ON COMMUTATIVITY OF REGULAR PRODUCTS

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Abstract. We study the one-sided regularity of matrices in upper triangular matrix rings in relation with the structure of diagonal entries. We next consider a ring theoretic condition that $ab$ being regular implies $ba$ being also regular for elements $a, b$ in a given ring. Rings with such a condition are said to be commutative at regular product (simply, CRP rings). CRP rings are shown to be contained in the class of directly finite rings, and we prove that if $R$ is a directly finite ring that satisfies the descending chain condition for principal right ideals or principal left ideals, then $R$ is CRP. We obtain in particular that the upper triangular matrix rings over commutative rings are CRP.

This article concerns a ring property related to directly finite (or Dedekind finite) condition, which extends the study of one-sided inverses (e.g., Baer [3] and Jacobson [9]) to one of one-sided regularity. In Section 1, the structure of diagonal entries of one-sided regular upper triangular matrices are investigated, and this gives useful information to observe the commutativity of regular products in upper triangular matrix rings. In Section 2, we study the structure of rings which satisfy the commutativity of regular products. In the procedure we observe directly finite rings which do not satisfy the commutativity of regular products, which provides interesting information to our study.

Throughout this note every ring is an associative ring with identity unless otherwise stated. Let $R$ be a ring. Use $U(R)$, $N^*(R)$, and $N(R)$ to denote the group of units, the upper nilradical (i.e., sum of all nil ideals), and the set of all nilpotent elements in $R$, respectively. Clearly $N^*(R) \subseteq N(R)$. The polynomial (resp., power series) ring with an indeterminate $x$ over $R$ is denoted by $R[x]$ (resp., $R[[x]]$). $\mathbb{Z} (\mathbb{Z}_n)$ denotes the ring of integers (modulo $n$). Denote the $n$ by $n$ full (resp., upper triangular) matrix ring over $R$ by $\text{Mat}_n(R)$ (resp., $T_n(R)$), and $D_n(R)$ denotes the subring $\{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$ of $T_n(R)$. Use $E_{ij}$ for the matrix with $(i, j)$-entry 1 and elsewhere 0. These notations are usually used in the literature.

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1. One-sided regularity of upper triangular matrices

In this section we investigate the right (left) regularity of matrices in upper triangular matrix rings. We follow the literature in using the next definitions. An element $u$ of a ring $R$ is right regular if $ur = 0$ implies $r = 0$ for $r \in R$. The left regular can be defined similarly. An element is regular if it is both left and right regular.

**Theorem 1.1.** Let $R$ be a ring, $n \geq 2$, and $(a_{ij}) \in T_n(R)$.

1. If $a_{ii}$ is right (resp. left) regular in $R$ for all $i \in \{1, \ldots, n\}$, then $(a_{ij})$ is right (resp. left) regular in $T_n(R)$.
2. If $(a_{ij})$ is right (resp., left) regular in $T_n(R)$, then $a_{11}$ is right regular (resp., $a_{nn}$ is left regular) in $R$.
3. Let $R$ be a commutative ring. Then $(a_{ij})$ is right or left regular in $T_n(R)$ if and only if $a_{ii}$ is regular in $R$ for all $i \in \{1, \ldots, n\}$ if and only if $(a_{ij})$ is regular in $T_n(R)$.

**Proof.** (1) Let $A = (a_{ij})$. Suppose that every $a_{ii}$ is right regular for $1 \leq i \leq n$, and let $AB = 0$ for $B = (b_{st}) \in T_n(R)$. Clearly $b_{ss} = 0$ for all $s$. Assume $B \neq 0$ on the contrary. Take $b_{ef} \neq 0$ so that $e$ and $f$ are largest respectively. Then $e < f$ and the $(e, f)$-entry of $AB$ is $a_{ef}b_{ef} \neq 0$, hence $AB \neq 0$, contrary to $AB = 0$. Thus $B = 0$. The proof for the case of left regular is similar.

(2) Suppose that $A = (a_{ij})$ is right regular in $T_n(R)$. Assume on the contrary that $a_{11}b = 0$ for some nonzero $b \in R$, and set $B = (b_{st})$ with $b_{11} = b$ and elsewhere zeros. Then $AB = 0$. But since $A$ is right regular, we get $B = 0$, contrary to $b \neq 0$. Thus $a_{11}$ is right regular. The proof for the case of left regular is similar.

(3) By (1) and the commutativity of $R$, it is enough to show that if $A = (a_{ij})$ is right or left regular in $T_n(R)$, then $a_{ii}$ is regular in $R$ for all $i \in \{1, \ldots, n\}$.

First, let $A = (a_{ij}) \in T_n(R)$ and suppose that $A$ is right (resp., left) regular in $T_n(R)$. Then $a_{11}$ (resp., $a_{nn}$) is regular by (2). We use this fact freely hereafter. We will proceed the proof by induction on $n$.

Consider first the case of $n = 2$. Assume on the contrary that $a_{22}$ is not regular. Say that $a_{22}b = 0$ for some $0 \neq b \in R$. Consider $B = \begin{pmatrix} 0 & b_{12} \\ -b_{11} & 0 \end{pmatrix}$ in $T_2(R)$. Then

$$AB = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} 0 & b_{12} \\ 0 & -b_{11} \end{pmatrix} = \begin{pmatrix} b(a_{11}a_{12} - a_{12}a_{11}) \\ -a_{22}b_{11} \end{pmatrix} = 0.$$ 

But since $a_{11}$ is regular, we have $b_{11} \neq 0$, implying $B \neq 0$. Thus $A$ is not right regular, contrary to $A$ being right regular. Therefore $a_{22}$ is also regular.

We next prove the case of $T_n(R)$ for $n \geq 3$. Assume that we use the matrix

$$E_0 = \begin{pmatrix} 0 & \cdots & 0 & e_0b_{11}^{2} \\ 0 & \cdots & 0 & e_0b_{22}^{2} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & e_0b_{n-1,n}^{2} \\ 0 & \cdots & 0 & e_0b_{n,n}^{2} \end{pmatrix}$$
with \( \beta_{n-1} = -a_{(n-2)(n-2)} \cdots a_{22} a_{11}, \beta_{n-2}, \cdots, \beta_1 \in R \), in the procedure to obtain that \( a_{ii} \) is regular for all \( i = 1, \ldots, n-1 \) in \( T_{n-1}(R) \), where \( a_{(n-1)(n-1)} e_0 = 0 \) for some \( 0 \neq e_0 \in R \).

Suppose that \( a_{ii} \) is regular in \( R \) for all \( i \in \{1, 2, \ldots, n-1\} \). Assume on the contrary that \( a_{nn} \) is not regular. Say that \( a_{nn} e = 0 \) for some \( 0 \neq e \in R \). Then, by help of the preceding matrix \( E_0 \) in \( T_{n-1}(R) \), we can find

\[
E = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & -e[a_{12} \beta_1 + e_{13} a_{22} + \cdots + a_{0} a_{13} a_{22} + a_{11} a_{13} a_{22}] \\
0 & 0 & 0 & \cdots & 0 & e_{13} a_{11} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & e_{13} a_{11} \\
0 & 0 & 0 & \cdots & 0 & e_{13} a_{11}
\end{pmatrix} \in T_n(R)
\]

such that \( AE = 0 \), where \( \beta_k \) is obtained from \( \beta_k' \) by replacing \( a_{st} \) by \( a_{(s+1)(t+1)} \) for all \( k = 1, \ldots, n-1 \). But every \( a_{ii} \) is regular for \( i = 1, 2, \ldots, n-1 \), so \( a_{11} \cdots a_{(n-1)(n-1)} \) is also regular. This implies \( e a_{(n-1)(n-1)} \cdots a_{11} \neq 0 \) because \( e \neq 0 \), entailing \( E \neq 0 \). Thus \( A \) is not right regular, contrary to \( A \) being right regular. Therefore \( a_{nn} \) is also regular.

Next we also claim that if \( A \) is left regular, then \( a_{ii} \) is regular in \( R \) for all \( i \).

Consider the case of \( n = 2 \). Assume on the contrary that \( a_{11} \) is not regular. Say that \( a_{11} b = 0 \) for some \( 0 \neq b \in R \). Consider \( B = (-a_{22} a_{12} b) \) in \( T_2(R) \). Then \( B \neq 0 \) since \( a_{22} b \neq 0 \), and

\[
BA = \begin{pmatrix}
-a_{22} b & a_{12} b \\
0 & 0
\end{pmatrix} \begin{pmatrix}
a_{11} & a_{12} \\
a_{11} & a_{22}
\end{pmatrix} = \begin{pmatrix}
a_{12} b a_{11} & -a_{22} b a_{12} + a_{12} b a_{22} \\
0 & 0
\end{pmatrix} = 0,
\]

contradicting \( A \) being left regular. Therefore \( a_{11} \) is also regular. We use this fact freely hereafter. We will proceed the proof by induction on \( n \).

We next prove the case of \( T_n(R) \) for \( n \geq 3 \). Assume that we use the matrix

\[
E_0 = \begin{pmatrix}
e_{01} \beta_1 & e_{02} \beta_2 & \cdots & e_{0n} \beta_{n-1} \\
e_{11} \beta_1 & e_{12} \beta_2 & \cdots & e_{1n} \beta_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0
\end{pmatrix}
\]

with \( \beta_1 = -a_{22} a_{33} \cdots a_{(n-1)(n-1)}, \beta_2, \cdots, \beta_{n-1} \in R \), in the procedure to obtain that \( a_{ii} \) is regular for all \( i = 1, \ldots, n-1 \) in \( T_{n-1}(R) \), where \( a_{ii} e_0 = 0 \) for some \( 0 \neq e_0 \in R \).

Suppose that \( a_{ii} \) is regular in \( R \) for all \( i \in \{2, \ldots, n\} \). Assume on the contrary that \( a_{11} \) is not regular. Say that \( a_{11} e = 0 \) for some \( 0 \neq e \in R \). Then,
by help of the preceding matrix $E_0$ in $T_{n-1}(R)$, we can find
\[
E = \left(\begin{array}{cccccc}
\epsilon \beta_1 a_{nn} & \epsilon \beta_2 a_{nn} & \epsilon \beta_3 a_{nn} & \cdots & \epsilon \beta_{n-1} a_{nn} & -\epsilon (\beta_1 a_{1n} + \beta_2 a_{2n} + \beta_3 a_{3n} + \cdots + \beta_{n-1} a_{(n-1)n}) \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0
\end{array}\right)
\]
in $T_n(R)$ such that $EA = 0$. Note $\epsilon \beta_1 a_{nn} = -ea_{22} a_{33} \cdots a_{(n-1)(n-1)} a_{nn}$. But every $a_{ii}$ is regular for $i = 2, \ldots, n$, so $a_{22} \cdots a_{nn}$ is also regular. This implies $ea_{22} \cdots a_{nn} \neq 0$ because $e \neq 0$, entailing $E \neq 0$. Thus $A$ is not left regular, contradicting $A$ being left regular. Therefore $a_{11}$ is also regular.

The converse of Theorem 1.1(1) need not be true by the following.

**Example 1.2.** Let $R$ be a ring which has right regular elements $a$ and $b$ satisfying $aR \cap bR = 0$. Consider
\[
\left(\begin{array}{cc}
a & b \\
0 & 0
\end{array}\right) \left(\begin{array}{cc}
c & d \\
0 & e
\end{array}\right) = \left(\begin{array}{cc}
ac & ad + be \\
0 & 0
\end{array}\right) = 0
\]
for $(c \ d) \in T_2(R)$. Since $a$ is right regular, we get $c = 0$. From $ad + be = 0$, we obtain $d = 0 = e$ by the condition that $ad = -be \in aR \cap bR = 0$, because $a$ and $b$ are right regular. Thus $(c \ d) = 0$, and so $(a \ b)$ is right regular in $T_2(R)$.

For example, let $K$ be a field and $R = K(x, y)$ be the free algebra generated by the non-commuting indeterminates $x, y$ over $K$. Then $(x \ y)$ is right regular in $T_2(R)$ by the argument above, because $xR \cap yR = 0$.

Considering Theorem 1.1(3), it is natural to ask whether if $(a_{ij})$ is regular in $T_n(R)$, then every $a_{ii}$ is regular in $R$ when $R$ is a noncommutative ring. The answer is negative by the following.

**Example 1.3.** (I) Then case of $T_2(R)$:

Let $R_0 = K(x, y)$ be the free algebra generated by the non-commuting indeterminate $x, y$ over a field $K$.

We use the matrix $(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix})$ in Example 1.2 that is right regular in $T_2(R_0)$. But this matrix is not left regular in $T_2(R_0)$ by Theorem 1.1(2), since its $(2,2)$-entry is not left regular. Write $R = T_2(R_0)$. Consider a matrix
\[
A = \left(\begin{array}{ccc}
x & 0 & 0 \\
0 & 0 & x \\
0 & x & 0 \\
0 & 0 & y
\end{array}\right) \in T_2(R).
\]

Then $(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix})$ is regular in $R$ by Theorem 1.1(1), and so $A$ is right regular in $T_2(R)$ also by Theorem 1.1(1). Suppose that $BA = 0$ for $B = (b_{ij}) \in T_2(R)$ with $b_{11} = (a_{11} \ b_{11}), b_{12} = (a_{21} \ b_{21})$. Since $(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix})$ is regular, we have $b_{22} = 0$. 

From \((a_1, b_1) \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = 0\), we get \(a_1 = 0\). So, from

\[
0 = \begin{pmatrix} 0 & b_1 \\ 0 & c_1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} a_2x + b_1y & b_1x + b_2y \\ 0 & c_1x + c_2y \end{pmatrix},
\]

we first get \(a_2 = 0\), and next obtain \(b_1 = 0 = b_2\) and \(c_1 = 0 = c_2\) because \(R_0x \cap R_0y = 0\). Thus \(B = 0\) and so \(A\) is left regular in \(T_2(R)\). Therefore \(A\) is regular in \(T_3(R)\), but the \((1,1)\)-entry of \(A\) is not left regular in \(R\).

(II) Then case of \(T_n(R)\) for \(n \geq 3\):

We first argue about the case of \(T_3(R)\). Let \(R = K(x, y)\) as in (I). Consider \(C = \begin{pmatrix} x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y \end{pmatrix}\) in \(T_3(R)\), and let \(CD = 0\) for \(D = (d_{ij}) \in T_3(R)\). Then \(d_{11} = 0 = d_{33}\), and we obtain \(d_{12} = d_{13} = d_{22} = d_{23} = 0\) from \(xd_{12} + yd_{22} = 0\) and \(xd_{13} + yd_{23} = 0\) because \(xR_0 \cap yR_0 = 0\). Let \(EC = 0\) for \((e_{ij}) \in T_3(R)\). Then \(e_{11} = 0 = e_{33}\), and we obtain \(e_{12} = e_{13} = e_{22} = e_{23} = 0\) from \(e_{12}x + e_{13}y = 0\) and \(e_{22}x + e_{23}y = 0\) because \(R_0x \cap R_0y = 0\). These imply that \(C\) is regular in \(T_3(R)\), but the \((2,2)\)-entry of \(C\) is zero.

We extend the preceding result to the general case. Let \(R\) be the free algebra \(K(x_1, x_2, \ldots, x_{n-1})\) generated by the non-commuting indeterminates \(x_1, x_2, \ldots, x_{n-1}\) over \(K\), where \(n \geq 3\). Consider a matrix

\[
M = \begin{pmatrix}
x_1 & x_2 & x_3 & \cdots & x_{n-2} & x_{n-1} & 0 \\
0 & x_2 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & x_3 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & x_{n-3} \\
0 & 0 & 0 & \cdots & 0 & 0 & x_{n-2} \\
0 & 0 & 0 & \cdots & 0 & 0 & x_{n-1} \\
\end{pmatrix}
\]

in \(T_n(R)\). Let \(MF = 0\) for \(F = (f_{ij}) \in T_n(R)\). Then \(f_{11} = f_{22} = \cdots = f_{(n-2)(n-2)} = f_{nn} = 0\), and we obtain \(f_{(n-1)(n-1)} = 0\) and \(f_{ij} = 0\) for all \(i, j\) with \(i \neq j\) from the equalities

\[
x_1f_{1k} + x_2f_{2k} + \cdots + x_{k-1}f_{(k-1)k} + x_kf_{kk} = 0 \text{ for } k = 2, \ldots, n-1
\]

and

\[
x_1f_{1n} + x_2f_{2n} + \cdots + x_{n-2}f_{(n-2)n} + x_{n-1}f_{(n-1)n} = 0,
\]

because \(x_sR \cap (\sum_{t \neq s} x_tR) = 0\) for all \(s = 1, \ldots, n-1\). Thus \(F = 0\). Let \(GM = 0\) for \(G = (g_{ij}) \in T_n(R)\). Then \(g_{11} = g_{22} = \cdots = g_{(n-2)(n-2)} = g_{nn} = 0\), and we obtain \(g_{(n-1)(n-1)} = 0\) and \(g_{ij} = 0\) for all \(i, j\) with \(i \neq j\) from the equality

\[
gh_1x_1 + gh_3x_2 + \cdots + gh_{(n-1)}x_{n-2} + ghnx_{n-1} = 0 \text{ for } h = 1, \ldots, n-1,
\]

because \(Rx_s \cap (\sum_{t \neq s} Rx_t) = 0\) for all \(s = 1, \ldots, n-1\). Thus \(G = 0\), entailing that \(M\) is regular in \(T_n(R)\). But the \((n-1, n-1)\)-entry of \(M\) is zero.
2. Commutative property at regular products

In this section we study the structure of rings with a kind of commutative property at regular products. Following the literature, a ring \( R \) is said to be directly finite (or Dedekind finite) if \( ab = 1 \) implies \( ba = 1 \) for \( a, b \in R \). A ring is usually called Abelian if every idempotent is central. A ring is usually called reduced if it has no nonzero nilpotent elements. It is easily checked that Abelian rings are directly finite and reduced rings are Abelian. The class of directly finite rings is obviously closed under subrings. Left or right Noetherian rings are directly finite by [9, Theorem 1]. It is well-known that the class of directly finite rings is obviously closed under subrings. Left or right Noetherian rings are directly finite by [9, Theorem 1].

We first observe several useful equivalent conditions to the direct finiteness in the following.

Lemma 2.1. Given a ring \( R \), the following conditions are equivalent:

1. \( R \) is directly finite.
2. If \( a_1 a_2 \cdots a_n \in U(R) \) for \( a_1, a_2, \ldots, a_n \in R \), then
   \[ a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)} \in U(R) \]
   for any permutation \( \sigma \) of the set \( \{1, 2, \ldots, n\} \), where \( n \geq 2 \).
3. \( ab \in U(R) \) implies \( ba \in U(R) \) for all \( a, b \in R \).
4. If \( a_1 a_2 \cdots a_n = 1 \) for \( a_1, a_2, \ldots, a_n \in R \), then
   \[ a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)} \in U(R) \]
   for any permutation \( \sigma \) of the set \( \{1, 2, \ldots, n\} \), where \( n \geq 2 \).
5. \( ab = 1 \) implies \( ba \in U(R) \) for all \( a, b \in R \).
6. \( ab = 1 \) implies that both \( a \) and \( b \) are regular for all \( a, b \in R \).

Proof. (1) \( \Rightarrow \) (2): Let \( R \) be directly finite, and suppose that \( a_1 a_2 \cdots a_n \in U(R) \) for \( a_1, a_2, \ldots, a_n \in R \), where \( n \geq 2 \). Say that \( a_1 a_2 \cdots a_n b = b a_1 a_2 \cdots a_n = 1 \) for \( b \in R \). Since \( R \) is directly finite and \( a_1 (a_2 \cdots a_n b) = 1 \), we have \( (a_2 \cdots a_n b) a_1 = 1 \), entailing \( a_1 \in U(R) \). Next since \( R \) is directly finite and \( a_2 (a_3 \cdots a_n b a_1) = 1 \), we have \( (a_3 \cdots a_n b a_1) a_2 = 1 \), entailing \( a_2 \in U(R) \). Inductively we have \( a_i \in U(R) \) for all \( i \). This yields that \( a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)} \in U(R) \) for any permutation \( \sigma \) of the set \( \{1, 2, \ldots, n\} \).

(2) \( \Rightarrow \) (3), (2) \( \Rightarrow \) (4), (3) \( \Rightarrow \) (5), and (4) \( \Rightarrow \) (5) are obvious.

(5) \( \Rightarrow \) (6): Let \( ab = 1 \) for \( a, b \in R \), and suppose that the condition (5) holds. If \( ar = 0 \) for \( r \in R \), then \( (ba)r = 0 \) and so \( r = 0 \) since \( ba \in U(R) \). If \( r'a = 0 \) for \( r' \in R \), then \( 0 = r'ab = r' \). These show that \( a \) is regular. Similarly, it can be obtained that \( b \) is also regular.

(6) \( \Rightarrow \) (1): Let \( ab = 1 \) for \( a, b \in R \), and suppose that the condition (6) holds. Then \( ab = 1 \) implies \( (ba)^2 = ba \), and \( ba(ba - 1) = 0 \). So \( ba = 1 \) since \( ba \) is regular by (6). \( \square \)
We can see a basic extension preserving the direct finiteness in the following. Recall the subring
\[ V_n(R) = \{ m = (m_{ij}) \in D_n(R) \mid m_{st} = m_{(s+1)(t+1)} \text{ for } s = 1, \ldots, n-2 \text{ and } t = 2, \ldots, n-1 \} \]
of \( D_n(R) \), where \( R \) is given a ring and \( n \geq 2 \). Note that \( R[x]/x^nR[x] \) is isomorphic to \( V_n(R) \), via \((a_0 + a_1x + \cdots + a_{n-1}x^{n-1}) + x^nR[x] \mapsto (a_0, a_1, \ldots, a_{n-1})\), where \((a_{ij}) \in V_n(R)\) is expressed by \((a_0, a_1, \ldots, a_{n-1})\) with \( a_{ij} = a_{ji} \). Consider a power series \( f(x) = \sum_{i=0}^{\infty} a_i x^i \) over given a ring \( R \). Note that \( a_0 \in U(R) \) if and only if \( f(x) \in U(R[[x]]) \).

**Corollary 2.2.** Let \( R \) be a ring and \( n \geq 2 \).

1. A ring \( R \) is directly finite if and only if \( T_n(R) \) is directly finite if and only if \( D_n(R) \) is directly finite if and only if \( V_n(R) \) is directly finite.

2. A ring \( R \) is directly finite if and only if so is \( R[[x]] \).

**Proof.** (1) It suffices to show that \( E = T_n(R) \) is directly finite when \( R \) is a directly finite ring, because the class of directly finite rings is closed under subrings. Let \((a_{ij})(b_{ij}) = 1\) for \((a_{ij}), (b_{ij}) \in E\). Then \( a_{ii} b_{ii} = 1 \) for all \( i = 1, \ldots, n \). Since \( R \) is directly finite, \( b_{ii} a_{ii} = 1 \) for all \( i \). This implies that \((b_{ij})(a_{ij}) \in U(E)\). Then \( E \) is directly finite by Lemma 2.1.

(2) It also suffices to show the necessity. Let \( f(x)g(x) = 1 \) for \( f(x), g(x) \in R[[x]] \), where \( f(x) = \sum_{i=0}^{\infty} a_i x^i \) and \( g(x) = \sum_{j=0}^{\infty} b_j x^j \). Then \( a_0 b_0 = 1 \). Since \( R \) is directly finite, we have \( b_0 a_0 = 1 \), entailing \( a_0, b_0 \in U(R) \). This implies that \( f(x), g(x) \in U(R[[x]]) \). Then \( g(x) f(x) = 1 \) by Lemma 2.1.

From the condition (3) in Lemma 2.1, we next introduce a new concept through the argument to follow. First for a given ring \( R \) consider the following:

\[(*)\]

If \( ab \) is regular for \( a, b \in R \), then \( ba \) is also regular.

A ring \( R \) satisfying the condition (*) is directly finite. For, letting \( ab = 1 \) for \( a, b \in R \), \( ba \) is regular because \( R \) satisfies (*). Then \( (ba)^2 = ba \) and so \( ba (ba - 1) = 0 \), implying \( ba = 1 \). So it is natural to ask whether directly finite rings satisfy the condition (*). But the answer is negative by the following.

**Example 2.3.** We use the ring in [2, Example 4.8] and the argument in [11, Proof of Theorem 1]. Let \( K \) be a field and \( A = K \langle a, b \rangle \) be the free algebra generated by the noncommuting indeterminates \( a, b \) over \( K \). Let \( I \) be the ideal of \( A \) generated by \( b^a \) and set \( R = A/I \), where \( n \geq 2 \). Write \( r = r + I \) for \( r \in A \). Then \( R \) is Abelian (hence directly finite) by [2, Theorem 4.7] and [7, Corollary 8]. We next claim that \( R \) does not satisfy the condition (*). Let \( \alpha, \beta \in R \setminus \{0\} \) satisfy \( \alpha \beta = 0 \). Then \( \alpha = \alpha' b^s \) and \( \beta = b^t \beta' \) for some \( \alpha', \beta' \in R \), where \( 1 \leq s, t \leq n - 1 \) and \( s + t \geq n \), by help of the argument in [11, Proof of Theorem 1]. Consider the element \( \alpha \beta a \) in \( R \). Then \( \alpha \beta a = a (\alpha \beta) a = 0 \) is regular by the preceding argument. But \( (b^a)a \) is not (left) regular as can be seen by \( b^{n-1} (b^a)a = 0 \), noting \( b^{n-1} \neq 0 \). So \( R \) does not satisfy the condition (*).
A ring $R$ shall be said to be commutative at regular-product (simply, CRP) if $R$ satisfies the condition ($*$), based on Lemma 2.1 and Example 2.3. Then CRP rings are directly finite by an argument above. Due to Bell [4], a ring $R$ is said to satisfy Insertion-of-Factors-Property (or simply, be an IFP ring) if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. It is easily checked that reduced rings are IFP, and the converse need not hold because any commutative ring is clearly IFP (e.g., $\mathbb{Z}_{mn}$ for $m, n \geq 2$). The proof for IFP rings to be Abelian is simple.

Following Huh et al. [7], a ring is called locally finite if every finite subset generates a finite multiplicative semigroup. It is shown that a ring is locally finite if and only if every finite subset generates a finite subring by [6, Theorem 2.2(1)]. Finite rings, infinite direct sums of finite rings, and algebraic closures of finite fields are basic examples of locally finite rings.

**Theorem 2.4.** (1) Left or right Artinian rings are CRP.
(2) IFP rings are CRP.
(3) Locally finite rings are CRP.
(4) A ring $R$ is CRP if and only if $ab$ being regular for $a, b \in R$ implies that both $a$ and $b$ are regular.

**Proof.** (1) Let $R$ be a right Artinian ring and suppose that $ab$ is regular for $a, b \in R$. Then $(ba)^k = (ba)^{k+1}c$ for some $k \geq 1$ and $c \in R$. This yields

$$0 = a(ba)^k - a(ba)^{k+1}c = (ab)^k(a - abac)$$

because $ab$ is regular. Multiplying $a - abac = 0$ by $b$ on the right-hand side, we get $ab(1 - acb) = 0$ and moreover $acb = 1$ because $ab$ is regular. Since right Artinian rings are directly finite, we obtain $b(ac) = 1$ and $(cb)a = 1$ from $acb = 1$. So both $a$ and $b$ are regular, implying that $ba$ is regular. Therefore $R$ is CRP. The proof for the case of a left Artinian ring is similar.

(2) Let $R$ be an IFP ring, and assume on the contrary that there exist $a, b \in R$. Then $(ba)^m = 0$ and $(ba)^n = 1$ for some $m, n \geq 1$ by the proof of [7, Proposition 16]. This forces $(ab)^m = 1$ and $(ba)^n = 1$ because both $ab$ and $ba$ are regular in $S$. So $a, b \in U(R)$, entailing that $ba$ is regular in $R$. Therefore $R$ is CRP.

(3) Let $R$ be a locally finite ring. Suppose that $ab$ is regular for $a, b \in R$. Consider the subring $S$ of $R$ generated by $a, b, 1$. Then $S$ is finite because $R$ is locally finite; hence $S$ is CRP by (1). But $ab$ is also regular in $S$, implying that $ba$ is regular in $S$. Since $R$ is locally finite, both $(ab)^m$ and $(ba)^n$ are idempotents for some $m, n \geq 1$ by the proof of [7, Proposition 16]. This forces $(ab)^m = 1$ and $(ba)^n = 1$ because both $ab$ and $ba$ are regular in $S$. So $a, b \in U(R)$, entailing that $ba$ is regular in $R$. Therefore $R$ is CRP.

(4) It suffices to show the necessity. Let $R$ be a CRP ring and suppose that $ab$ is regular for $a, b \in R$. Then $a$ is left regular and $b$ is right regular. Since $R$ is CRP, $ba$ is regular and so this implies that $a$ is right regular and $b$ is left regular. □
By Theorem 2.4(1), a CRP ring need not be Abelian. In fact, Mat\(_n(R)\) is CRP but non-Abelian over a right Artinian ring \(R\) for \(n \geq 2\). So the concepts of regular-commutativity and Abelian are independent of each other, considering Example 2.3. Thus direct finiteness is a ring theoretic property which unifies the concepts of regular-commutativity and Abelian.

**Proposition 2.5.** For a ring \(R\), the following are equivalent:

1. \(R\) is CRP.
2. \(abc\) being regular for \(a, b, c \in R\) implies that \(acb\) is regular.
3. \(abc\) being regular for \(a, b, c \in R\) implies that \(bac\) is regular.
4. \(a_1a_2\cdots a_n\) being regular for \(a_1, a_2, \ldots, a_n \in R\) implies that \(a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)}\) is regular for any permutation \(\sigma\) of the set \(\{1, 2, \ldots, n\}\).

**Proof.** (1) \(\Leftrightarrow\) (2) and (1) \(\Leftrightarrow\) (3): Let \(R\) be a CRP ring. Suppose that \(abc\) is regular for \(a, b, c \in R\). Then both \((bc)a\) and \(c(ab)\) are regular and so \(a, b\) and \(c\) are regular by Theorem 2.4(4). This implies that both \(acb\) and \(bac\) are regular.

Each converse is clear since \(ab = 1 \cdot ab = ab \cdot 1\).

(2) \(\Rightarrow\) (4): This follows from the same argument as the proof of [1, Theorem I.1], using regular in place of 0.

(4) \(\Rightarrow\) (1): It is obvious. \(\square\)

If a ring is not directly finite, then it has infinitely many matrix units by [9, page 1]. We see a similar result for non-regular elements as follows.

**Theorem 2.6.** Let \(R\) be a directly finite ring. If \(R\) is not CRP, then we have the following results:

1. \(R\) has an infinite set of nonzero non-regular elements which intersects with \(N(R)\) at emptiness.
2. \(R\) has both an infinite properly descending chain of principal right ideals and an infinite properly descending chain of principal left ideals.

**Proof.** (1) Since \(R\) is not CRP, there exist \(a, b \in R\) such that \(ab\) is regular but \(ba\) is not regular. Since \(ab\) is regular, \(a\) is left regular and \(b\) is right regular.

Since \(ba\) is not regular, \((ba)\alpha = 0\) (hence \(\alpha a = 0\)) for some \(\alpha \in R \setminus \{0\}\), or \(\beta(ba) = 0\) (hence \(\beta b = 0\)) for some \(\beta \in R \setminus \{0\}\).

We first note that \((ba)^k\) is nonzero for any \(k \geq 1\). For, the regularity of \(a(ba)^k b = (ab)^{k+1}\) implies \((ba)^k \neq 0\). We next claim that \(ab\) is not a unit. If \((ca)b = a(bc) = 1\) for some \(c \in R\), then \(a\) and \(b\) are regular by Lemma 2.1(6). This is contrary to that \(ba\) is not regular. Thus \(ab\) is not a unit.

Assume that \((ba)^m = (ba)^n\) for some \(1 \leq m < n\). Then

\[(ab)^{m+1} = a[(ba)^m]b = a[(ba)^m]b = (ab)^{n+1}\] and \((ab)^{m+1}[1 - (ab)^{n-m}] = 0\).

This implies \((ab)^{n-m} = 1\) because \((ab)^{m+1}\) is regular. So \(ab\) is a unit, contrary to the result above. Thus \((ba)^m \neq (ba)^n\) for all \(1 \leq m < n\). Therefore we now obtain an infinite set

\[\{(ba)^n \mid n = 1, 2, \ldots\}\]
of nonzero non-regular elements in \(R\). Since every \((ba)^n\) is nonzero, the intersection of \(\{(ba)^n \mid n = 1, 2, \ldots\}\) and \(N(R)\) is an empty set.

(2) Recall the elements \(a\) and \(b\) in \(R\) such that \(ab\) is regular but \(ba\) is not regular, in the proof of (1). Consider the descending chain

\[
\begin{align*}
(ba)R & \supseteq (ba)^2R \supseteq \cdots \supseteq (ba)^nR \supseteq (ba)^{n+1}R \supseteq \cdots
\end{align*}
\]

of principal right ideals in \(R\), where \(n = 1, 2, \ldots\). Assume that \((ba)^kR = (ba)^{k+1}R\) for some \(k \geq 1\). Then \((ba)^k = (ba)^{k+1}c\) for some \(c \in R\). Multiplying this equality by \(a\) on the left-hand side, we get

\[
(ab)^ka = a(ba)^k = a(ba)^{k+1}c = (ab)^k abac \quad \text{and} \quad 0 = a(ba)^k - a(ba)^{k+1}c = (ab)^k(a - abac).
\]

This yields \(a - abac = 0\) because \(ab\) is regular. Multiplying \(a - abac = 0\) by \(b\) on the right-hand side, we get

\[
ab - abacb = 0 \quad \text{and} \quad ab(1 - abac) = 0.
\]

This yields \(acb = 1\) because \(ab\) is regular. Then \(a(cb) = 1 = (ac)b\) imply that both \(a\) and \(b\) are regular by Lemma 2.1(6), contrary to \(ba\) being not regular. So the descending chain \((†)\) is not stationary.

The argument for the chain principal left ideals is similar to the preceding one. \(\square\)

Recall the ring \(R\) in Example 2.3 which is Abelian but not CRP. The element \(aba\) of \(R\) is regular but \(ba^2\) is not left regular (in fact, \(b^{n-1}ba^2 = 0\)). So, by the proof of Theorem 2.6(1), we get an infinite set \(\{(ba^2)^k \mid k = 1, 2, \ldots\}\) of nonzero non-regular elements. We see an application of Theorem 2.6(1) in the following.

Example 2.7. We use the construction in [10, Theorem 2.2(2)]. Let \(F\) be a domain. Define a map \(\sigma : D_{2^n}(F) \rightarrow D_{2^{n+1}}(F)\) by \(B \mapsto \left( \begin{array}{cc} 0 & B \\ B & 0 \end{array} \right)\). Then \(D_{2^n}(F)\) can be considered as a subring of \(D_{2^{n+1}}(F)\) via \(\sigma\) (i.e., \(B = \sigma(B)\) for \(B \in D_{2^n}(F)\)). Set \(R = \bigcup_{n=1}^{\infty} D_{2^n}(F)\). Then \(R\) is Abelian (hence directly finite) by [5, Lemma 2].

Since \(F\) is a domain, we have \(N(R) = \{A \in R \mid \text{the diagonal entries of } A \text{ are all zero}\} = N^*(R)\). So every matrix in \(R \setminus N(R)\) is regular by [8, Lemma 2.1] because the diagonal entries are nonzero and \(F\) is a domain. Consequently the set of non-nilpotent non-regular elements is empty. Thus \(R\) is CRP by Theorem 2.6(1).

In the following we see an application of Theorem 2.6(2).

Proposition 2.8. If \(R\) is a directly finite ring that satisfies the descending chain condition for principal right ideals or principal left ideals, then \(R\) is CRP.

Proof. Let \(R\) be a directly finite ring that satisfies the descending chain condition for principal right ideals or principal left ideals. Assume on the contrary that \(R\) is not CRP. Then \(R\) has both an infinite properly descending chain
of principal right ideals and an infinite properly descending chain of principal left ideals by Theorem 2.6(2), a contradiction to the condition. Therefore $R$ is CRP. □

We can also obtain Theorem 2.4(1) as a corollary of Proposition 2.8. In what follows, we elaborate upon the structure of descending chains in the proof of Theorem 2.6(2).

**Note.** Let $R$ be a directly finite ring that is not CRP. Then there exist $a, b \in R$ such that $ab$ is regular but $ba$ is not regular. So $a$ is not right regular or $b$ is not left regular. Suppose that $b$ is not left regular. Consider a descending chain 

$$(\delta) \quad bR \supseteq b^2R \supseteq \cdots \supseteq b^nR \supseteq b^{n+1}R \supseteq \cdots$$

of principal right ideals in $R$, where $n = 1, 2, \ldots$. Assume that $b^kR = b^{k+1}R$ for some $k \geq 1$. Then $b^k = b^{k+1}d$ for some $d \in R$. Thus $b^k(1 - bd) = 0$ implies $1 = bd$ because $b$ is right regular. Then $b$ is regular by Lemma 2.1(6), a contradiction. Therefore the chain $(\delta)$ is non-stationary.

Next suppose that $a$ is not right regular. Consider a descending chain 

$$(\gamma) \quad Ra \supseteq Ra^2 \supseteq \cdots \supseteq Ra^n \supseteq Ra^{n+1} \supseteq \cdots$$

of principal left ideals in $R$, where $n = 1, 2, \ldots$. Assume that $Ra^h = Ra^{h+1}$ for some $h \geq 1$. Then $a^h = ea^{h+1}$ for some $e \in R$. Thus $(1 - ea)a^h = 0$ implies $1 = ea$ because $a$ is left regular. Then $a$ is regular by Lemma 2.1(6), a contradiction. Therefore the chain $(\gamma)$ is non-stationary.

We see in the following some sorts of matrix rings through which the CRP property passes.

**Theorem 2.9.** (1) For a ring $R$ the following conditions are equivalent:

(a) $R$ is CRP.
(b) $D_n(R)$ is CRP for $n \geq 2$.
(c) $V_n(R)$ is CRP for $n \geq 2$.

(2) If $R$ is a commutative ring, then $T_n(R)$ is a CRP ring for all $n \geq 2$.
(3) If $R$ is a domain, then $T_2(R)$ is a CRP ring.
(4) Let $R$ be a ring and $n \geq 2$. If $T_n(R)$ is CRP, then so is $R$.

**Proof.** (1) (a) $\implies$ (b): Let $R$ be CRP and suppose that $AB$ is regular for $A = (a_{ij}), B = (b_{st}) \in D_n(R)$. Then $a_{uu}b_{uu}$ is regular in $R$ for all $u = 1, \ldots, n$ by [8, Lemma 2.1]. Since $R$ is CRP, $a_{uu}$ and $b_{uu}$ are regular for any $u$ by Theorem 2.4(4). Then both $A$ and $B$ are regular by [8, Lemma 2.1], and thus $D_n(R)$ is CRP by Theorem 2.4(4).

Conversely, suppose that $D_n(R)$ is CRP and let $ab$ be regular for $a, b \in R$. Consider two matrices $D = (d_{ij})$ and $E = (e_{st})$ in $D_n(R)$ such that $d_{ii} = a$, $e_{ss} = b$, and $d_{ij} = 0 = e_{st} = 0$ for $i \neq j$ and $s \neq t$. Then $DE$ is regular in $D_n(R)$ by [8, Lemma 2.1]. Since $D_n(R)$ is CRP, both $D$ and $E$ are regular by Theorem 2.4(4). Hence both $a$ and $b$ are regular by [8, Lemma 2.1]. Therefore $R$ is CRP.
The proof of (a) ⇔ (c) is almost the same as one of (a) ⇔ (b).

(2) Let \( R \) be a commutative ring and suppose that \( AB \) is regular for \( A = (a_{ij}), B = (b_{st}) \in T_n(R) \). Then \( a_{uu}b_{uu} \) is regular in \( R \) for all \( u = 1, \ldots, n \), implying that \( a_{uu} \) and \( b_{uu} \) are regular in \( R \) because \( R \) is commutative. Hence both \( A \) and \( B \) are regular in \( T_n(R) \) by Theorem 1.1(3). Therefore \( T_n(R) \) is CRP.

(3) Let \( R \) be a domain and suppose that \( (a_{ij})(b_{st}) \) is regular in \( T_2(R) \) for \( (a_{ij}), (b_{st}) \in T_2(R) \). Then \( a_{11}b_{11} \) is right regular and \( a_{22}b_{22} \) is left regular in \( R \) by Theorem 1.1(2). This implies that \( a_{11}, a_{22}, b_{11} \) and \( b_{22} \) are regular because \( R \) is a domain. Thus both \( (a_{ij}) \) and \( (b_{st}) \) are regular by Theorem 1.1(1), entailing that \( T_2(R) \) is CRP.

(4) Suppose that \( ab \) is regular for \( a, b \in R \). Consider two matrices \( D = (d_{ij}) \) and \( E = (e_{st}) \) in \( T_n(R) \) such that \( d_{ii} = a, e_{ss} = b \), and \( d_{ij} = 0 = e_{st} \) for \( i \neq j \) and \( s \neq t \). Then \( DE \) is regular by Theorem 1.1(1). If \( T_n(R) \) is CRP, then both \( D \) and \( E \) are regular in \( T_n(R) \). Hence both \( a \) and \( b \) are regular in \( R \) by Theorem 1.1(2). Thus \( R \) is CRP. \( \square \)

Example 2.10. There exists a domain (hence CRP by Theorem 2.4(2)) \( R \) such that \( \text{Mat}_n(R) \) is not directly finite (hence not CRP) by [12, Theorem 1.0]. We extend this argument to the general case of \( n \geq 2 \). \( \text{Mat}_2(R) \) is isomorphic to a subring of \( \text{Mat}_n(R) \) via the corresponding \( (a_{ij}) \mapsto (a'_{ij}) \) where \( a'_{ij} = a_{ij} \) for all \( 1 \leq i, j \leq 2 \), \( a'_{ii} = 1 \) for all \( 3 \leq i \leq n \), and \( a'_{ij} = 0 \) for all \( 3 \leq i, j \leq n \) with \( i \neq j \). Thus \( \text{Mat}_n(R) \) is not directly finite (hence not CRP) because the class of directly finite rings is closed under subrings.

Considering Theorem 2.9(3), one may ask whether \( T_n(R) \) is also CRP over a domain \( R \) for \( n \geq 3 \). But we see a negative answer in what follows.

Example 2.11. Let \( R = K\langle x, y \rangle \) be the free algebra generated by the non-commuting indeterminate \( x, y \) over a field \( K \). Note that \( \begin{pmatrix} x & y & 0 \\ 0 & 0 & x \\ 0 & 0 & y \end{pmatrix} \) is regular in \( T_3(R) \) by (II) in Example 1.3.

\[
\begin{pmatrix} x & y & 0 \\ 0 & 0 & x \\ 0 & 0 & y \end{pmatrix} = \begin{pmatrix} x & 1 & 0 \\ 0 & 0 & x \\ 0 & 0 & y \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

and

\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x & 1 & 0 \\ 0 & 0 & x \\ 0 & 0 & y \end{pmatrix} = \begin{pmatrix} x & 1 & 0 \\ 0 & 0 & yx \\ 0 & 0 & y \end{pmatrix}.
\]

The former is regular but the latter is not regular in \( T_3(R) \) as can be seen by

\[
\begin{pmatrix} x & 1 & 0 \\ 0 & 0 & yx \\ 0 & 0 & y \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]
So $T_3(R)$ is not CRP, in spite of $R$ being a domain (hence CRP). This result is also shown by Theorem 2.4(4) because $\begin{pmatrix} x & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -x \\ 0 & 0 & 0 \end{pmatrix} = 0$. This result can be extended to the case of $n \geq 4$.

A ring $R$ is usually called right Ore if given $a, b \in R$ with $b$ regular there exist $a_1, b_1 \in R$ with $b_1$ regular such that $ab_1 = ba_1$. It is well-known that right Noetherian domains are right Ore. The ring $R$ in Example 2.11 is not right Ore.

**Proposition 2.12.** Let $R$ be a domain. If $R$ is right Ore, then $T_3(R)$ is CRP.

**Proof.** Let $R$ be right Ore and suppose that $AB$ is regular in $T_3(R)$ for $A = (a_{ij})$, $B = (b_{ij}) \in T_3(R)$. Write $C = AB = (c_{ij})$. Then both $c_{11}$ and $c_{33}$ are nonzero by Theorem 1.1(2), entailing that $a_{11}, a_{33}, b_{11}, b_{33} \in R \setminus \{0\}$, since $R$ is a domain.

Assume $c_{22} = 0$ (i.e., $a_{22} = 0$ or $b_{22} = 0$). Since $R$ is right Ore and $c_{11} \neq 0$, there exist $\alpha \neq 0$ and $\beta$ in $R$ such that $c_{11}\beta + c_{12}\alpha = 0$. This implies

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ 0 & 0 & c_{23} \\ 0 & 0 & c_{33} \end{pmatrix} \begin{pmatrix} 0 & 0 & \beta \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & c_{11}\beta + c_{12}\alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$  

Thus $C$ is not right regular because $\begin{pmatrix} 0 & 0 & \beta \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{pmatrix}$ is nonzero, contrary to $C$ being regular. Hence we have $c_{22} \neq 0$, and this implies that both $a_{22}$ and $b_{22}$ are nonzero. Therefore both $A$ and $B$ are regular by Theorem 1.1(1), and hence $T_3(R)$ is CRP by Theorem 2.4(4).  

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