FOURTH HANKEL DETERMINANT FOR THE FAMILY OF FUNCTIONS WITH BOUNDED TURNING

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Abstract. The main aim of this paper is to study the fourth Hankel determinant for the class of functions with bounded turning. We also investigate for 2-fold symmetric and 3-fold symmetric functions.

1. Introduction and definitions

Let $\mathcal{A}$ denote the family of all functions $f$ that are analytic in the open unit disc $D = \{ z \in \mathbb{C} : |z| < 1 \}$ having the Taylor series expansions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in D),$$

while $S$ represents a family of functions $f \in \mathcal{A}$ that are univalent in $D$. Let $S^*$, $C$ and $R$ denote the classes of starlike, convex and bounded turning functions respectively and are defined as:

$$S^* = \left\{ f : f \in \mathcal{A} \text{ and } \Re \left( \frac{zf'(z)}{f(z)} \right) > 0, \quad z \in D \right\},$$

$$C = \left\{ f : f \in \mathcal{A} \text{ and } \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0, \quad z \in D \right\},$$

and

$$R = \left\{ f : f \in \mathcal{A} \text{ and } \Re (f'(z)) > 0, \quad z \in D \right\}.$$

Let $P$ denote the family of all analytic functions $p$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

in $D$ whose real parts are positive in $D$. It is known that the $n$th coefficient for the functions belong to the family $S^*$, is bounded by $n$ and this bound helps to study its geometric properties. In particular, the growth and distortion...
properties of a normalized univalent function \( f \in S \) are determined by the bound of its second coefficient.

The Hankel determinant \( H_{q,n}(f) \) \( (q, n \in \mathbb{N} = \{1, 2, \ldots \}) \) for a function \( f \in S \) of the form (1.1) was defined by Pommerenke [21, 22], (see also [2, 3]) as

\[
H_{q,n}(f) := \begin{vmatrix}
    a_n & a_{n+1} & \ldots & a_{n+q-1} \\
    a_{n+1} & a_{n+2} & \ldots & a_{n+q} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}
\end{vmatrix}.
\]

(1.3)

For fixed integer \( q \) and \( n \), the growth of \( H_{q,n}(f) \) has been studied for different subfamilies of univalent functions. We include here a few of them. The sharp bounds of \(|H_{2,2}(f)|\) for the subfamilies \( S^* \), \( C \) and \( R \) of the set \( S \) were investigated by Janteng et al. [10, 11]. They proved the bounds

\[
|H_{2,2}(f)| \leq \begin{cases} 
1 & \text{for } f \in S^*, \\
\frac{1}{8} & \text{for } f \in C, \\
\frac{4}{9} & \text{for } f \in R.
\end{cases}
\]

For the family of Bazilevič functions, the exact estimate of \(|H_{2,2}(f)|\) was obtained by Krishna et al. [13]. For more works on \( H_{2,2}(f) \) for subfamilies of \( S \) see the references [5, 9, 12, 14, 17, 19, 20].

Unfortunately, the sharp bound of \(|H_{2,2}(f)|\) for the whole class \( S \) is still not known. In [26], Thomas conjectured that if \( f \in S \), then \(|H_{2,2}(f)| \leq 1\). As it was shown by Li and Srivastava in [15], this conjecture is not true for \( n \geq 4 \). Similarly, Răducanu and Zaprawa in [23] proved that it is also false for \( n = 2 \). In fact, they showed that \( \max\{|H_{2,2}(f)| : f \in S\} \geq 1.175 \ldots \).

The estimation of \(|H_{3,1}(f)|\) is much more difficult than the case of \(|H_{2,2}(f)|\). The first paper on \( H_{3,1}(f) \) appears in 2010 by Babalola [4] in which he obtained the upper bound of \( H_{3,1}(f) \) for the families of \( S^* \), \( C \) and \( R \). Later on some other authors [1, 6, 8, 24, 25, 27] published their works concerning \(|H_{3,1}(f)|\) for different subfamilies of analytic and univalent functions. Recently in 2016, Zaprawa [28] improved the results of Babalola [4] by proving

\[
|H_{3,1}(f)| \leq \begin{cases} 
1 & \text{for } f \in S^*, \\
\frac{49}{450} & \text{for } f \in C, \\
\frac{41}{60} & \text{for } f \in R,
\end{cases}
\]

and claimed that these bounds are still not sharp. Further for the sharpness, he considered the subfamilies of \( S^* \), \( C \) and \( R \) consisting of functions with \( m \)-fold symmetry and obtained the sharp bounds. In this paper, we contribute to the fourth Hankel determinant for the class of functions with positive real part.

### 2. A set of lemmas

In order to find the bound of the fourth Hankel determinant, we need the following sharp estimates for the class \( S^* \) of starlike functions and \( P \) of functions with positive real part.
Lemma 2.1. If \( p \in P \), then, for \( n, k \in \mathbb{N} = \{1, 2, \ldots \} \), the following sharp inequalities hold

\[
|c_{n+k} - \lambda c_n c_k| \leq 2 \quad \text{for } 0 \leq \lambda \leq 1, \tag{2.1}
\]

\[
|c_n| \leq 2. \tag{2.2}
\]

The inequalities (2.1) and (2.2) are proved in [7] and [18] respectively.

Lemma 2.2. Let \( p \in P \) of the form (1.2). Then

\[
2c_2 = c_1^2 + x (4 - c_1^2)
\]

for some \( x \) with \( |x| \leq 1 \).

This result is due to Libera and Zlotkiewicz [16].

Let \( g \in S^* \) of the form

\[
g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (z \in \mathbb{D}). \tag{2.3}
\]

Then for the real number \( \lambda \), consider the functional

\[
\Phi_g(\lambda) = |b_2^2 (b_3 - \lambda b_2^2)|. 
\]

Now we prove the upper bound of \( \Phi_g(\lambda) \) as follows.

Theorem 2.3. Let \( g \in S^* \) of the form (2.3). Then

\[
\Phi_g(\lambda) \leq \begin{cases} 
4 (3 - 4\lambda), & \lambda \leq 5/8, \\
\frac{10}{(1 - \lambda)}, & \lambda \in [5/8, 3/4], \\
\frac{4 (1 - \lambda)}, & \lambda \in [3/4, 7/8], \\
\frac{4 (4\lambda - 3)}, & \lambda \geq 7/8.
\end{cases}
\]

Proof. Let \( g \in S^* \) of the form (2.3). Then

\[
\frac{z g'(z)}{g(z)} = p(z),
\]

where \( p \) is in class \( P \) of functions with positive real part. Then it is easy to see that

\[
b_2 = c_1, \quad 2b_3 = c_2 + c_1^2.
\]

Hence by applying Lemma 2.2, and the above relations, we get

\[
\Phi_g(\lambda) = \frac{1}{4} |c_1^2 [x (4 - c_1^2) + (3 - 4\lambda) c_1^2]| 
\]

for some \( x \) such that \( |x| \leq 1 \). Taking into account of the invariance of \( \Phi_g \) under rotation, we may assume that \( c_1 \) is a non negative real number such that \( c_1 = 2r, r \in [0, 1] \). Therefore

\[
\Phi_g(\lambda) = 4r^2 |(1 - r^2) x + (3 - 4\lambda) r^2|.
\]

1. Now we suppose that \( \lambda \leq 3/4 \). Then

\[
\Phi_g(\lambda) \leq 4r^2 [2 (1 - 2\lambda) r^2 + 1].
\]
Let \( q_1 (r) = 4r^2 \left[ 2 (1 - 2\lambda) r^2 + 1 \right] \). Then for \( \lambda \leq 1/2 \) and \( r \in [0, 1] \), \( q_1 (r) \) is an increasing function. Hence \( q_1 (r) \leq q_1 (1) \). For \( \lambda \in (1/2, 3/4] \), we have

\[
q_1 (r) \leq \begin{cases} 
q_1 (1), & \lambda \in (1/2, 5/8], \\
q_1 \left(1/\sqrt{4(2\lambda - 1)}\right), & \lambda \in [5/8, 3/4]. 
\end{cases}
\]

2. For the case \( \lambda \geq 3/4 \), we have

\[
\Phi_3 (\lambda) \leq 4r^2 \left[ 4 (\lambda - 1) r^2 + 1 \right].
\]

Again, letting \( q_2 (r) = 4r^2 [4 (\lambda - 1) r^2 + 1] \) and using similar arguments, we have

\[
q_2 (r) \leq \begin{cases} 
q_2 \left(1/\sqrt{8(1-\lambda)}\right), & \lambda \in [3/4, 7/8], \\
q_2 (1), & \lambda \geq 7/8.
\end{cases}
\]

Hence, we have the required result.

3. Bounds of \(|H_{4,1} (f)|\) for the set \( \mathcal{R} \)

First, for any \( f \in \mathcal{A} \) of the form (1.1), we can write \( H_{4,1} (f) \) in the form

\[
H_{4,1} (f) := a_7 H_3 (1) - a_6 \Delta_1 + a_5 \Delta_2 - a_4 \Delta_3,
\]

where \( \Delta_1, \Delta_2 \) and \( \Delta_3 \) are determinants of order 3 given by

\[
\Delta_1 = (a_3 a_6 - a_4 a_5) - a_2 (a_2 a_6 - a_3 a_5) + a_4 (a_2 a_4 - a_3^2),
\]

\[
\Delta_2 = (a a_6 - a_2^2) - a_2 (a a_6 - a_4 a_5) + a_3 (a_3 a_5 - a_2^2),
\]

\[
\Delta_3 = a_2 (a_4 a_6 - a_2^2) - a_3 (a_3 a_6 - a_4 a_5) + a_4 (a_3 a_5 - a_4^2).
\]

From (1.3), we observe that \( H_{4,1} (f) \) is a polynomial of six successive coefficients \( a_2, a_3, a_4, a_5, a_6 \) and \( a_7 \) of a function \( f \) in a given class. However, in many problems these coefficients are connected to the coefficients of the function \( p \) in the set \( \mathcal{P} \).

Assume now that \( f \in \mathcal{R} \). We have

\[
f'(z) = p(z),
\]

where \( p \in \mathcal{P} \) of the form (1.2). From (3.5), we can easily obtain

\[
a_{n-1} = c_n.
\]

Using (3.6) in (3.2), (3.3) and (3.4), it follows that

\[
\Delta_1 = \frac{1}{18} c_2 c_5 - \frac{1}{20} c_3 c_4 - \frac{1}{24} c_2^2 c_3 + \frac{1}{36} c_1 c_2 c_4 + \frac{1}{32} c_1^2 c_3 - \frac{1}{36} c_2 c_3^2,
\]

\[
\Delta_2 = \frac{1}{24} c_3 c_5 - \frac{1}{25} c_4 c_4 + \frac{1}{40} c_1 c_3 c_4 - \frac{1}{36} c_1 c_2 c_5 + \frac{1}{45} c_2 c_4 - \frac{1}{48} c_2 c_3^2,
\]

\[
\Delta_3 = \frac{1}{48} c_1 c_3 c_5 - \frac{1}{50} c_1 c_3^2 + \frac{1}{30} c_2 c_3 c_4 - \frac{1}{64} c_3^3 - \frac{1}{54} c_2 c_5.
\]

Now we can prove our main result.
Theorem 3.1. If \( f \in \mathcal{R} \), then
\[
|H_{4,1}(f)| \leq \frac{73757}{94500} \approx 0.78050.
\]

Proof. Let \( f \in \mathcal{R} \). Then we can rewrite (3.7), (3.8) and (3.9) in the following ways
\[
\Delta_1 = \frac{c_5 (c_2 - c_1^2)}{24} + \frac{c_3 (c_4 - c_2^2)}{36} - \frac{c_3 (c_4 - c_1 c_3)}{32} - \frac{67 c_4 (c_3 - c_1 c_2)}{1440}
\]
\[
+ \frac{19 c_2 (c_5 - c_1 c_4)}{1440} + \frac{e_2 c_5}{1440},
\]
\[
\Delta_2 = \frac{c_5 (c_3 - c_1 c_2)}{36} - \frac{c_4 (c_4 - c_2^2)}{45} + \frac{c_3 (c_5 - c_2 c_3)}{48} - \frac{4 c_4 (c_4 - c_1 c_3)}{225}
\]
\[
- \frac{13 c_3 (c_5 - c_1 c_4)}{1800} + \frac{e_3 c_5}{3600},
\]
\[
\Delta_3 = \frac{c_5 (c_4 - c_2^2)}{54} - \frac{c_5 (c_4 - c_1 c_3)}{48} + \frac{c_3 (c_6 - c_3^2)}{64} - \frac{c_3 (c_6 - c_2 c_4)}{64}
\]
\[
+ \frac{c_4 (c_5 - c_1 c_4)}{50} - \frac{17 c_4 (c_5 - c_2 c_3)}{960} + \frac{e_4 c_5}{43200}.
\]

Using the triangle inequality along with the inequalities (2.1) and (2.2), we obtain
\[
|\Delta_1| \leq \frac{1}{6} + \frac{1}{9} + \frac{1}{8} + \frac{67}{360} + \frac{19}{360} + \frac{1}{360} = \frac{29}{45},
\]
\[
|\Delta_2| \leq \frac{1}{9} + \frac{4}{45} + \frac{1}{12} + \frac{16}{225} + \frac{26}{900} + \frac{1}{900} = \frac{173}{450},
\]
and
\[
|\Delta_3| \leq \frac{2}{27} + \frac{1}{12} + \frac{1}{16} + \frac{1}{16} + \frac{2}{25} + \frac{17}{240} + \frac{1}{10800} = \frac{13}{30}.
\]

Now putting the values \(|H_{3,1}(f)| \leq \frac{41}{108}, |\Delta_1| \leq \frac{29}{45}, |\Delta_2| \leq \frac{123}{450}, |\Delta_3| \leq \frac{13}{30}\) along with the inequality \(|a_n| \leq \frac{2}{n}\) for \( n \geq 2 \) in (3.1), we obtain
\[
|H_{4,1}(f)| \leq |a_7| |H_3(1)| + |a_6| |\Delta_1| + |a_5| |\Delta_2| + |a_4| |\Delta_3|
\]
\[
\leq \frac{2}{7} + \frac{1}{6} + \frac{1}{2} + \frac{1}{6} + \frac{17}{3} + \frac{1}{2} + \frac{1}{3} + \frac{11}{2}
\]
\[
= \frac{73757}{94500} \approx 0.78050
\]
and this completes the proof. \( \square \)

4. Bounds of \( |H_{4,1}(f)| \) for the sets \( \mathcal{R}^{(2)} \) and \( \mathcal{R}^{(3)} \)

Let \( m \in \mathbb{N} = \{1, 2, \ldots\} \). A domain \( \Lambda \) is said to be \( m \)-fold symmetric if a rotation of \( \Lambda \) about the origin through an angle \( 2\pi/m \) carries \( \Lambda \) on itself. A function \( f \) is said to be \( m \)-fold symmetric in \( \mathbb{D} \), if
\[
f(e^{2\pi i/m} z) = e^{2\pi i/m} f(z), \quad (z \in \mathbb{D}).
\]
By $S^{(m)}$, we mean the set of $m$-fold univalent functions having the following Taylor series form

\begin{equation}
 f(z) = z + \sum_{k=1}^{\infty} a_{mk+1}z^{mk+1}, \ (z \in \mathbb{D}).
\end{equation}

The sub-family $R^{(m)}$ of $S^{(m)}$ is the set of $m$-fold symmetric bounded turning functions. More intuitively, an analytic function $f$ of the form (4.1) belongs to the family $R^{(m)}$ if and only if

\[ f'(z) = p(z) \]

with $p \in P^{(m)}$, where the set $P^{(m)}$ is defined by

\begin{equation}
 P^{(m)} = \left\{ p \in P : p(z) = 1 + \sum_{k=1}^{\infty} c_{mk}z^{mk}, \ (z \in \mathbb{D}) \right\}.
\end{equation}

**Theorem 4.1.** If $f \in f \in R^{(3)}$, then

\[ |H_{4,1}(f)| \leq \frac{1}{49}. \]

**Proof.** Now, let $f \in R^{(3)}$. Then there exists a function $\tilde{g}(z) = z + d_4z^4 + d_7z^7 + \cdots \in S^*(3)$ such that $\frac{\tilde{g}'(z)}{\tilde{g}(z)} = f'(z)$. Since $f \in R^{(3)}$, using the series form (4.1) for $m = 3$, we get

\[ 1 + 3d_4z^3 + (6d_7 - 3d_4^2)z^6 + \cdots = 1 + 4a_4z^3 + 7a_7z^6 + \cdots. \]

Comparing the coefficients of $z^3$ and $z^6$ on both sides, we obtain

\begin{equation}
 3d_4 = 4a_4, \quad 6d_7 - 3d_4^2 = 7a_7.
\end{equation}

Since $\tilde{g} \in S^*(3)$, there exists a function $g$ in $S^*$ of the form (2.3) such that $\tilde{g}(z) = \sqrt[3]{g(z^3)}$. Therefore

\[ z + d_4z^4 + d_7z^7 + \cdots = z + \frac{1}{3}b_2z^4 + \left( \frac{1}{3}b_3 - \frac{1}{9}b_2^2 \right)z^7 + \cdots. \]

Comparing the coefficients of $z^4$ and $z^7$, we get

\begin{equation}
 d_4 = \frac{1}{3}b_2, \quad d_7 = \frac{1}{3}b_3 - \frac{1}{9}b_2^2.
\end{equation}

Now from (4.3) and (4.4), it follows that

\begin{equation}
 a_4 = \frac{b_2}{4}, \quad a_7 = \frac{1}{7}(2b_3 - b_2^2).\]

We observe that $a_2 = a_3 = a_5 = a_6 = 0$ for the function $f \in R^{(3)}$. Also it is clear that $H_{4,1}(f) = a_4^2(a_4^2 - a_7)$. This implies that

\[ |H_{4,1}(f)| = \frac{1}{56} \left| b_2^2 \left( b_3 - \frac{23}{32}b_2^2 \right) \right|. \]

Using Theorem 2.3 for $\lambda = \frac{23}{32} \in \left[ \frac{5}{8}, \frac{3}{4} \right]$, we have the required result. \qed
Theorem 4.2. If $f \in f \in \mathcal{R}^{(2)}$, then

$$|H_{4,1}(f)| \leq \frac{368}{2625}.$$ 

Proof. It is clear that for $f \in \mathcal{R}^{(2)}$ we have $a_2 = a_4 = a_6 = 0$. Consequently

$$H_{4,1}(f) := a_3a_5a_7 - a_3^3a_7 + a_3^2a_5^2 - a_3^3.$$ 

Since $f \in \mathcal{R}^{(2)}$, there exists a function $p \in \mathcal{P}^{(2)}$ such that $f'(z) = p(z)$. For $f \in \mathcal{R}^{(2)}$, using the series form (4.1) and (4.2) when $m = 2$, we can write

$$3a_3 = c_2, \quad 5a_5 = c_4, \quad 7a_7 = c_6.$$ 

Therefore

$$H_{4,1}(f) = \frac{1}{105}(c_2c_4c_6 - \frac{1}{189}c_2^3c_6 + \frac{1}{225}c_2^2c_4^2 - \frac{1}{125}c_4^3)$$

$$= \frac{1}{105}(c_2c_6 - \frac{21}{25}c_2^2)(c_4 - \frac{5}{9}c_2^2).$$

Using Lemma 2.1 and the triangle inequality, we get

$$|H_{4,1}(f)| \leq \frac{368}{2625}.$$ 

Hence the proof is complete. □

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