeness of \(R\) implies semiprimeness of \(R\), we obtain that
\[
[F_1(x), y]zd_2(F_1(t)) = 0 \text{ for all } x, y, z, t \in I.
\]

Since \(I\) is \(\ast\)-ideal and \(\ast\) commutes with \(F_1\), we get
\[
*((F_1(x), y))zd_2(F_1(t)) = 0 \text{ for all } x, y, z, t \in I.
\]

Also, \(R\) is \(\ast\)-prime, we get either \([F_1(x), y] = 0\) for all \(x, y \in I\) or \(d_2(F_1(t)) = 0\) for all \(t \in I\). Take \([F_1(x), y] = 0\) for all \(x, y \in I\). Replace \(y\) by \(yr\) to get \(y[F_1(x), r] = 0\) for all \(x, y \in I\) and \(r \in R\) and hence \(yz[F_1(x), r] = 0\) for all \(x, y, z \in I\) and \(r \in R\).

Since \(I\) is a \(\ast\)-ideal, we get \(*y)z[F_1(x), r] = 0\) for all \(x, y, z \in I\) and \(r \in R\). By Lemma 2.1, we get \([F_1(x), r] = 0\) for all \(x \in I\) and \(r \in R\). Further, replacing \(x\) by \(xr\), we find that
\[
[x, r]d_4(r) + x[d_4(r), r] = 0 \text{ for all } x \in I \text{ and } r \in R.
\]

Replacing \(x\) by \(xz\) in (2.5) and using (2.5), we get \([x, r]zd_4(r) = 0\) for all \(x, z \in I\) and \(r \in R\). Substituting \(r\) by \(r + r_1\), we find that \([x, r]zd_4(r_1) + [x, r_1]zd_4(r) = 0\) for all \(x, z \in I\) and \(r, r_1 \in R\). Therefore \([x, r]zd_4(r_1)R[x, r]zd_4(r_1) = -[x, r_1]zd_4(r)R[x, r]zd_4(r_1) = \{0\}\). Hence, we get \([x, r]zd_4(r_1) = 0\) for all \(x, z \in I\) and \(r, r_1 \in R\). Since \(I\) is a \(\ast\)-ideal, by Lemma 2.1, we have either \([x, r] = 0\) or \(d_4(r_1) = 0\). But \([x, r] = 0\) implies that \([x, s, r] = 0\) for all \(x \in I\), \(r, s \in R\). Since \(I\) is nonzero \(\ast\)-ideal, we find that \(xt[s, r] = *x[t[s, r] = 0\) for all \(x \in I\), \(r, s, t \in R\). Since \(R\) is \(\ast\)-prime and \(I\) is nonzero, we find that \(R\) is commutative which leads to a contradiction. Also, \(d_4(r_1) = 0\) implies \(d_4 = 0\) which is again a contradiction to the assumption that \(d_4F_1 \neq 0\).

Now, take \(d_2(F_1(x)) = 0\) for all \(x \in I\). Replace \(x\) by \(xr\), we get
\[
F_1(x)d_2(r) + d_2(x)d_4(r) + xzd_2(d_4(r)) = 0 \text{ for all } x \in I \text{ and } r \in R.
\]

Replacing \(r\) by \(F_1(y)\), we get
\[
d_2(x)d_4(F_1(y)) + xzd_2(d_4(F_1(y))) = 0 \text{ for all } x, y \in I.
\]

Replace \(x\) by \(xz\) in (2.6) and using (2.6), we find that \(d_2(x)zd_4(F_1(y)) = 0\) for all \(x, y, z \in I\). Since \(I\) is a \(\ast\)-ideal and \(\ast\) commutes with \(d_2\), we get \(*d_2(x)zd_4(F_1(y)) = 0\) for all \(x, y, z \in I\). Using \(\ast\)-primeness of \(R\) and Lemma 2.1, we find that either \(d_2 = 0\) or \(d_1F_1 = 0\) on \(I\), a contradiction. But \(d_2 = 0\) on \(I\) implies that \(d_2 = 0\) on \(R\).

Replace \(y\) by \(yr\) in \([F_1(x), F_2(y)] = 0\) to get
\[
F_2(y)[F_1(x), r] + [F_1(x), y[d_2(r)] + y[F_1(x), d_2(r)] = 0 \text{ for all } x, y \in I \text{ and } r \in R.
\]
Now, replacing \( r \) by \( F_1(x) \), we get

\[
[ F_1(x), y ]d_2(F_1(x)) + y[F_1(x), d_2(F_1(x))] = 0 \quad \text{for all} \quad x, y \in I.
\]

Replace \( y \) by \( yz \) in (2.7) and use (2.7), to find that \([ F_1(x), y ]zd_2(F_1(x)) = 0 \) for all \( x, y, z \in I \). Using similar arguments as used after (2.2), we get that \( d_2 = 0 \). \( \square \)

**Theorem 2.3.** Let \( R \) be a noncommutative *-prime ring and \( I \) be a nonzero *-ideal in \( R \). Suppose \( F_1, F_2 : R \rightarrow R \) are two generalized derivations with associated derivations \( d_1, d_2 : R \rightarrow R \) respectively such that \( d_1F_1 \neq 0 \) on \( I \) and * commutes with \( d_2, F_1 \). If \( F_1(x) \circ F_2(y) = x \circ y \) for all \( x, y \in I \) or \([ F_1(x), F_2(y) ] = [ x, y ] \) for all \( x, y \in I \), then \( d_2 = 0 \).

**Proof.** It is given that \( F_1(x) \circ F_2(y) = x \circ y \) holds for all \( x, y \in I \). Replace \( y \) by \( yr \) for \( y \in I \) and \( r \in R \), to get

\[
(F_1(x) \circ F_2(y))r + F_2(y)[r, F_1(x)] + [F_1(x), y]d_2(r) + y(F_1(x) \circ d_2(r)) = (x \circ y)r + yr + [r, x].
\]

Application of given condition yields that

\[
F_2(y)[r, F_1(x)] + [F_1(x), y]d_2(r) + y(F_1(x) \circ d_2(r)) = yr + [r, x] \quad \text{for all} \quad x, y \in I \quad \text{and} \quad r \in R.
\]

Replacing \( r \) by \( F_1(x) \), we get

\[
[F_1(x), y]d_2(F_1(x)) + y(F_1(x) \circ d_2(F_1(x))) = [F_1(x), x] \quad \text{for all} \quad x, y \in I.
\]

Replacing \( y \) by \( yz \) in (2.9) and using (2.9), we have

\[
[F_1(x), y]zd_2(F_1(x)) = 0 \quad \text{for all} \quad x, y, z \in I.
\]

Arguing with similar lines as used after (2.2), we get the required result.

Take \([ F_1(x), F_2(y) ] = [ x, y ] \) for all \( x, y \in I \). Replace \( y \) by \( yr \) for \( y \in I \), \( r \in R \) and use similar arguments as used above with necessary variations, we get the required result. \( \square \)

**Theorem 2.4.** Let \( R \) be a noncommutative *-prime ring and \( I \) a nonzero *-ideal in \( R \). Suppose \( F_1, F_2 : R \rightarrow R \) are two generalized derivations with associated derivations \( d_1, d_2 : R \rightarrow R \) respectively and * commutes with \( d_2, F_1 \). If \( F_1(x)F_2(y) \pm xy \in Z(R) \) for all \( x, y \in I \), then either \( d_1 = 0 \) or \( d_2 = 0 \).

**Proof.** Replace \( y \) by \( yz \) in \( F_1(x)F_2(y) \pm xy \in Z(R) \) for \( y, z \in I \), we find that

\[
(F_1(x)F_2(y) \pm xy)z + F_1(x)yd_2(z) \in Z(R) \quad \text{for all} \quad x, y, z \in I.
\]

On commuting with \( z \), we get

\[
[F_1(x)F_2(y) \pm xy, z]z + [F_1(x), z]yd_2(z) + F_1(x)[yd_2(z), z] = 0 \quad \text{for all} \quad x, y, z \in I.
\]

Using the given hypothesis we obtain

\[
[F_1(x), z]yd_2(z) + F_1(x)[yd_2(z), z] = 0 \quad \text{for all} \quad x, y, z \in I.
\]
Replacing $y$ by $F_1(x)y$ in (2.10), we find that
\[(2.11)\]
\[F_1(x), z|F_1(x)yd_2(z)+F_1(x)(F_1(x)|yd_2(z), z|+[F_1(x), z|yd_2(z)] = 0 \text{ for all } x, y, z \in I.\]

Application of (2.10) in (2.11) yields that
\[(2.12)\]
\[F_1(x), z|F_1(x)yd_2(z) = 0 \text{ for all } x, y, z \in I.\]

Replacing $z$ by $z+t$ in (2.12) and using (2.12), we find that
\[(2.13)\]
\[F_1(x), z|F_1(x)yd_2(t) + [F_1(x), t|F_1(x)yd_2(z) = 0 \text{ for all } x, y, z, t \in I.\]

Application of (2.12) and (2.13) gives that
\[(2.14)\]
\[F_1(x), z|F_1(x)yd_2(t) R[F_1(x), z|F_1(x)yd_2(t) = -[F_1(x), t|F_1(x)yd_2(t)R[F_1(x), z|F_1(x)yd_2(z) = \{0\} \text{ for all } x, y, z, t \in R.\]

As a $*$-prime ring is also semiprime, (2.14) implies that $[F_1(x), z|F_1(x)yd_2(t) = 0$ for all $x, y, z, t \in I$. Since $*$ commutes with $d_2$ and $I$ is a $*$-ideal, we have
\[F_1(x), z|F_1(x)yd_2(t) = 0 = [F_1(x), z|F_1(x)y * (d_2(t)) \text{ for all } x, y, z, t \in I.\]

Since $R$ is $*$-prime and $I$ is $*$-ideal, by Lemma 2.1, we find that either $[F_1(x), z|F_1(x) = 0$ or $d_2(t) = 0$ for all $x, y, z, t \in I$. But $d_2 = 0$ on $I$ implies that $d_2 = 0$ on $R$. Now, take $[F_1(x), z|F_1(x) = 0$ for all $x, y, z \in I$. On replacing $z$ by $yz$ for $z, y \in I$, we find that $[F_1(x), y|zF_1(x) = 0$ for all $x, y, z \in I$. Replacing $x$ by $x+t$ for $x, t \in I$ in $[F_1(x), y|zF_1(x) = 0$ and using the similar arguments as used after (2.14), we get
\[F_1(x), z|zF_1(x) = 0 \text{ for all } x, y, z, t \in I.\]

Since $I$ is $*$-ideal and $*$ commutes with $F_1$, we find that $[F_1(x), y|zF_1(t) = [F_1(x), y|z * (F_1(t)) = 0$ for all $x, y, z, t \in I$. By Lemma 2.1, we have either $[F_1(x), y| = 0$ for all $x, y \in I$ or $F_1(t) = 0$ for all $t \in I$. If $F_1(t) = 0$ for all $t \in I$, then $F_1(ty) = 0$ for all $t, y \in I$. This yields that $td_1(y) = 0$ for all $y \in I$. Replace $t$ by $xt$ for $x, t \in I$, we find that $xtd_1(y) = 0$. Since $I$ is $*$-ideal, $xtd_1(y) = *x(td_1(y) = 0$ for all $t, x, y \in I$. By Lemma 2.1, we find that either $I = \{0\}$ or $d_1 = 0$ on $I$. But $I \neq \{0\}$ implies that $d_1 = 0$ on $I$ which yields that $d_1 = 0$ on $R$. Now, consider $[F_1(x), z| = 0$ for all $x, z \in I$. Replace $x$ by $xz$ in $[F_1(x), z| = 0$ to find that
\[(2.15)\]
\[x, z|d_1(z) + x[d_1(z), z| = 0 \text{ for all } x, z \in I.\]

Again replacing $x$ by $xy$ in (2.15) and using (2.15), we get $[x, z|yd_1(z) = 0$ for all $x, y, z \in I$. Use the similar steps as used after (2.14) with necessary variations, we find that $[x, z|yd_1(s) = 0$ for all $x, y, z, s \in I$. But, since $I$ is a $*$-ideal, we find that $[x, z|yd_1(s) = 0 = *([x, z]|yd_1(s)$ for all $x, y, z, s \in I$. By Lemma 2.1, we get that either $I$ is commutative or $d_1 = 0$ on $I$. But, in a semiprime ring if an ideal is commutative then it is central and since $*$-prime ring is semiprime, we get $I$ is central and hence $R$ is commutative which is a contradiction. \[\square\]
Theorem 2.5. Let $R$ be a noncommutative $*$-prime ring and $I$ a nonzero $*$-ideal in $R$. Suppose $F_1, F_2 : R \rightarrow R$ are two generalized derivations with associated derivations $d_1, d_2 : R \rightarrow R$ respectively. If $F_1(xy) + F_2(yx) = \pm [x, y]$ for all $x, y \in I$, then $d_1 = d_2 = 0$.

Proof. Let $F_1(xy) + F_2(yx) = \pm [x, y]$ for all $x, y \in I$. Replace $y$ by $yx$ in $F_1(xy) + F_2(yx) = \pm [x, y]$ for all $x, y \in I$, to find that

\[(2.16) \quad (F_1(xy) + F_2(yx))x + yxd_1(x) + yxd_2(x) = \pm [x, y]x \text{ for all } x, y \in I.
\]

Using the given hypothesis, we get

\[xyd_1(x) + yxd_2(x) = 0 \text{ for all } x, y \in I.
\]

Replace $y$ by $yz$ for $y, z \in I$ to get $0 = xyzd_1(x) + yzxd_2(x) = [x, y]zd_1(x)$ for all $x, y, z \in I$. Using the similar steps as used after (2.14) with necessary variations, we find that $[x, z]yd_1(s) = 0$ for all $x, y, z, s \in I$. As $I$ is a $*$-ideal, we find that $[x, y]zd_1(s) = *([x, y])d_1(s)$ for all $x, y, z, s \in I$. By Lemma 2.1, we get that either $d_1 = 0$ on $I$ or $I$ is commutative. But, if $d_1 = 0$ on $I$, then $d_1 = 0$ on $R$. Also, if $I$ is a commutative ideal of $*$-prime ring $R$, then $I$ is central ideal of $R$. But if $I$ is central, then $R$ is commutative which leads to a contradiction.

If we replace $x$ by $xy$ in $F_1(xy) + F_2(yx) = \pm [x, y]$ and use the similar arguments as used above, then we find that $d_2 = 0$.

Now, take $F_1(xy) - F_2(yx) = \pm [x, y]$ for all $x, y \in I$. Using the similar arguments as used above, we get the required result. \qed

Theorem 2.6. Let $R$ be a noncommutative $*$-prime ring and $I$ a nonzero $*$-ideal in $R$. Suppose $F_1, F_2 : R \rightarrow R$ are two generalized derivations with associated derivations $d_1, d_2 : R \rightarrow R$ respectively. If $F_1(xy) + yF_2(x) = 0$ for all $x, y \in I$, then $d_1 = d_2 = 0$.

Proof. Replacing $y$ by $yz$ in $F_1(x)y - yF_2(x) = 0$, we find that $[F_1(x), y]z = 0$ for all $x, y, z \in I$. As $I$ is a nonzero $*$-ideal and $R$ is $*$-prime ring, by Lemma 2.1, we find that $[F_1(x), y] = 0$ for all $x, y \in I$. Replacing $x$ by $xy$ in $[F_1(x), y] = 0$, we get

\[(2.17) \quad [x, y]d_1(y) + z[d_1(y), y] = 0 \text{ for all } x, y \in I.
\]

Replacing $x$ by $xz$ in (2.17) and using (2.17), we find that $[x, y]zd_1(y) = 0$ for all $x, y, z \in I$. Using the similar steps as used after (2.14) with necessary variations, we find that $[x, z]yd_1(s) = 0$ for all $x, y, z, s \in I$. Hence either $d_1 = 0$ on $I$, and hence on $R$ or $I$ is commutative, hence $I$ is central. But if $I$ is central, then $R$ is commutative which leads to a contradiction. Further, replacing $x$ by $xy$ in $F_1(x)y - yF_2(x) = 0$, we find that

\[(2.18) \quad x[d_1(y), y] - yxd_2(y) = 0 \text{ for all } x, y \in I.
\]

Replacing $x$ by $xz$ in (2.18) and using (2.18), we find that $[x, y]zd_2(y) = 0$. Again, using the similar steps as used after (2.14) with necessary variations, we find that...
[x, z]yd_2(s) = 0 for all x, y, z, s ∈ I. This implies that either d_2 = 0 on I and hence on R or I is central i.e., R is commutative, a contradiction.

Now, take F_1(x)y + yF_2(x) = 0 for all x, y ∈ I and use the similar arguments as used above with necessary variations, we get the required result.

**Theorem 2.7.** Let R be a noncommutative *-prime ring and I a nonzero *-ideal in R. Suppose F_1, F_2 : R → R are two generalized derivations with associated derivations d_1, d_2 : R → R respectively and * commutes with d_1. If F_1(xy) + F_2(yx) = F_1(x)y + yF_2(x) for all x, y ∈ I, then either d_1 = 0 or d_2 = 0.

**Proof.** Let F_1(xy) + F_2(yx) = F_1(x)y + yF_2(x) for all x, y ∈ I. Replace y by xy in F_1(xy) + F_2(yx) = F_1(x)y + yF_2(x), to get

\[(F_1(xy) + F_2(yx))x + xyd_1(x) + yxd_2(x) = F_1(x)yx + yxF_2(x)\] for all x, y ∈ I.

Using the given hypothesis, we find that

\[yF_2(x)x + xyd_1(x) + yxd_2(x) = yxF_2(x)\] for all x, y ∈ I.

This implies that

\[(2.19) \quad y[F_2(x), x] + xyd_1(x) + yxd_2(x) = 0\] for all x, y ∈ I.

Replacing y by d_2(z)y in (2.19) and using (2.19), we find that \([x, d_2(z)]yd_1(x) = 0\) for all x, y, z ∈ I. This implies that either d_1 = 0 on I and hence d_1 = 0 on R or \([x, d_2(z)] = 0\) for all x, z ∈ I. Take \([x, d_2(z)] = 0\) for all x, y ∈ I. Replacing z by yx in \([x, d_2(z)] = 0\), we find that \([x, y]d_2(x) = 0\) for all x, y, z ∈ I and hence either I is central or d_2 = 0 on I i.e., d_2 = 0 on R. If I is central, then R is commutative, which is a contradiction.

**Theorem 2.8.** Let R be a noncommutative *-prime ring and I be a nonzero *-ideal in R. Suppose F_1, F_2 : R → R are two generalized derivations with associated derivations d_1, d_2 : R → R respectively. If F_1(xy) ± F_2(yx) = 0 for all x, y ∈ I, then d_1 = d_2 = 0.

**Proof.** It is given that F_1(xy) ± F_2(yx) = 0 for all x, y ∈ I. Replacing y by yx, we find that

\[F_1(xy)x + xyd_1(x) ± (F_2(yx)x + yxd_2(x)) = 0\] for all x, y ∈ I.

Using the given condition, we get

\[(2.20) \quad xyd_1(x) ± yxd_2(x) = 0\] for all x, y ∈ I.

Replacing y by yz for y, z ∈ I, we find that 0 = xyzd_1(x) ± yzxd_2(x) = [x, y]zd_1(x) for all x, y, z ∈ I. Replacing x by x + s for x, s ∈ I, we get [x, y]zd_1(s) + [s, y]zd_1(x) =
0. Hence, we find that $[x, y]zd_1(s)R[x, y]zd_1(s) = -[s, y]zd_1(s)R[x, y]zd_1(x) = \{0\}$ for all $x, y, z, s \in I$. Now, since $*$-prime ring is semiprime, the above yields that $[x, y]zd_1(s) = 0$ for all $x, y, z, s \in I$. Since $I$ is $*$-prime ring, we find that $*([x, y])zd_1(s) = [x, y]zd_1(s) = 0$ for all $x, y, z, s \in I$. By, Lemma 2.1, either $I$ is commutative, hence central which implies that $R$ is commutative, a contradiction or $d_1 = 0$ on $I$ and hence $d_1 = 0$ on $R$.

Similarly, we can show that $d_2 = 0$.

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