ON THE MINIMUM ORDER OF 4-LAZY COPS-WIN GRAPHS

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Abstract. We consider the minimum order of a graph $G$ with a given lazy cop number $c_L(G)$. Sullivan, Townsend and Werzanski [7] showed that the minimum order of a connected graph with lazy cop number 3 is 9 and $K_3 \sqcup K_3$ is the unique graph on nine vertices which requires three lazy cops. They conjectured that for a graph $G$ on $n$ vertices with $\Delta(G) \geq n - k^2$, $c_L(G) \leq k$. We proved that the conjecture is true for $k = 4$. Furthermore, we showed that the Petersen graph is the unique connected graph $G$ on 10 vertices with $\Delta(G) \leq 3$ having lazy cop number 3 and the minimum order of a connected graph with lazy cop number 4 is 16.

1. Introduction

The game of Cops and Robbers is a well-known two-player game played on a finite connected undirected graph. It was independently introduced by Quilliot [6], and by Nowakowski and Winkler [4]. The first player occupies some vertices with some number of cops (multiple cops may occupy a single vertex) and the second player occupies a vertex with a single robber. After that they move alternately along the edges of the graph. On the cops' turn, each of the cops may remain stationary or move to an adjacent vertex. On the robber's turn, he may remain stationary or move to an adjacent vertex. A round of the game is a cop move together with the subsequent robber move. The cops win if after a finite number of rounds, one of them can move to catch the robber, that is, the cop and the robber occupy the same vertex.

The main object of study in the game of Cops and Robbers on a graph $G$ is the cop number $c(G)$, the minimum number of cops required to catch the robber, introduced by Aigner and Fromme [1]. For a fixed positive integer $k$, we say a graph $G$ is $k$-cop-win if $c(G) = k$. For example, a path is 1-cop-win and the Petersen graph is 3-cop-win [2]. We define $M_k$ to be the minimum order of a connected $k$-cop-win graph and $m_k$ to be the minimum order of a connected graph $G$ satisfying $c(G) \geq k$. Clearly, we have $m_k \leq M_k$. The exact values

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of these parameters are only known for first three values of \( k \). Baird et al. [2] showed that \( m_1 = M_1 = 1 \), \( m_2 = M_2 = 4 \) and \( m_3 = M_3 = 10 \). Moreover, they proved that the Petersen graph is the unique 3-cop-win graph with order 10. They also used a computer search to calculate the cop number of every connected graph on 10 or fewer vertices. They performed this categorization by checking for cop-win orderings [4] and using an algorithm provided in [3].

We are interested in a variant of cops and robber introduced by Offner and Ojakian [5], where at most one cop moves in any round. It is called the game of Lazy Cops and Robbers and the lazy cop number is the minimum number of cops required to catch the robber in this setting. We write \( c_L(G) \) for the lazy cop number of a graph \( G \). Let \( P_n \) and \( C_n \) be the \( n \)-path and \( n \)-cycle, respectively. It is straightforward that \( c(P_n) = c_L(P_n) = 1 = c(C_3) = c_L(C_3) \) for \( n \geq 1 \) and \( c(C_n) = c_L(C_n) = 2 \) for \( n \geq 4 \). A graph satisfying \( c_L(G) = k \) is \( k \)-lazy cop-win. Define \( M_k^L \) to be the minimum order of a connected \( k \)-lazy cop-win graph and define \( m_k^L \) to be the minimum order of a connected graph \( G \) with \( c_L(G) \geq k \). It is easy to see that \( m_1^L = M_1^L = 1 \). For \( k = 2 \), we must have \( m_2^L = M_2^L = 4 \) since the only connected graphs with three vertices are \( P_3 \) and \( C_3 \), both of which are 1-lazy cop-win graphs.

For a graph \( G \), the degree of a vertex \( u \in V(G) \) is denoted as \( \deg_G(u) \). The minimum degree and the maximum degree of \( G \) are denoted as \( \delta(G) \) and \( \Delta(G) \), respectively. Given two graphs \( G \) and \( H \), their Cartesian product \( G \square H \) is a graph with vertex set \( V(G) \times V(H) \) and two vertices \((u_1, v_1)\) and \((u_2, v_2)\) are adjacent in \( G \square H \) if and only if either

(i) \( u_1 = u_2 \) and \( v_1 \) is adjacent to \( v_2 \) in \( H \), or

(ii) \( v_1 = v_2 \) and \( u_1 \) is adjacent to \( u_2 \) in \( G \).

Sullivan, Townsend and Werzanski [7] proved that for the game of lazy cops and robber, \( K_3 \square K_3 \) is the unique 3-lazy cop-win graph on nine vertices. In addition, all other graphs on 9 or fewer vertices have lazy cop number at most two. Hence \( m_3^L = M_3^L = 9 \). They also showed that \( c_L(K_n \square K_n) = n \) and interestingly, noted that \( K_n \square K_n \) can be interpreted as an \( n \times n \) grid with edges representing a Rook’s move in chess. Furthermore, they conjectured that for a graph \( G \) on \( n \) vertices with \( \Delta(G) \geq n - k^2 \), we must have \( c_L(G) \leq k \).

In this paper, we compute the exact values for \( m_4^L \) and \( M_4^L \) and prove some related results, including the above conjecture for the case \( k = 4 \) (see Corollary 4.7).

**Theorem 1.1.** If \( G \) is a connected graph with 10 vertices and \( \Delta(G) \leq 3 \), then \( c_L(G) \leq 3 \). Furthermore, equality holds if and only if \( G \) is the Petersen graph.

**Theorem 1.2.** If \( G \) is a connected graph with at most 15 vertices, then \( c_L(G) \leq 3 \).

The exact values for \( m_4^L \) and \( M_4^L \) can be deduced easily from Theorem 1.2 and the fact that \( K_4 \square K_4 \) is a 4-lazy cop-win graph [7].

**Corollary 1.3.** \( m_4^L = M_4^L = 16 \).
Given a vertex $v \in V(G)$, its neighborhood $N_G(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and $N_G[u]$ is the set $\{v\} \cup N_G(v)$. Furthermore, for any subset $U \subseteq V(G)$, $N_G(U) = \bigcup_{u \in U} N_G(u)$ and $N_G[U] = \bigcup_{u \in U} N_G[u]$. We write neighborhood of the vertex set $U = \{v_1, v_2, v_3, \ldots, v_i\}$ as $N_G[v_1, v_2, v_3, \ldots, v_i]$. If the graph in question is clear, we shall write $N(v)$, $N[v]$, $N(U)$, $N[U]$ and $N[v_1, v_2, v_3, \ldots, v_i]$. A vertex occupied by a cop or robber is also called a position.

Let the cops $c_i$, $i = 1, 2, \ldots$, be at our disposal to play on a graph $G$. A winning strategy of the cops on $G$ refers to a set of instructions for the cops $c_i$, $i = 1, 2, \ldots$, if followed, guarantees that the cops can win any game played on $G$, regardless of how the robber $r$ moves throughout the game. If $c$ is a cop or a robber and is at position $u \in V(G)$, we shall write $N_G(c)$ instead of $N_G(u)$. Similarly, $N_G[e] = N_G[u]$.

When we say a cop $c$ moves one step at a time to a vertex $w$, we mean that $c$ will move towards $w$ in all cop's turn regardless of the movement of $r$ in each robber's turn. So $c$ will occupy $w$ in finite steps.

**Lemma 1.4** ([7, Theorem 2.5]). Assume $G = (V, E)$ has a vertex $v \in V$ with $\text{deg}(v) = 1$; say $uv \in E$ is the unique edge incident to $v$. Define $G'$ to be the graph with vertex set $V' = V - \{v\}$ and edge set $E' = E - \{uv\}$. Then $c_L(G') = c_L(G)$.

By virtue of Lemma 1.4, we may ignore graphs that have a vertex of degree 1. By removing vertices of degree 1, we obtain a graph with the same lazy cop number but with smaller number of vertices.

In Section 2, we will show that $c_L(P(n, 2)) = 3$ for $n \geq 5$ (Lemma 2.1). This result is of interest on its own. Then we prove Theorem 1.1 and Theorem 1.2 in Section 3 and Section 4 respectively.

### 2. $c_L(P(n, 2))$

The generalized Petersen graph $P(n, 2)$ is the graph with vertex set

$$V(P(n, 2)) = \{u_1, \ldots, u_n, v_1, \ldots, v_n\}$$

and edge set

$$E(P(n, 2)) = \{u_i v_i, u_i u_{i+1}, v_i v_{i+2} : i \geq 1\},$$

where the subscripts are taken modulo $n$. Note that $P(5, 2)$ is the Petersen graph.

**Lemma 2.1.** For $P(n, 2)$ of girth $\geq 5$, we have $c_L(P(n, 2)) = 3$.

**Proof.** [1] shows that for any graph $G$ with girth at least 5, $c(G) \geq \delta(G)$. Since $P(n, 2)$ is 3-regular and $c(G) \leq c_L(G)$ (see [8]), this indicates that $c_L(P(n, 2)) \geq 3$.

Now, it is left to show that $c_L(P(n, 2)) \leq 3$. Here we describe a winning strategy for three cops. Suppose we have 3 cops at our disposal, say $c_1, c_2$ and
c3. The robber will be denoted by r. If at round t, the robber is at position u_j or v_j, we set W_t(r) = j. We do the same for the cop c_i. We may consider W_t as a weight of a cop or the robber at round t.

Initially we place c_1 at position u_n, c_2 at position v_1 and c_3 at position v_2 (see Figure 1).

Therefore W_1(c_1) = n, W_1(c_2) = 1 and W_1(c_3) = 2. Note that r cannot be placed at positions \{u_1, v_1, u_2, v_2, v_3, u_{n-1}, u_n, v_{n-1}, v_n\}. So, initially we must have

\[
\max (W_1(c_2), W_1(c_3)) = 2 < W_1(r) < n - 1 = W_1(c_1) - 1,
\]

and W_1(c_2) and W_1(c_3) are consecutive integers. The size of the interval that W_1(r) can lie within is W_1(c_1) - 1 - \max (W_1(c_2), W_1(c_3)) = n - 3.

We prove this by induction on t. Suppose that at round t, we have

\[
\max (W_t(c_2), W_t(c_3)) < W_t(r) < W_t(c_1) - 1,
\]

and W_t(c_2) and W_t(c_3) are consecutive integers. The size of the interval that W_t(r) can lie within is s = W_t(c_1) - 1 - \max (W_t(c_2), W_t(c_3)). Now we shall give a strategy depending on the value of W_t(r) that will reduce the size of the interval that W_{t+1}(r) can lie within.

**Scenario 1.** Suppose W_t(r) = W_t(c_1) - 3 (see Figure 2). So r is at position z or w. We move the cop c_1 to position x. At robber’s turn, if r is at position z, he cannot move to a, otherwise he will be caught in the next round. Similarly,
if \( r \) is at position \( w \), he cannot move to \( y \). Thus, at round \( t + 1 \), we must have \( W_{t+1}(c_1) < W_{t+1}(c_1) - 1 = W_t(c_1) - 2 \). Note that \( W_{t+1}(c_2) = W_t(c_2) \), \( W_{t+1}(c_3) = W_t(c_3) \) and \( W_{t+1}(r) = W_t(r) \). \( W_t(r) = 1 \) or \( W_t(r) = 2 \). So, \( W_{t+1}(c_2) \) and \( W_{t+1}(c_3) \) are still consecutive integers. We now consider two cases.

First, we suppose \( W_{t+1}(r) > \max(W_{t+1}(c_2), W_{t+1}(c_3)) \), then we have achieved our objective for the size of the interval that \( W_t(r) \) can lie within is \( W_{t+1}(c_3) - \max(W_{t+1}(c_2), W_{t+1}(c_3)) = W_t(c_1) - 2 - \max(W_t(c_2), W_t(c_3)) = s - 1 \). Recall that the size of the interval that \( W_t(r) \) can lie within is \( s \).

Next, we suppose \( W_{t+1}(r) \leq \max(W_{t+1}(c_2), W_{t+1}(c_3)) \). We may assume that \( W_{t+1}(c_2) = W_{t+1}(c_3) - 1 \) because \( W_{t+1}(c_2) \) and \( W_{t+1}(c_3) \) are consecutive integers. Since \( \max(W_t(c_2), W_t(c_3)) < W_t(r) \), this can only happen if \( W_{t+1}(r) = W_t(r) - 1 \) or \( W_t(r) = 2 \). If \( W_{t+1}(r) = W_t(r) - 1 \), then \( r \) must be at position \( w \) or \( f \) at round \( t \) (see Figure 3), and at his turn, he moves to the position a cop is occupying. This is absurd. If \( W_{t+1}(r) = W_t(r) - 1 \), then \( r \) must be at position \( z \) at round \( t \), and at his turn, he moves to \( e \). The robber will be caught at round \( t + 1 \) by the cop \( c_3 \).

\[ \text{Scenario 2.} \] Suppose \( W_t(r) \neq W_t(c_1) - 3 \). Assume that \( W_t(c_2) = W_t(c_3) - 1 \) (see Figure 3). We move the cop \( c_2 \) to position \( w \). At the robber’s turn, he cannot move to \( z \), otherwise he will be caught at round \( t + 1 \) by the cop \( c_2 \). Moreover, if the robber was already at \( z \), he cannot remain there, nor can he move to \( e \), so he must move to \( g \) (the vertex above \( f \)). So we must have \( W_t(c_3) + 1 = \max(W_{t+1}(c_2), W_{t+1}(c_3)) < W_{t+1}(r) \). Since \( W_t(r) < W_t(c_1) - 1 \) and \( W_t(r) \neq W_t(c_1) - 3 \), either \( W_t(r) = W_t(c_1) - 2 \) or \( W_t(r) < W_t(c_1) - 3 \). If \( W_t(r) = W_t(c_1) - 2 \), then \( r \) is at position \( a \) or \( b \) (see Figure 2). If \( r \) is at \( a \), he cannot move to \( x \), otherwise he will be caught at round \( t + 1 \) by the cop \( c_1 \). Similarly, if \( r \) is at \( b \), he cannot move to \( d \). Thus, \( W_{t+1}(r) < W_{t+1}(c_1) - 1 = W_t(c_1) - 1 \). If \( W_t(r) < W_t(c_1) - 3 \), then \( W_{t+1}(r) < W_t(c_1) - 1 \) for \( W_{t+1}(r) \leq W_t(r) + 2 \). Hence we must have \( W_{t+1}(r) < W_{t+1}(c_1) - 1 \). We have achieved our objective for the size of the interval that \( W_t(r) \) can lie within is \( W_{t+1}(c_1) - 1 - \max(W_{t+1}(c_2), W_{t+1}(c_3)) = W_t(c_1) - 1 - (W_t(c_3) + 1) = s - 1 \).

From Scenario 1 and 2, we see that either the robber is caught or the interval is getting smaller and smaller. This process cannot go on indefinitely. So the robber will be caught eventually.
This completes the proof of the lemma. \hfill \qed

3. Proof of Theorem 1.1

Lemma 3.1. Let $G$ be a connected graph on 10 vertices with $\Delta(G) = 3$. If $G - N[v]$ is not a 6-cycle for all $v \in V(G)$ with $\deg(v) = 3$, then $c_L(G) \leq 2$.

Proof. Let $c_1$ and $c_2$ be the two cops at our disposal to catch the robber $r$ in $G$. Recall that for a vertex $u \in V(G)$, we write $\deg_G(u) = k$ to mean the degree of $u$ in the graph $G$, as a whole, is $k$, and that $\Delta(G - N[u])$ is the maximum degree of the subgraph $G - N[u]$.

**Case 1.** Suppose there is a vertex $u_0 \in V(G)$ with $\deg_G(u_0) = 3$ such that $\Delta(G - N[u_0]) \leq 2$.

Since $\Delta(G - N[u_0]) \leq 2$, every component in $G - N[u_0]$ is a path or a cycle. Initially, we place the two cops at position $u_0$. Then $r$ can only be placed at a component $H$ in $G - N[u_0]$. As long as there is a cop occupying $u_0$, $r$ will have to remain in $H$.

- If $H$ is a path, then we keep $c_1$ at $u_0$ and move $c_2$ to a vertex in $H$. Since $c_L(H) = 1$, $r$ will be caught by $c_2$ eventually.
- Suppose $H$ is a cycle. By the hypothesis of the lemma, $H$ cannot be a 6-cycle. We shall assume $H$ is a 5-cycle. The case $H$ is a 4-cycle or a 3-cycle can be proved similarly.
  - Assume there is a vertex $w_0 \in V(H)$ with $\deg_G(w_0) = 2$. Then $w_0$ is not adjacent to any vertices in $N[u_0]$. There are two possibilities (see Figure 4). We keep $c_1$ at $u_0$ and move $c_2$ into position as in Figure 4.

![Figure 4](image)

Since $\deg_G(b) \leq 3$ and $\deg_H(b) = 2$, $b$ is not adjacent to any vertices in $N(u_0)$ (Figure 4(a)) or $b$ is adjacent to $a \in N(u_0)$ (Figure 4(b)). In either case, $r$ can only stay at positions $b$ or $w_0$. 
In Figure 4(a), we keep $c_2$ at his position and move $c_1$ to position $w_0$ one step at a time. In Figure 4(b), we keep $c_2$ at his position and move $c_1$ to position $b$ via $a$. In either case, $r$ will be caught.

Assume $\deg_G(w) = 3$ for all $w \in V(H)$.

![Figure 5](image)

Since $\deg_H(w) = 2$, $N(w) \cap N(u_0) = 1$ for all $w \in V(H)$. This means there is a vertex $a \in N(u_0)$ with $|N(a) \cap V(H)| = 2$. We keep $c_1$ at $u_0$ and move $c_2$ into position as in Figure 5. Note that $r$ can only stay at positions $w_1$ or $w_2$. In Figure 5(a), we keep $c_2$ at his position and move $c_1$ to $a$. The robber will be caught. In Figure 5(b), we move $c_2$ to $w_3$. Then $r$ can be at positions $w_1$ or $w_4$ only. Now move $c_1$ to $b$. At robber’s turn, he can only remain at $w_1$. In the next round, we move $c_1$ from $b$ to $w_4$. The robber will be caught.

**Case 2.** Suppose $\Delta(G - N[u]) = 3$ for all $u \in V(G)$ with $\deg_G(u) = 3$.

Pick a vertex $u_0 \in V(G)$ with $\deg_G(u_0) = 3$ and pick a $v_0 \in V(G - N[u_0])$ with $\deg_{G - N[u_0]}(v_0) = 3$. Initially we place $c_1$ at $u_0$ and $c_2$ at $v_0$. Note that $G - N[u_0] - N[v_0]$ is a disjoint union of 2 vertices or a 2-path. Let $V(G - N[u_0] - N[v_0]) = \{w_1, w_2\}$.

Suppose $G - N[u_0] - N[v_0]$ is a disjoint union of 2 vertices. We may assume $r$ is at position $w_1$. Since $\deg(w_1) \leq 3$, there is a $c_i$ such that $|N(c_i) \cap N(w_1)| \leq 1$ for some $i = 1, 2$. We may assume $|N(c_1) \cap N(w_1)| \leq 1$ (see Figure 6).

In Figure 6(a), we keep $c_2$ at his position and move $c_1$ to $w_1$ one step at a time. Note that $r$ can only remain at $w_1$ for $c_2$ is occupying $v_0$. So the robber will be caught. In Figure 6(b), we keep $c_2$ at his position and move $c_1$ to $a$. The robber will also be caught.

Suppose $G - N[u_0] - N[v_0]$ is a 2-path.

(i) $|N(w_2) \cap N(c_1)| = 0$ and $|N(w_3) \cap N(c_1)| \leq 1$. 

![Figure 6](image)
This situation is quite similar like the one in Figure 6 except that \( w_1 \) and \( w_2 \) are adjacent. So we use the same cop-winning strategy, that is, keep \( c_2 \) at his position and move \( c_1 \) towards \( w_1 \). The robber will be caught.

(ii) \[ |N(w_2) \cap N(c_1)| = 0 \text{ and } |N(w_1) \cap N(c_1)| = 2. \]

Since \( \deg_G(w_1) = 3 \), \( w_1 \) is not adjacent to any vertices in \( N(c_2) \), i.e., \[ |N(w_1) \cap N(c_2)| = 0. \]

If \( |N(w_2) \cap N(c_2)| \leq 1 \), then the cops will have a winning strategy similar to (i). So we may assume \( |N(w_2) \cap N(c_2)| = 2 \) (see Figure 7). If \( r \) is at \( w_1 \), then we move \( c_2 \) to \( b \) and in the next round from \( b \) to \( w_2 \). The robber will be caught. If \( r \) is at \( w_2 \), then we move \( c_1 \) to \( a \) and in the next round from \( a \) to \( w_1 \). The robber will also be caught.

From (i) and (ii), we may assume that \[ |N(w_i) \cap N(c_1)| = 1 = |N(w_i) \cap N(c_2)| \]
for \( i = 1, 2 \) (see Figure 8). There are two possibilities. In Figure 8(a), we have the case where \( w_1 \) and \( w_2 \) have a common neighbor in (without loss of
In that case, we move $c_1$ to $z_1$ and the robber will be caught. In Figure 8(b), we have the case where $w_1$ and $w_2$ have no common neighbors at all. Now, from the graph in Figure 8(b), we remove $N[w_1]$ from $G$ (see Figure 9).

Let $J_1 = G - N[w_1]$. From what we assume in Case 2, there is a vertex of degree 3 in $J_1$. Note that $u_0$, $v_0$, $z_2$ and $z_4$ are at most of degree 2 in $J_1$. We may assume $a$ is of degree 3 in $J_1$.

- Suppose $a$ is adjacent to vertices $z_2$ and $b$ (see Figure 10(a)). We move $c_1$ to $z_1$. Then $r$ can only move to $w_2$ or $z_2$. Next, move $c_2$ to $z_4$. Then $r$ can only move to $z_2$ or $a$. Next, move $c_1$ back to $u_0$. Then $r$ can only move to $b$. Now move $c_2$ back to $v_0$. Since the robber’s potential moves are $N(b) \subseteq \{v_0, a, z_1, z_3, z_4\}$, the robber cannot move back to $w_1$ or $w_2$. Hence the robber will be caught.

- Suppose $a$ is adjacent to $z_4$ (see Figure 10(b)). Note that $a$ cannot be adjacent to $z_1$ or $z_3$ since $\deg_{J_1}(a) = 3$. It may be adjacent to $z_2$ or $b$. We move $c_2$ to $z_3$. Then $r$ can only move to $w_2$ or $z_4$. Next, move $c_1$ to $z_2$. Then $r$ can only move to $z_4$ or $a$. Next, move $c_2$ back to $v_0$. Then...
Figure 10

Theorem 1.1. If \( G \) is a connected graph with 10 vertices and \( \Delta(G) \leq 3 \), then \( c_L(G) \leq 3 \). Furthermore, equality holds if and only if \( G \) is the Petersen graph.

Proof. By Lemma 2.1, \( c_L(P(5, 2)) = 3 \). So it is sufficient to show that if \( G \) is not the Petersen graph \( P(5, 2) \), then \( c_L(G) \leq 2 \). If \( \Delta(G) \leq 2 \), then \( G \) is a path or a cycle, and thus, \( c_L(G) \leq 2 \). So we may assume that \( \Delta(G) = 3 \) and \( G \) is not the Petersen graph. By Lemma 3.1, we may further assume that there is a vertex \( u_0 \in V(G) \) with \( \deg(u_0) = 3 \) and \( J = G - N[u_0] \) is a 6-cycle. Note that each vertex in \( V(J) \) is adjacent to at most one vertex in \( N(u_0) \). Initially we may place two cops \( c_1 \) and \( c_2 \) at \( u_0 \). Note that the robber \( r \) can only remain in \( J \) as long as a cop is occupying \( u_0 \).

Case 1. Suppose there are two vertices \( a, b \in V(J) \) such that \( a \) and \( b \) are not adjacent to any vertices in \( N(u_0) \). We consider three cases where (i) \( a \) is adjacent to \( b \) in \( J \), (ii) \( a \) and \( b \) are separated by a vertex in \( J \) or (iii) \( a \) and \( b \) are separated by two vertices in \( J \).

(i) Suppose \( a \) is adjacent to \( b \) in \( J \). We keep \( c_1 \) at \( u_0 \) and move \( c_2 \) into position as in Figure 11.

Note that \( r \) can only stay at \( a, b \) or \( v \). In Figure 11(a), \( v \) is not adjacent to any vertices in \( N(u_0) \). So we move \( c_1 \) towards \( v \), one step at a time. We keep \( c_2 \) at his position. At each robber’s turn, he can only remain at \( a, b \) or \( v \). Thus he will be caught by \( c_1 \). In Figure 11(b),

\( r \) can only move to \( a \). Now move \( c_1 \) back to \( u_0 \). Since \( a \) is adjacent to \( b \) or \( z_i \), the robber cannot move back to \( w_1 \) or \( w_2 \). Hence the robber will be caught.

This completes the proof of the lemma. \( \square \)
v is adjacent to the vertex $z \in N(u_0)$. So we move $c_1$ to $z$ and then from $z$ to $v$. The robber will also be caught.

(ii) The case where $a$ and $b$ are separated by a vertex is almost identical to case (i). The cop’s winning strategy can be argued analogously, starting with keeping $c_1$ at $u_0$ and moving $c_2$ into position as in Figure 12.

(iii) Suppose $a$ and $b$ are separated by two vertices. Each $w_i \in V(J) \setminus \{a, b\}$ is adjacent to a vertex in $N(u_0)$, or else we would be in the situation of Case (i) or (ii). Thus there is a vertex $z \in N(u_0)$ that is adjacent to two vertices $w_1$ and $w_2$ in $J$. There are three possibilities (see Figure 13). We keep $c_1$ at $u_0$ and move $c_2$ into position as in Figure 13.

Note that $r$ can only stay in $a, x$ or $y$. In Figure 13(a), $w_1$ and $w_2$ are adjacent. We move $c_1$ to $z$ and then from $z$ to $x$. The robber will
be caught. In Figure 13(b), \(w_1\) and \(w_2\) are separated by one vertex in \(J\). We move \(c_2\) to \(w_2\), and so \(r\) can only stay in \(\{x, y, b\}\). Then we move \(c_1\) to \(s\) and then from \(s\) to \(y\). The robber will be caught. In Figure 13(c), \(w_1\) and \(w_2\) are separated by two vertices in \(J\). We move \(c_1\) to \(s\), and then from \(s\) to \(y\). Now \(r\) can only stay in \(a\). We move \(c_1\) from \(y\) to \(x\) and \(r\) will be caught.

Henceforth, we may assume there is at most one \(v \in J\) such that \(N(v) \cap N(u_0) = \emptyset\). So, with 5 or 6 vertices in \(J\), each having exactly 1 neighbor in \(N(u_0)\), there must be at least one \(z \in N(u_0)\) which has 2 neighbors in \(J\).

**Case 2.** Suppose there is a \(z \in N(u_0)\) with \(N(z) \cap V(J) = \{a, b\}\) such that (i) \(a\) is adjacent to \(b\) in \(J\) or (ii) \(a\) and \(b\) are separated by a vertex in \(J\).

(i) Suppose \(a\) is adjacent to \(b\) in \(J\). We keep \(c_1\) at \(u_0\) and move \(c_2\) into position as in Figure 14.

- Figure 14(a), \(v\) is not adjacent to any vertices in \(N(u_0)\). So we move \(c_1\) to \(z\) and then from \(z\) to \(y\).
to \(b\). The robber will be caught. In Figure 14(b), \(v\) is adjacent to the vertex \(w \in N(u_0)\). So we move \(c_1\) to \(w\). Note that \(r\) can only stay at \(a\), \(b\) or \(z\). Next, move \(c_2\) to \(d\) and then from \(d\) to \(a\). The robber will be caught.

(ii) Suppose \(a\) and \(b\) are separated by a vertex. We keep \(c_1\) at \(u_0\) and move \(c_2\) into position as in Figure 15.

\[\text{Figure 15}\]

In Figure 15(a), \(v\) is not adjacent to any vertices in \(N(u_0)\). So, we move \(c_1\) to \(z\). Note that \(r\) can only remain at \(v\). Now move \(c_2\) to \(d\) and then from \(d\) to \(a\). The robber will be caught.

In Figure 15(b), \(v\) is adjacent to the vertex \(w \in N(u_0)\).
- Suppose \(w\) is not adjacent to any vertices in \(V(J)\) except \(v\). By Case 1, we may assume that \(x\) is adjacent to exactly 2 vertices in \(\{d, e, f\}\). Therefore, \(\deg_{G}(w) = 2\). We move \(c_1\) to \(z\). Note that \(r\) can only remain at \(v\) or \(w\). Next, move \(c_2\) to \(f\) and then from \(f\) to \(b\) and from \(b\) to \(v\). The robber will be caught.

- Suppose \(w\) is adjacent to a vertex \(y \in V(J)\). Note that \(y \in \{d, e, f\}\). We move \(c_1\) to \(z\). Note that \(r\) can only remain at \(v\) or \(w\). Next, move \(c_2\) to \(y\) and then from \(y\) to \(w\). The robber will be caught.

By Case 1 and 2, we may assume that if there is a \(z \in N(u_0)\) with \(N(z) \cap V(J) = \{a, b\}\), then \(a\) and \(b\) are separated by exactly 2 vertices in \(J\) (\(a\) and \(b\) are of distance 3 in \(J\)). We deduce that \(G\) is isomorphic to one of the three graphs shown in Figure 16. Note that Figure 16(c) is the Petersen graph.

In Figure 16(a), \(w_3\) is not adjacent to \(z_3\). We move \(c_1\) to \(z_1\). Then \(r\) can only stay at \(w_2\) or \(w_3\). Next, move \(c_2\) to \(z_2\) and then from \(z_2\) to \(w_2\). The robber will be caught. In Figure 16(b), \(w_3\) is adjacent to \(z_3\). Suppose \(z_3\) is not adjacent to \(w_6\). We move \(c_1\) to \(z_1\). Then \(r\) can only stay at \(w_2, w_3\) or \(z_3\). Next, move \(c_2\) to
Then \( r \) can only stay at \( w_3 \) or \( z_3 \). Next, move \( c_2 \) from \( z_2 \) to \( w_2 \) and then from \( w_2 \) to \( w_3 \). The robber will be caught.

This completes the proof of the theorem. \( \square \)

4. Proof of Theorem 1.2

Here, we provide some known results and prove the following lemmas which will be useful in proving Theorem 1.2.

**Theorem 4.1** ([7, Theorem 3.1]). If \( G \) is a connected graph on at most 8 vertices, then \( c_L(G) \leq 2 \).

**Theorem 4.2** ([7, Theorem 2.4]). The graph \( G = K_3 \circ K_3 \) is the unique graph on 9 vertices with \( c_L(G) = 3 \). All other graphs \( H \) on 9 vertices have \( c_L(H) \leq 2 \).

**Lemma 4.3.** If \( G \) is a connected graph with \( \Delta(G) \leq 2 \), then \( c_L(G) \leq 2 \).

**Proof.** Since \( \Delta(G) \leq 2 \), \( G \) is a path or a cycle. Hence \( c_L(G) \leq 2 \). \( \square \)

**Lemma 4.4.** If \( G \) is a connected graph on \( n \) vertices with \( \Delta(G) \leq 3 \), then \( c_L(G) \leq \max(3, \lfloor \frac{n}{4} \rfloor) \).

**Proof.** Let \( \lfloor \frac{n}{4} \rfloor = t \). We shall show that the lemma holds by using induction on \( t \). If \( t = 1 \), then \( n \leq 7 \) and the lemma follows from Theorem 4.1. Assume that the lemma holds for all \( 1 \leq t < m \). We shall show that the lemma also holds for \( t = m \); that is, we shall show it holds for \( n = 4m + q \), where \( 0 \leq q \leq 3 \).

Let \( u \in V(G) \) be of degree 3. If \( \Delta(G - N[u]) \leq 2 \), then by Lemma 4.3, \( c_L(G - N[u]) \leq 2 \). Thus, \( c_L(G) \leq 3 \). So we may assume \( \Delta(G - N[u]) = 3 \).

The number of vertices in \( G - N[u] \) is \( n' = 4(m - 1) + q \). If \( m - 1 \geq 3 \), then by induction, \( c_L(G - N[u]) \leq m - 1 \), and hence \( c_L(G) \leq m \), the lemma holds.

So we may assume \( m \leq 3 \), i.e., \( G \) is a graph with at most 15 vertices. We shall show that 3 cops are enough to catch the robber.

Let \( S \subseteq V(G) \) be the set of all vertices of degree 3. A subset \( M \subseteq S \) is said to be independent if \( N[s] \cap N[s'] = \emptyset \) for all \( s, s' \in M \). \( M \subseteq S \) is a maximal independent set if \( |M| \) is of the largest size. Note that \( |M| \leq 3 \), as \( |V(G)| \leq 15 \).
Case 1. Suppose $|M| = 3$.

Let $u_1, u_2, u_3 \in M$. Initially, we place $c_i$ at $u_i$ for $i = 1, 2, 3$ (see Figure 17).

![Figure 17](image-url)

Let $r$ be in a component $J$ in $G - N[u_1, u_2, u_3]$. Let $\{w_j\} \in V(J)$ for some $j = 1, 2, 3$. Since $\deg_G(w_j) \leq 3$, for any possible graph of $J$, $|N(J) \cap N(u_i)| \leq 1$ for some $i = 1, 2, 3$. We may assume $|N(J) \cap N(u_1)| \leq 1$. Now we move $c_1$ to $r$ in $J$ one step at a time or via the vertex of $N(J) \cap N(u_1)$ if exists. In the latter scenario, as $c_1$ moves into $V(J)$ via that common neighbor, the robber cannot sneak around and then escape via that neighbor. This is because $|V(J)| \leq 3$. The robber will have to remain in $J$ as long as $c_2$ and $c_3$ are occupying $u_2$ and $u_3$, respectively. The robber will be caught.

Case 2. Suppose $|M| = 2$.

Let $u_1, u_2 \in M$. Initially we place $c_1$ at $u_1$ and $c_2, c_3$ at $u_2$. Let $r$ be in a component $J$ in $G - N[u_1] - N[u_2]$.

(i) Suppose $J$ is a path or a 3-cycle. Then we keep $c_1$ and $c_2$ at their positions and use $c_3$ to catch the robber in $J$. The robber will be caught because $c_L(J) = 1$.

(ii) Suppose $J$ is a $t$-cycle, $t = 4, 5, 6$ with vertex set $\{w_1, w_2, \ldots, w_t\}$ and edge set $\{w_j, w_{j+1}\}$ where the subscripts are taken modulo $t$. We move $c_3$ to a vertex $w_{i-1}$ in $J$ as in Figure 18.

Note that $|N(w_1, w_2, w_3) \cap N(u_i)| \leq 1$ for some $i = 1, 2$. We may assume $|N(w_1, w_2, w_3) \cap N(u_1)| \leq 1$. If $|N(w_1, w_2, w_3) \cap N(u_1)| = 0$, we move $c_1$ towards $w_2$ one step at a time. If $|N(w_1, w_2, w_3) \cap N(u_1)| = 1$, we move $c_1$ towards $w_2$ through that common vertex. We keep $c_2$ and $c_3$ at their positions all the while. The robber can only remain at $w_1, w_2$ or $w_3$. So he will be caught.

(iii) Suppose $J$ is a 7-cycle with vertex set $\{w_1, w_2, \ldots, w_7\}$ and edge set $\{w_j, w_{j+1}\}$ where the subscripts are taken modulo 7.

Suppose there exists a vertex in the 7-cycle that is not adjacent to $N(u_1, u_2)$. We may assume $w_4$ is not adjacent to $N(u_1, u_2)$. Then we move $c_3$ to $w_6$ in $J$ as in Figure 19. Note that $|N(w_1, w_2, w_3) \cap N(u_1)| \leq 1$ for some $i = 1, 2$. We may assume $|N(w_1, w_2, w_3) \cap N(u_1)| \leq 1$. Then
we move $c_1$ similarly as in Case 2(ii). Then, we move $c_1$ to $w_3$ if the robber is at $w_4$. The robber can only remain at $w_1, w_2, w_3$ or $w_4$ as $c_2$ and $c_3$ remain throughout the game. So he will be caught.

Now suppose each vertex in $J$ is adjacent to $N(u_1, u_2)$. By the Pigeonhole Principle, $N(u_1)$ or $N(u_2)$ has at least 4 neighbors in $J$; suppose, without loss of generality, that $N(u_1)$ does. Since $J$ is a 7-cycle, then the Pigeonhole Principle further tells us that there are four consecutive vertices on the 7-cycle such that at least three of them have neighbors in $N(u_1)$. Let $\{w_1, w_2, w_3, w_4\}$ be those four consecutive vertices. This means $|N(w_1, w_2, w_3, w_4) \cap N(u_2)| \leq 1$; let $x$ be that common neighbor, if it exists. We now move $c_3$ to $w_6$. The robber must be in $\{w_1, w_2, w_3, w_4\}$. Then we move $c_2$ towards $r$ one step at a time, via that vertex $x$ (if it exists), while $c_1$ and $c_3$ remain still. The robber must be caught.

**Case 3.** Suppose $|M| = 1$. Let $u \in M$. Then $\Delta(G - N[u]) \leq 2$. By Lemma 4.3, $c_L(G - N[u]) \leq 2$. Hence $c_L(G) \leq 3$.

This completes the proof.
The following lemma is a direct modification of Lemma 3.2 in [7], and the proof is essentially the same.

**Lemma 4.5.** If $G$ is a connected graph on $n$ vertices with $\Delta(G) \geq n - 9$, then $c_L(G) \leq 3$.

**Proof.** Place a cop at a vertex $u$ with $\text{deg}(u) = \Delta(G)$ and keep it stationary at all time. Then by Theorem 4.1, two cops are sufficient to catch the robber in any component of $G - N[u]$. □

**Lemma 4.6.** Let $G$ be a connected graph with 15 vertices and there is at least one vertex of degree 4. If $G - N[u]$ is the Petersen graph for all $u \in V(G)$ with $\text{deg}_G(u) = 4$, then $\Delta(G) \geq 5$.

**Proof.** It is sufficient to show that there is a vertex in $V(G)$ with degree 5. Let $u_1 \in V(G)$ with $\text{deg}_G(u_1) = 4$. Since $G$ is connected, there is a vertex $u_2$ in $N(u_1)$ adjacent to a vertex $v_1$ in $V(G - N[u_1])$. We may assume the graph is as in Figure 20.

![Figure 20](image1.png)

**Figure 20**

Now consider $G - N[v_1]$ (see Figure 21). Since the resulting graph must be the Petersen graph, we may assume $u_2$ is adjacent to $w_2$ and $w_4$, $u_4$ is adjacent to $w_2$ and $w_5$, and $u_5$ is adjacent to $w_3$ and $w_6$ (see Figure 22).

![Figure 21](image2.png)

**Figure 21**
Now consider $G - N[w_1]$ (see Figure 23). Since the resulting graph is the Petersen graph, $w_4$ must be adjacent to $u_2$. Hence, $\deg_G(w_4) = 5$. □

We are now ready to prove Theorem 1.2.

**Theorem 1.2.** If $G$ is a connected graph with at most 15 vertices, then $c_L(G) \leq 3$.

**Proof.** We first consider the case when $|V(G)| \leq 14$. By Lemmas 4.3, 4.4 and 4.5, we shall only need to deal with the case when $\Delta(G) = 4$. Let $u$ be a vertex in $G$ with degree 4. Observe that $G - N[u]$ has at most 9 vertices and that $G - N[u]$ is not the graph $K_3 \square K_3$. So by Theorems 4.1 and 4.2, $c_L(G - N[u]) \leq 2$, implying $c_L(G) \leq 3$.

We now assume that $|V(G)| = 15$. If $\Delta(G) \leq 3$, then by Lemmas 4.3 and 4.4, $c_L(G) \leq 3$. If $\Delta(G) \geq 6$, then by Lemma 4.5, $c_L(G) \leq 3$. Suppose $\Delta(G) = 5$. Let $u \in V(G)$ with $\deg_G(u) = 5$. Initially, place all the three cops $c_1, c_2$ and $c_3$ at $u$. Then the robber $r$ must be at one of the components in $G - N[u]$, say $H$. Note that $r$ has to remain in $H$ as long as there is a cop occupying $u$. If $H$ has at most 8 vertices, then by Theorem 4.1, $c_L(H) \leq 2$. If $H$ has 9 vertices and $H \neq K_3 \square K_3$, then by Theorem 4.2, $c_L(H) \leq 2$. In either case, we keep $c_1$ at $u$ and use $c_2$ and $c_3$ to catch the robber in $H$. Suppose $H = K_3 \square K_3$. There is a vertex $w \in V(H)$ with $\deg_G(w) = 5$ because $H$ is 4-regular and $G$ is...
connected. Since $K_3 \sqcup K_3$ is vertex transitive, we may assume $w$ is the vertex at the top left of $K_3 \sqcup K_3$. Now keep $c_1$ at $u$ and move $c_2$ to the center vertex of $K_3 \sqcup K_3$. After that, move $c_3$ to the vertex at the bottom right vertex of $K_3 \sqcup K_3$. Note that $r$ can only stay at $w$. Now move $c_1$ to $w$ through the only vertex in $N(u) \cap N(w)$. The robber will be caught.

So we are only left with the case when $\Delta(G) = 4$. Let $S \subseteq V(G)$ be the set of all vertices of degree 4. A subset $M \subseteq S$ is said to be independent if $N[w] \cap N[w'] = \emptyset$ for all $w, w' \in M$ with $w \neq w'$. $M \subseteq S$ is a maximal independent set if $|M|$ is of the largest size. Note that $|M| \leq 3$. We shall show that three cops $c_1, c_2, c_3$ are sufficient to catch the robber for each of the possible size of $M$.

**Case 1.** Suppose $|M| = 3$.

Let $w_1, w_2, w_3 \in M$. Place $c_1$ at $w_1$ for $i = 1, 2, 3$. Since $V(G) = N[w_1] \cup N[w_2] \cup N[w_3]$, the robber will be caught.

**Case 2.** Suppose $|M| = 2$.

Let $w_1, w_2 \in M$. Place $c_1$ at $w_1$ and $c_2$ at $w_2$. The robber $r$ must be at one of the components in $G - N[w_1] - N[w_2]$, say $J$. Since $|M| = 2$, $\Delta(J) \leq 3$. Note that $r$ has to remain in $J$ as long as $w_1$ and $w_2$ are occupied by cops.

(i) $\Delta(J) = 3$.

Let $a \in V(J)$ with $\deg_J(a) = 3$. We keep $c_1$ and $c_2$ at $w_1$ and $w_2$, respectively, and move $c_3$ to $a$. Since $|N[w_1] \cup N[w_2] \cup N[a]| = 14$, $r$ must be at the remaining vertex, say $b$. Since $\deg_G(b) \leq 4$, there is a $c_i$ ($1 \leq i \leq 3$) such that $|N(b) \cap N(c_i)| \leq 1$. Now move $c_1$ to $b$ one step at a time or via the vertex in $N(b) \cap N(c_i)$ (if exists). The robber will be caught.

(ii) $\Delta(J) \leq 2$. Then $J$ is a path or a $s$-cycle where $s \leq 5$.

- If $J$ is a path or a 3-cycle, then we keep $c_1$ and $c_2$ at $w_1$ and $w_2$, respectively, and use $c_3$ to catch the robber in $J$.
- If $J$ is a 4-cycle, then we keep $c_1$ and $c_2$ at $w_1$ and $w_2$, respectively, and move $c_3$ to a vertex in $J$. Note that $r$ must be at the remaining vertex in $J$, say $b$ (see Figure 24).

![Figure 24](image-url)

Since $\deg_G(b) \leq 4$, there is a $c_i$ ($1 \leq i \leq 2$) such that $|N(b) \cap N(c_i)| \leq 1$. Now move $c_1$ to $b$ one step at a time (if $|N(b) \cap N(c_i)| = 0$) or via the vertex in $N(b) \cap N(c_i)$ (if $|N(b) \cap N(c_i)| = 1$). The robber will be caught.

- If $J$ is a 5-cycle, then we keep $c_1$ and $c_2$ at $w_1$ and $w_2$, respectively, and move $c_3$ to a vertex in $J$ (see Figure 25).
If there is no edge connecting a vertex in \( \{a, b\} \) with a vertex in \( N(c_1) \), then we move \( c_1 \) to \( a \) one step at a time. Note that \( r \) can stay at \( a \) or \( b \) only, as long as \( c_2 \) and \( c_3 \) are at their positions. So the robber will be caught. Hence we may assume there is an edge connecting a vertex in \( \{a, b\} \) with a vertex in \( N(c_1) \). Without loss of generality, we may assume \( a \) is adjacent to a vertex \( z \) in \( N(c_1) \) (see Figure 26).

Suppose \( N(a) \cap N(c_2) = \emptyset \).

i. If \( |N(b) \cap N(c_2)| \leq 1 \), then keep \( c_1 \) and \( c_3 \) at their positions and move \( c_2 \) to \( b \) one step at a time (if \( |N(b) \cap N(c_2)| = 0 \)) or via the vertex in \( N(b) \cap N(c_2) \) (if \( |N(b) \cap N(c_2)| = 1 \)). The robber will be caught.

ii. If \( |N(b) \cap N(c_2)| = 2 \), then \( N(b) \cap N(c_1) = \emptyset \). If \( |N(a) \cap N(c_1)| = 1 \), move \( c_1 \) to \( a \) via \( z \). The robber will be caught. For if \( |N(a) \cap N(c_1)| = 2 \) (see Figure 27), we move \( c_2 \) to \( b \) via \( x \) if the robber is at \( a \) and move \( c_1 \) to \( a \) via \( z \) if the robber is at \( b \). In either case, the robber will be caught.

So we may assume \( |N(a) \cap N(c_2)| = 1 \) (see Figure 28).
If $|N(b) \cap N(c_2)| = 0$, then keep $c_1$ and $c_3$ at their positions and move $c_2$ to $a$ via $x$. If $|N(b) \cap N(c_1)| = 0$, then keep $c_2$ and $c_3$ at their positions and move $c_1$ to $a$ via $z$. In either case, the robber will be caught. So we may assume $|N(b) \cap N(c_i)| = 1$ for $i = 1, 2$. If $b$ is adjacent to $z$, then keep $c_2$ and $c_3$ at their positions and move $c_1$ to $z$. The robber will be caught. So we may assume $b$ is not adjacent to $z$. Similarly, we may assume $b$ is not adjacent to $x$ (see Figure 29).

Note that $r$ can be at $a$ or $b$. We shall assume $r$ is at $a$. The case $r$ is at $b$ is similar. Move $c_2$ to $x$. Then $r$ will have to move to $b$. Next, move $c_1$ to $e$. Then $r$ will have to move to $f$. Now, move $c_2$ back to $w_2$. If $f$ is not adjacent to a vertex in $N(w_1) \setminus \{e\}$, then $r$ will be caught in the next cop’s turn. If $f$ is adjacent to a vertex in $N(w_1) \setminus \{e\}$, then $r$ will have to move from $f$ to a vertex in $N(w_1) \setminus \{e\}$. Now, move $c_1$ back to $w_1$. At robber’s turn, if $r$ is not at $z$, he will be caught in the next cop’s turn. So $r$ must be at $z$ and $f$ is adjacent to $z$. 

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure27}
\caption{Figure 27}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure28}
\caption{Figure 28}
\end{figure}
Now, reset the movements and assume $r$ is at $a$. Move $c_1$ to $z$. Then $r$ will have to move to $b$. Next, move $c_2$ to $f$. Then $r$ will have to move to $e$. Now, move $c_1$ back to $w_1$. If $e$ is not adjacent to a vertex in $N(w_2) \{ f \}$, then $r$ will be caught in the next cop’s turn. If $e$ is adjacent to a vertex in $N(w_2) \{ f \}$, then $r$ will have to move from $e$ to a vertex in $N(w_2) \{ f \}$. Now, move $c_2$ back to $w_2$. At robber’s turn, if $r$ is not at $x$, he will caught in the next cop’s turn. So $r$ must be at $x$ and $e$ is adjacent to $x$ (see Figure 30).

![Figure 30](image-url)

**Case 3.** Suppose $|M| = 1$. Then $\Delta(G - N[u]) \leq 3$ for all $u \in V(G)$ with $\deg_G(u) = 4$. Suppose there is a vertex $w \in V(G)$ with $\deg_G(w) = 4$ such that $G - N[w]$ is not connected. Place all the cops at $w$. The robber $r$ must be at one of the
components in $G - N[w]$, say $J$. If $\Delta(J) \leq 2$, then by Lemma 4.3, $c_L(J) \leq 2$. So we keep one cop at $w$ and use the other two cops to catch the robber in $J$. Similarly, by Theorems 4.1 and 4.2, we may assume $J = K_3 \Box K_3$ or $|V(J)| = 10$. The former cannot happen because $|M| = 1$. The latter also cannot happen because $G - N[w]$ is not connected.

So we may assume that $G - N[u]$ is connected for all $u \in V(G)$ with $\deg_G(u) = 4$. If there is a vertex $v \in V(G)$ with $\deg_G(v) = 4$ such that $G - N[v]$ is not the Petersen graph, then by Theorem 1.1, $c_L(G - N[v]) \leq 2$. Hence we keep one cop at $v$ and use the other two cops to catch the robber in $G - N[v]$.

Now we may assume that $G - N[u]$ is the Petersen graph for all $u \in V(G)$ with $\deg_G(u) = 4$. By Lemma 4.6, $\Delta(G) \geq 5$, a contradiction.

Hence $c_L(G) \leq 3$ and this completes the proof of the theorem. □

**Corollary 4.7.** If $G$ is a connected graph with $n$ vertices and $\Delta(G) \geq n - 16$, then $c_L(G) \leq 4$.

**Proof.** Let $u \in V(G)$ with $\deg(u) = \Delta(G)$. Place all the four cops at $u$. The robber $r$ must be at a component in $G - N[u]$, say $H$. Note that $|V(H)| \leq 15$. By Theorem 1.2, $c_L(H) \leq 3$. So we keep one cop at $u$ and use the other three cops to catch the robber in $H$. □

5. Remarks

In this paper, we proved in Theorem 1.2 that if a connected graph has lazy cop number 4, then it must have at least 16 vertices. We also know from [7] that the graph $K_4 \Box K_4$ has lazy cop number 4. However we are unable to show that there is no other connected graph of order 16 with lazy cop number 4.

**Question 5.1.** Is it true that $K_4 \Box K_4$ the unique graph of order 16 with lazy cop number 4?

In fact, the general case of the above question is conjectured in [7], and is still an open problem.

**Conjecture 5.2 ([7], Conjecture 5.1).** The graph $K_n \Box K_n$ is the unique small-
est graph with $c_L = n$.

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