Elliptic Linear Weingarten Surfaces

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Abstract. We establish some characterizations of isoparametric surfaces in the three-dimensional Euclidean space, which are associated with the Laplacian operator defined by the so-called $II$-metric on surfaces with non-degenerate second fundamental form and the elliptic linear Weingarten metric on surfaces in the three-dimensional Euclidean space. We also study a Ricci soliton associated with the elliptic linear Weingarten metric.

1. Introduction

Surfaces in the Euclidean 3-space $E^3$ with constant mean curvature $H$ which are often called $H$-surfaces and those with constant Gauss curvature $K$ are called $K$-surfaces.

As Bonnet pointed out that $K$-surfaces and $H$-surfaces are big classes hard to classify, the so-called Weingarten condition is considered. We call a surface $S$ in $E^3$ linear Weingarten if a linear combination of the mean curvature $H$ and the Gauss curvature $K$ is a constant, i.e.,

$$2aH + bK = c$$

for some real numbers $a$, $b$ and $c$ which are not all zero. If $a^2 + bc > 0$, the local graph of the surface satisfies the elliptic condition for differential equation relative to the principal curvatures [3, 4, 5].

On the other hand, the eigenvalue problem of an isometric immersion $x : M \to E^m$ of a Riemannian manifold $M$ into a Euclidean space $E^m$ is a nice tool to determine a geometric character for a sphere, i.e., if $\Delta x = kx$ is satisfied for a non-zero real number $k$, then $M$ is part of a sphere [6]. Generalizing this notion, B.-Y. Chen defined the notion of order and type for the immersion of $M$ into $E^m$. By definition, a finite-type immersion $x : M \to E^m$ of a submanifold $M$ into a Euclidean space $E^m$ means $x$ is decomposed as a finite sum of the eigenvectors of
the Laplace operator $\Delta$ of $M$ in the following

$$\Delta = \xi_0 + x_1 + \cdots + x_k,$$

where $x_0$ is a constant vector and $x_1, \ldots, x_k$ are non-constant vectors satisfying $\Delta x_i = \lambda_i x_i$, $i = 1, 2, \ldots, k$. In particular, if all of $\lambda_1, \ldots, \lambda_k$ are different, it is called $k$-type or the submanifold $M$ is said to be of $k$-type (cf. [1, 2]). Thus, if a submanifold $M$ of $E^m$ is 1-type, then its immersion $x$ satisfies

$$\Delta x = kx + C$$

for some non-zero real number $k$ and a constant vector $C$.

All surfaces under consideration is smooth and connected unless otherwise stated.

2. Preliminaries

Let $S$ be an oriented surface in the 3-dimensional Euclidean space $E^3$ and $x : S \to E^3$ an isometric immersion. Then a unit vector field $N$ called the Gauss map is well-defined on $S$.

We now assume that the immersion $x$ satisfies (1.1) with $a^2 + bc > 0$. In this case, the surface $S$ is called the elliptic linear Weingarten or shortly ELW surface. An ELW surface with $b = 0$ has constant mean curvature and that with $a = 0$ has constant Gauss curvature.

We put $E_1 = \langle x_s, x_s \rangle$, $F_1 = \langle x_s, x_t \rangle$, $G_1 = \langle x_t, x_t \rangle$, $E_2 = \langle x_{st}, N \rangle$, $F_2 = \langle x_{sv}, N \rangle$, and $G_2 = \langle x_{vv}, N \rangle$, where $x = x(s, t)$ for some coordinate system $(s, t)$ of $S$. We then have the first and second fundamental forms, respectively,

$$I = E_1 ds^2 + 2F_1 ds dt + G_1 dt^2,$$

$$II = E_2 ds^2 + 2F_2 ds dt + G_2 dt^2.$$

Then, similarly to Lemma 1 in [3], we have

**Lemma 2.1.** Let $x : M \to E^3$ be an ELW immersion of a surface $S$ in $E^3$ satisfying (1.1). Then,

$$(2.1) \quad \sigma = aI + bII$$

defines a Riemannian metric on $M$.

The Riemannian metric $\sigma$ defined in Lemma 2.1 is called the ELW metric. Then we have the Gauss map $\eta$ relative to the Riemannian metric $\sigma$, which is called the associated Gauss map.

Let $(u, v)$ be the isothermal coordinates for $\sigma$. If we adopt the same notations by $E_1 = \langle x_u, x_u \rangle$, $F_1 = \langle x_u, x_v \rangle$, $G_1 = \langle x_v, x_v \rangle$, $E_2 = \langle x_{uu}, N \rangle$, $F_2 = \langle x_{uv}, N \rangle$, and $G_2 = \langle x_{vv}, N \rangle$ as above relative to the isothermal coordinates $(u, v)$, we have

$$(2.2) \quad \sigma = (aE_1 + bE_2)du^2 + 2(aF_1 + bF_2)du dv + (aG_1 + bG_2)dv^2 = \lambda(du^2 + dv^2)$$
for some positive function $\lambda$. Without loss of generality, we may assume that

$$(2.3) \quad a^2 + bc = 1 \quad \text{and} \quad c \geq 0$$

by taking the appropriate direction for the Gauss map if necessary. In particular, (2.3) shows that if $b = 0$, then we may assume that $a = 1$. In this case, the ELW metric $\sigma$ is nothing but the first fundamental form $I$. The ELW metric $\sigma$ is said to be non-trivial if $b \neq 0$.

We then have the Laplacian $\Delta^\sigma$ with respect to the Riemannian metric $\sigma$ by

$$\Delta^\sigma = -\frac{1}{\sqrt{\det \sigma}} \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right)$$

$$= -\frac{1}{\lambda} \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right).$$

If we compute $\lambda^2$ by using (2.2), we have

$$\lambda^2 = (aE_1 + bE_2)(aG_1 + bG_2) - (aF_1 + bF_2)^2,$$

from which,

$$\lambda^2 = \{a^2 + b(2aH + bK)\}(E_1G_1 - F_1^2).$$

Since $2aH + bK = c \geq 0$, we get

$$\lambda^2 = (a^2 + bc)(E_1G_1 - F_1^2) = (E_1G_1 - F_1^2),$$

or, equivalently

$$\lambda = \sqrt{E_1G_1 - F_1^2}.$$ 

Then, we get

**Lemma 2.2.** Let $S$ be an ELW surface in $E^3$ satisfying (1.1) with $a^2 + bc = 1$. Then, the associated Gauss map $\eta$ and the Gauss map $N$ are the same.

From the first and second fundamental forms $I$ and $II$, we have the shape operator $A$ of the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where

$$A_{11} = \frac{1}{\lambda^2}(G_1E_2 - F_1F_2), \quad A_{12} = \frac{1}{\lambda^2}(G_1F_2 - F_1G_2),$$

$$A_{21} = \frac{1}{\lambda^2}(-E_2F_1 + E_1F_2), \quad A_{22} = \frac{1}{\lambda^2}(E_1G_2 - F_1F_2).$$
As is given in [3], we have

**Theorem 2.3.** Let \( x : S \to E^3 \) be an ELW immersion satisfying (1.1) with \( a^2 + bc = 1 \). Then, we have

\[
\begin{align*}
\Delta^\sigma x &= (c + bK)\eta, \\
\Delta^\sigma \eta &= 2(aK - cH)\eta.
\end{align*}
\]  

(2.9)

3. Harmonic and Bi-harmonic ELW Surfaces

In this section, we characterize harmonic and bi-harmonic ELW surfaces in \( E^3 \) with respect to the ELW metric \( \sigma \).

Let \( S \) be an ELW surface with the metric \( \sigma \) defined by (2.1) satisfying \( a^2 + bc = 1 \).

**Definition 3.1.** An ELW surface \( S \) in \( E^3 \) is said to be \( \sigma \)-harmonic or ELW harmonic if its immersion \( x \) satisfies \( \Delta^\sigma x = 0 \). It is said to be \( \sigma \)-biharmonic or ELW biharmonic if its immersion \( x \) satisfies \( (\Delta^\sigma)^2 x = 0 \).

First of all, we prove

**Theorem 3.2.** Let \( S \) be an ELW surface in \( E^3 \) with the ELW metric \( \sigma \). Then, \( S \) is \( \sigma \)-harmonic if and only if \( S \) is minimal or part of a plane.

**Proof.** Suppose that the ELW surface \( S \) is \( \sigma \)-harmonic. From (2.9), we have

\[
(3.1) \\
\begin{align*}
c + bK &= 0.
\end{align*}
\]

If \( b = 0 \), then \( c = 0 \). From the Weingarten condition (1.1) between the mean curvature \( H \) and Gauss curvature \( K \), we see that \( H = 0 \), i.e., \( S \) is minimal.

If \( b \neq 0 \), (3.1) gives \( K = -c/b \). Thus, the mean curvature is given by \( H = c/a \). Since \( H^2 - K \geq 0 \), \( c/b \geq 0 \) and hence the Gauss curvature \( K \leq 0 \). In this case, if \( c = 0 \), \( S \) is part of a plane. If \( c \neq 0 \), the Gauss curvature satisfies \( K < 0 \). Since the mean curvature \( H \) and the Gauss curvature \( K \) are constant, \( S \) is part of an isoparametric surface in \( E^3 \) which is one of a plane, a sphere or a circular cylinder. Thus, this case cannot occur.

Conversely, suppose that the ELW surface \( S \) is minimal. Then, (2.3) gives \( bK = c \). If \( b = 0 \), we get automatically \( c = 0 \) and (2.9) shows that \( S \) is \( \sigma \)-harmonic.

We now suppose \( b \neq 0 \). Then, the Gauss curvature \( K \) is given by \( K = c/b \). In this case, if \( c = 0 \), \( S \) is totally geodesic. In case of \( c > 0 \), there exists no possible isoparametric surface with \( H = 0 \) and \( K \neq 0 \) in \( E^3 \).

If \( S \) is totally geodesic, (1.1) shows that the ELW surface \( S \) is \( \sigma \)-harmonic. \( \Box \)

We now compute \((\Delta^\sigma)^2 x\). From (2.9), we get

\[
(3.2) \\
(\Delta^\sigma)^2 x &= \frac{-1}{\lambda} \{ b(K_{uu} + K_{vv})\eta + 2bK_u\eta_u + (c + bK)\eta_{uu} \} \\
&+ 2bK_v\eta_v + (c + bK)\eta_{vv} \}.
\]
Using Lemma 2.2 with (2.6)-(2.8) and (2.9), we obtain

\[(\Delta^\sigma)^2 x = -\frac{1}{\lambda}\left\{b\Delta^\sigma K + 2(c + bK)(aK - cH)\right\}x + \frac{2b}{\lambda}(K_u A_{11} + K_v A_{12})x_u + \frac{2b}{\lambda}(K_u A_{21} + K_v A_{22})x_v\].

We now prove

**Theorem 3.3.** Let \( S \) be an ELW surface in \( E^3 \) with the ELW metric \( \sigma \). Then, if \( S \) is \( \sigma \)-biharmonic, then \( S \) is part of either a minimal surface or an isoparametric surface in \( E^3 \), i.e., \( S \) is part of a sphere, a plane or a circular cylinder. Conversely, if we take appropriate real numbers \( a, b \) and \( c \), then a sphere, a plane or a minimal surface is \( \sigma \)-biharmonic.

**Proof.** Suppose that the ELW surface \( S \) is \( \sigma \)-biharmonic. Then, from (3.3), we get

\[(3.4) \quad b\Delta^\sigma K + 2(c + bK)(aK - cH) = 0,\]

\[(3.5) \quad b \left( \begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right) \left( \begin{array}{c} K_u \\ K_v \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right).\]

Case 1: \( b \neq 0 \).

Suppose the open subset \( M_0 = \{ p \in S | K(p) \neq 0 \} \) of \( S \) is not empty. Let \( U \) be a component of \( M_0 \). It follows from (3.5) that the Gauss curvature \( K \) is non-zero constant on \( U \). Together with (3.4), the mean curvature \( H \) is constant. Thus, \( U \) is contained in a plane, a circular cylinder or a sphere. Since the Gauss curvature \( K \) is non-zero, \( U \) is part of a sphere. By continuity, \( U \) is the whole surface \( S \).

Suppose \( K \) is vanishing, i.e., \( S \) is flat. By the condition of (1.1), we get \( 2aH = c \). Since \( a^2 + bc = 1 \), \( a \) cannot be zero.

If \( c = 0 \), \( H = 0 \) and thus the surface \( S \) is part of a plane.

In case of \( c \neq 0 \), the only possible case for the surface \( S \) as an isoparametric surface in \( E^3 \) is contained in a circular cylinder.

Case 2: \( b = 0 \).

(3.3) implies

\[(3.6) \quad c(aK - cH) = 0.\]

If \( c = 0 \), the ELW condition gives the mean curvature \( H \) is vanishing.

Suppose \( c \neq 0 \). Then, \( H = c/2 \) and \( K = c^2/2 \). However, this case cannot occur because \( H^2 - K \geq 0 \).
Conversely, it is easy to show that if we take appropriate real numbers $a$, $b$ and $c$, a minimal surface, a plane or a sphere is $\sigma$-biharmonic. \hfill \qed

4. Surfaces with $\text{II}$-metrics

Let $S$ be a surface in $E^3$ with non-degenerate second fundamental form via an isometric immersion $x : S \to E^3$.

Let $\tilde{\nabla}$ be the Levi-Civita connection on $E^3$ and $\nabla$ the induced connection on $S$. Then, the Gauss and Codazzi equations of $S$ in $E^3$ are respectively given by

\begin{align}
\tilde{\nabla} X Y &= \nabla X Y + \langle AX, Y \rangle N, \\
(\nabla X) A Y &= (\nabla Y) A X,
\end{align}

where $A$ is the shape operator of $S$ and $X$, $Y$ and $Z$ are the vector fields tangent to $S$.

Since $A$ is non-degenerate, we can choose a coordinate patch $x(u, v)$ on a neighborhood $U$ around $p$ such that $x_u, x_v$ are in the principal directions. Then, we have

$$A = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

with respect to the coordinate frame $\{x_u, x_v\}$ so that the mean curvature and the Gaussian curvature are respectively given by $H = (\kappa_1 + \kappa_2)/2$ and $K = \kappa_1 \kappa_2$. Define a symmetric tensor $h$ by

$$(4.3) \quad h(X, Y) = \langle AX, Y \rangle$$

for tangent vector fields $X$ and $Y$ to $S$.

Since $h$ is non-degenerate, $h$ is regarded as a non-degenerate metric on $M$, which is called the $\text{II}$-metric with representation given by

$$(4.4) \quad h = \begin{pmatrix} \kappa_1 E_1 & 0 \\ 0 & \kappa_2 G_1 \end{pmatrix}.$$

On the other hand, it is easy to derive

$$(4.5) \quad \nabla_{x_u} x_u = \frac{(E_1)_s}{2E_1} x_u - \frac{(E_1)_t}{2G_1} x_v,$$

$$(4.6) \quad \nabla_{x_u} x_v = \frac{(E_1)_t}{2E_1} x_u + \frac{(G_1)_s}{2G_1} x_v,$$

$$(4.7) \quad \nabla_{x_v} x_v = -\frac{(G_1)_s}{2E_1} x_u + \frac{(G_1)_t}{2G_1} x_v.$$
Without loss of generality, we may regard as \( \kappa_1 > 0 \). We put
\[
(4.8) \quad h_{11} = \kappa_1 E_1 = a^2, h_{12} = h_{21} = 0, h_{22} = \kappa_2 G_1 = \varepsilon b^2
\]
for some positive functions \( a \) and \( b \), where \( \varepsilon = \pm 1 \) depending upon the signature of \( h_{22} \). Then, we have the equations of Gauss
\[
(4.9) \quad x_{uu} = \nabla_{x_u} x_{u u} = \frac{(E_1)_u}{2E_1} x_u - \frac{(E_1)_v}{2G_1} x_v + a^2 N,
\]
\[
(4.10) \quad x_{uv} = \nabla_{x_u} x_{v u} = \frac{(E_1)_v}{2E_1} x_u + \frac{(G_1)_u}{2G_1} x_v,
\]
\[
(4.11) \quad x_{vv} = \nabla_{x_v} x_{v v} = -\frac{(G_1)_u}{2E_1} x_u + \frac{(G_1)_v}{2G_1} x_v + \varepsilon b^2 N.
\]

We then define the \( II \)-Laplace operator \( \Delta^{II} \) with respect to the metric \( h \) by
\[
(4.12) \quad \Delta^{II} = -\frac{1}{\sqrt{|\det h|}} \{ \frac{\partial}{\partial u} (\sqrt{|\det h|} \frac{1}{a^2} \frac{\partial}{\partial u}) + \frac{\partial}{\partial v} (\varepsilon \sqrt{|\det h|} \frac{1}{b^2} \frac{\partial}{\partial v}) \}
\]
\[= -\frac{1}{ab} \left( \frac{b}{a} \frac{\partial}{\partial u} \right) + \varepsilon \frac{1}{b} \frac{\partial}{\partial v} \left( \frac{a}{b} \frac{\partial}{\partial v} \right). \]

If we put \( f = b/a \), then (4.12) can be written as
\[
(4.13) \quad \Delta^{II} = -\frac{1}{ab} \left( f_u \partial / \partial u + f \partial^2 / \partial u^2 + \varepsilon (1/f)_v \partial / \partial v + \varepsilon (1/f) \partial^2 / \partial v^2 \right).
\]

Using (4.9), (4.10) and (4.11), we have

**Lemma 4.1.** Let \( M \) be a surface of \( S^3(1) \) with non-degenerate second fundamental form. Then, we have
\[
(4.14) \quad \Delta^{II} x = -\frac{1}{ab} \left( (f_u + f (E_1)_v) \frac{(E_1)_u}{2E_1} - \varepsilon (G_1)_u \right) x_u + \left( -\frac{f (E_1)_v}{2G_1} + \varepsilon (G_1)_v \right) x_v + 2ab N.
\]

We then have immediately from Lemma 4.1

**Proposition 4.2.** There do not exist \( II \)-harmonic surfaces of \( S^3(1) \) with non-degenerate second fundamental form satisfying \( \Delta^{II} x = 0 \).

Suppose that the surface \( S \) satisfies \( \Delta^{II} x = k x + C \) for some real number \( k \neq 0 \) and a constant vector \( C \), that is, \( S \) is of 1-type with respect to \( II \)-metric. From equation (4.14), we see that \( k x + C \) is in the normal direction, i.e., \( k x + C = \rho N \) for some function \( \rho \). It follows that
\[
\langle k x + C, N \rangle = -2,
\]
from which, we get
\[ x + \frac{1}{k} C = \frac{\rho}{k} N \]
and \( \rho \) is a constant. Thus, the surface \( S \) is part of a sphere.

Conversely, suppose that the surface \( S \) is part of sphere with radius \( r \). Without loss of generality, we may assume that the center of \( S \) is the origin. It is straightforward to compute
\[ \Delta \Pi x = -\frac{2}{r} x. \]
Therefore, \( S \) is of 1-type with respect to \( \Pi \)-metric. Therefore, we have

**Theorem 4.3.** Let \( S \) be a surface of \( E^3 \) with non-degenerate fundamental form. Then, \( S \) is of 1-type with respect to \( \Pi \)-metric if and only if \( S \) is part of a sphere.

5. Compact ELW Surfaces

In this section, we discuss about the geometric meaning of the Gauss curvature \( K^\sigma \) on the ELW surface \( M \) defined by the Riemannian metric \( \sigma \). We call \( K^\sigma \) the ELW-Gauss curvature. Let \( \nabla^\sigma \) be the Levi-Civita connection compatible with the Riemannian metric \( \sigma \) on \( M \).

By straightforward computation, we have the following

**Lemma 5.1.** Let \( M \) be an ELW surface with the metric \( \sigma \). Then, the Christoffel symbols \( \bar{\Gamma}^h_{ji} \) are given by

\[
\bar{\Gamma}^1_{11} = \bar{\Gamma}^2_{12} = \bar{\Gamma}^2_{21} = -\bar{\Gamma}^1_{22} = \frac{\lambda}{2\lambda},
\]

\[
-\bar{\Gamma}^1_{11} = \bar{\Gamma}^1_{12} = \bar{\Gamma}^1_{21} = \bar{\Gamma}^2_{22} = \frac{\lambda}{2\lambda}.
\]

Making use of Lemma 5.1, we have the ELW-Gauss curvature \( K^\sigma \)

\[
K^\sigma = \frac{R^\sigma(x_u, x_v, x_u, x_v)}{\sigma(x_u, x_u)\sigma(x_v, x_v) - \sigma(x_u, x_v)^2}
\]

\[
= -\frac{1}{\lambda} \left\{ \left( \frac{\lambda}{2\lambda} \right)_u + \left( \frac{\lambda}{2\lambda} \right)_v \right\}
\]

\[
= -\frac{1}{2\lambda} \{ (\ln \lambda)_{uu} + (\ln \lambda)_{vv} \}
\]

\[
= \frac{1}{2} \Delta^\sigma (\ln \lambda),
\]

where \( R^\sigma \) is the curvature tensor defined by \( \nabla^\sigma \) and we have put

\[
R^\sigma(X, Y, Z, W) = \sigma(\nabla^\sigma_X \nabla^\sigma_Y Z - \nabla^\sigma_Y \nabla^\sigma_X Z - \nabla^\sigma_{[X,Y]} Z, W)
\]
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for tangent vector field $X, Y, Z, W$ on $M$.

We now consider the non-trivial ELW metric $\sigma$ with a constant $\lambda$, i.e., $S$ satisfies $a^2 + bc = 1$ with $b \neq 0$.

**Theorem 5.2.** Let $S$ be a non-trivial ELW surface of $E^3$ with the ELW metric $\sigma$. Then, $S$ is flat if and only if $\lambda$ is constant.

**Proof.** Suppose that the surface $S$ is non-trivial ELW and flat. Then,

$$E_2G_2 - F_2^2 = 0.$$  

Since $S$ is ELW, the mean curvature $H$ is constant and thus $S$ is part of a plane or a circular cylinder.

If $S$ is part of plane, the metric $\sigma$ is nothing but $\sigma = aI$ and $\lambda = aE_1$. We may take an isothermal coordinate system so that $E_1 = G_1$ are constant and hence $\lambda$ is constant.

If $S$ is part of a circular cylinder, the second fundamental form is given by

$$II = \begin{pmatrix} 0 & 0 \\ 0 & kG_1 \end{pmatrix}$$

for some non-zero constant $k$. Thus, we have $\lambda = aE_1 = (a+bk)G_1$ and $F_1 = 0$, from which we get $k = c/a$ and $\lambda = G_1/a$. In this case, we also can take an isothermal coordinate system so that $E_1$ and $G_1$ are constant and hence $\lambda$ is constant.

Conversely, suppose that $\lambda$ is constant. Then, we get

$$0 = a((E_1)_u G_1 + E_1(G_1)_u - 2F_1(F_1)_u),$$

$$0 = a((E_1)_v G_1 + E_1(G_1)_v - 2F_1(F_1)_v).$$

By using the relationships $aE_1 + bE_2 = \lambda$ and $aF_1 + bF_2 = 0$, we have

$$0 = -b((E_2)_u G_1 + E_1(G_2)_u - 2F_1(F_2)_u),$$

$$0 = -b((E_2)_v G_1 + E_1(G_2)_v - 2F_1(F_2)_v).$$

Multiplying the last two equation with $a$, we obtain

$$0 = -ab((E_2)_u G_1 + E_1(G_2)_u - 2F_1(F_2)_u),$$

$$= b^2((E_2)_u G_2 + E_2(G_2)_u - 2F_2(F_2)_u)$$

$$= b^2(E_2G_2 - F_2^2)_u,$$

$$0 = -ab((E_2)_v G_1 + E_1(G_2)_v - 2F_1(F_2)_v),$$

$$= b^2((E_2)_v G_2 + E_2(G_2)_v - 2F_2(F_2)_v)$$

$$= b^2(E_2G_2 - F_2^2)_v.$$
Therefore, the Gauss curvature $K$ is constant. Since $S$ is ELW, the mean curvature is also constant. Thus, $S$ is part of a plane, a circular cylinder or a sphere.

Suppose that $S$ is a part of a sphere. Then, $S$ is totally umbilic and thus its second fundamental form $II$ is given by

$$II = \begin{pmatrix} kE_1 & 0 \\ 0 & kG_1 \end{pmatrix}$$

for some non-zero constant $k$. Therefore, the function $\lambda$ is represented by $\lambda = (a + bk)E_1 = (a + bk)G_1$ and $E_1 = G_1$ is constant. It follows that the Gauss curvature $K$ vanishes, which is a contradiction.

Hence, the ELW surface $S$ is flat.

**Corollary 5.3.** Let $S$ be a non-trivial ELW surface with the ELW metric $\sigma$. If $S$ is flat, the ELW-Gauss curvature $K^\sigma$ of $S$ vanishes.

### 6. Ricci Soliton ELW Surfaces

A complete Riemannian manifold $(M, g)$ is a **Ricci soliton** if there exists a smooth function $f$ on $M$ satisfying

$$\text{Ric} + \nabla^2 f = \rho g$$

for some constant $\rho$, where $\nabla^2$ is the Hessian defined by $(\nabla^2 f)(X, Y) = \nabla_X \nabla_Y f - (\nabla_X Y) f$ for vector fields $X, Y$ on $M$. In this case, $f$ is called a potential function of the Ricci soliton. The Ricci soliton is called **steady** if $\rho = 0$, **shrinking** if $\rho > 0$ and **expanding** if $\rho < 0$. If $f$ is constant, $(M, g)$ is Einstein. Thus, a Ricci soliton is a natural extension of Einstein manifolds.

Let $(S, \sigma)$ be a Ricci soliton and ELW surface in $E^3$ with $\lambda$ as a potential function, i.e., $M$ satisfies

$$Ric^\sigma + \nabla^\sigma \nabla^\sigma \lambda = \rho I,$$

for a constant $\rho$, where $Ric^\sigma$ is the Ricci tensor associated with the metric $\sigma$ and $\nabla^\sigma$ is the Levi-Civita connection on $S$ compatible with $\sigma$. The Ricci tensor is given by

$$Ric^\sigma(X, Y) = \sum_{i=1}^{2} R^\sigma(e_i, X, Y, e_i),$$

where $e_1, e_2$ are orthonormal frame along $S$ with respect to the metric $\sigma$ and $R^\sigma$ is the curvature tensor defined by (5.4).

Let $x : S \to E^3$ be an immersion of a Ricci soliton and ELW surface $S$ into $E^3$ with the isothermal coordinate system $(u, v)$ with respect to $\sigma$. Then, we have a natural orthonormal frame $e_1 = x_u/\lambda$ and $e_2 = x_v/\lambda$. Using (6.2), we get

$$R^\sigma(x_u, x_v) = \frac{\lambda_u}{2\lambda} v + \frac{\lambda_v}{2\lambda} u.$$
Therefore, equation (6.1) gives
\[
\lambda \lambda_{uv} = \lambda_{u} \lambda_{v},
\]
from which, we get
\[
\lambda_{u} = \Phi(u) \lambda \quad \text{and} \quad \lambda_{v} = \Psi(v) \lambda
\]
for some non-zero functions $\Phi = \Phi(u)$ and $\Psi = \Psi(v)$. Therefore, we have
\[
\lambda_{u} \lambda_{v} = \lambda \lambda_{uv} = \lambda \lambda_{u}.
\]
Suppose $\lambda_{u} \neq 0$ on an open subset $U$ on $S$. Then, $\lambda_{u} = \lambda_{v}$ and the functions $\Phi$ and $\Psi$ are constant on $U$. Thus, on $U$, $\lambda_{u} = \lambda_{v} = \lambda C$ for some non-zero constant $C$. It follows $u = v$, which is a contradiction. Hence, $\lambda_{u} = 0$ on $S$. Similarly, we can have $\lambda_{v} = 0$ on $S$. Thus, $\lambda$ is a constant. According to Corollary 5.3, $S$ is flat and its ELW Gauss curvature $K^{\sigma}$ is also vanishing.

Conversely, if the function $\lambda$ is constant, it is trivial that $S$ is a Ricci soliton satisfying (6.1).

Thus, we have

**Theorem 6.1.** Let $S$ be an ELW surface in $E^3$ with the ELW metric $\sigma$. Then, $S$ is a Ricci soliton with respect to $\sigma$ with $\lambda$ as a potential function if and only if $S$ is part of a plane or a circular cylinder.

**References**


