LOGHARMONIC MAPPINGS WITH TYPICALLY REAL ANALYTIC COMPONENTS

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Abstract. This paper treats the class of normalized logharmonic mappings $f(z) = zh(z)g(z)$ in the unit disk satisfying $\varphi(z) = zh(z)g(z)$ is analytically typically real. Every such mapping $f$ admits an integral representation in terms of its second dilatation function and a function of positive real part with real coefficients. The radius of starlikeness and an upper estimate for arclength are obtained. Additionally, it is shown that $f$ maps the unit disk into a domain symmetric with respect to the real axis when its second dilatation has real coefficients.

1. Introduction

Let $\mathcal{H}(U)$ be the linear space of analytic functions defined in the unit disk $U = \{z : |z| < 1\}$ of the complex plane $\mathbb{C}$. Let $B$ denote the set of self-maps $a \in \mathcal{H}(U)$, and $B_0$ its subclass consisting of $a \in B(U)$ with $a(0) = 0$. A logharmonic mapping in $U$ is a solution of the nonlinear elliptic partial differential equation

$$\left( \frac{f_z(z)}{f(z)} \right) = a(z) \frac{f_z(z)}{f(z)},$$

where the second dilatation function $a$ lies in $B$. Thus the Jacobian

$$J_f = |f_z|^2 (1 - |a|^2)$$

is positive, and all non-constant logharmonic mappings are sense-preserving and open in $U$.

If $f$ is a non-constant logharmonic mapping which vanishes only at $z = 0$, then [4] shows that $f$ admits the representation

$$f(z) = z^m |z|^{2g(z)h(z)},$$

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where \( m \) is a positive integer, \( \Re \beta > -1/2 \), and \( h, g \in \mathcal{H}(U) \) satisfy \( g(0) = 1 \) and \( h(0) \neq 0 \). The exponent \( \beta \) in (1.2) depends only on \( a(0) \) and is given by
\[
\beta = \frac{a(0)}{1 - |a(0)|^2}.
\]

Note that \( f(0) \neq 0 \) if and only if \( m = 0 \), and that a univalent logharmonic mapping vanishes at the origin if and only if \( m = 1 \), that is, \( f \) has the form
\[
f(z) = z|z|^{2\beta}h(z)\overline{g(z)},
\]
where \( 0 \not\in (hg)(U) \). This class has been studied extensively over recent years in [1–8].

As further evidence of its importance, note that \( F(\zeta) = \log f(e^{\zeta}) \) are univalent harmonic mappings of the half-plane \( \{ \zeta : \Re \zeta < 0 \} \), which are closely related to the theory of minimal surfaces (see [19, 20]).

An analytic function \( \varphi \) in \( U \) is typically real if \( \varphi(z) \) is real whenever \( z \) is real and nonreal elsewhere. Similarly, a logharmonic mapping \( f \) in \( U \) is typically real if \( f(z) \) is real whenever \( z \) is real and nonreal elsewhere. Investigations into typically real logharmonic mappings was initiated by Abdulhadi in [2].

Denote by \( HG \) the class of analytic functions \( \varphi(z) = zh(z)g(z) \), where \( h \) and \( g \) in \( \mathcal{H}(U) \) are normalized by \( h(0) = 1 = g(0) \), and \( 0 \not\in (hg)(U) \). This paper treats the class \( T_{Ra} \) of logharmonic mappings \( f(z) = zh(z)g(z) \) satisfying \( \varphi(z) = zh(z)g(z) \in HG \) is analytically typically real in \( U \).

In Section 2, every mapping \( f \in T_{Ra} \) is shown to admit an integral representation in terms of its second dilatation function and a function of positive real part with real coefficients. The radius of starlikeness is also obtained for the class \( T_{Ra} \), as well as an upper estimate for its arclength.

For an analytic univalent function \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \), it is known [2] that \( f \) is typically real if and only if the image \( f(U) \) is a domain symmetric with respect to the real axis. However, this characterization no longer holds for logharmonic maps, that is, it is not true that a univalent logharmonic mapping \( F(z) = zh(z)\overline{g(z)} \in T_{Ra} \) if and only if the image \( F(U) \) is a symmetric domain with respect to the real axis.

As an illustration, Figure 1 shows the mapping \( F(z) = z(1+iz/3)(1+i\zeta/3) \in T_{Ra} \) but \( F(U) \) is not a symmetric domain with respect to the real axis. On the other hand, Figure 2 shows the mapping
\[
F(z) = z \exp \{ \Re (4z/(1 - z)) \} (1 - \zeta)/(1 - z)
\]
which does not belong to the class \( T_{Ra} \), but yet maps \( U \) onto a symmetric domain with respect to the real axis \( F(U) \).

In Section 3 we explore conditions on the dilatation \( a \) that would ensure a logharmonic mapping \( f(z) = zh(z)\overline{g(z)} \in T_{Ra} \) necessarily satisfies \( f(U) \) is symmetric with respect to the real axis. Sufficient conditions for univalent
logharmonic mappings to belong to the class $T_{Ra}$ in some subdisk of $U$ are also determined.

2. An integral representation and radius of starlikeness

Let us denote by $\mathcal{P}_R$ the class of normalized analytic functions with positive real part and with real coefficients in $U$. The following result gives a representation of $f \in T_{Ra}$ in terms of the dilatation $a$ and $p \in \mathcal{P}_R$.

**Theorem 1.** Let $f = zh(z)\overline{g(z)}$ belongs to $T_{Ra}$ with respect to $a \in B_0$. Then

$$f(z) = \frac{z p(z)}{1 - z^2} \exp \left( -2i \text{Im} \int_0^z \frac{a(s)}{1 + a(s)} \left( \frac{1 + s^2}{s(1 - s^2)} + \frac{p'(s)}{p(s)} \right) ds \right)$$

for some $p \in \mathcal{P}_R$. 
Proof. Let $\varphi(z) = zh(z)g(z)$, and

\begin{equation}
(2.1) \quad f(z) = \varphi(z) \frac{g(z)}{g(z)}.
\end{equation}

It follows from (1.1) that

\[ g'(z)g(z) = a(z) \varphi'(z) + a(z) \varphi(z), \]

which readily yields

\begin{equation}
(2.2) \quad g(z) = \exp \int_{0}^{z} \frac{a(s)}{1 + a(s)} \varphi'(s) ds.
\end{equation}

Substituting (2.2) into (2.1) yields

\[ f(z) = \varphi(z) \exp \left(-2i \text{Im} \int_{0}^{z} \frac{a(s)}{1 + a(s)} \varphi'(s) ds \right). \]

It is known [21] that every typically real analytic function $\varphi$ has the form

\[ (1 - z^2)\varphi(z) = zp(z) \]

for some $p \in \mathbb{P}_R$, which yields the desired result. \qed

In the next result, we obtain an estimate on the radius of starlikeness for the class $T_{R_a}$.

**Theorem 2.** Let $f(z) = zh(z)g(z) \in T_{R_a}$ with respect to $a \in B_0$. Then $f$ maps the disk $|z| < 3 - 2\sqrt{2}$ onto a starlike domain.

**Proof.** The function $f$ maps the circle $|z| = r$ onto a starlike curve provided

\[ \frac{\partial}{\partial \theta} \arg f(re^{i\theta}) = \text{Im} \left( \frac{\partial}{\partial \theta} \log f(re^{i\theta}) \right) = \text{Re} \left( \frac{zf_z - \overline{zf_{\overline{z}}}}{f} \right) > 0. \]

With $\varphi(z) = zh(z)g(z)$, a short computation gives

\[ \text{Re} \left( \frac{zf_z - \overline{zf_{\overline{z}}}}{f} \right) = \text{Re} \left( \frac{1 - a(z) z \varphi'(z)}{1 + a(z) \varphi(z)} \right) \]

for some $a \in B_0$.

Next let

\[ q(z) = \frac{1 - a(z) z \varphi'(z)}{1 + a(z) \varphi(z)}, \]

and $\sigma(z) = \rho_0 z$. Kirwan [18] has shown that the radius of starlikeness for typically real analytic functions $\varphi$ is $\rho_0 = \sqrt{2} - 1$. Thus $\text{Re} \left\{ (\zeta \varphi'(\zeta)/\varphi(\zeta) |_{\sigma(z)}) \right\} > 0$, and so $q(\sigma(z))$ is subordinated to $((1 + z)/(1 - z))^2$ in $U$.

Writing $p(z) = (1 + z)/(1 - z)$, it follows from [14, p. 84] that

\[ \left| p(z) - \frac{1 + r^2}{1 - r^2} \right| \leq \frac{2r}{1 - r^2}. \]

Thus $|\arg(p(z))| < \pi/4$ provided $|z| < \rho_0$, where $\rho_0$ is a smallest positive root of the equation $r^2 - 2\sqrt{2}r + 1 = 0$. The function $f(z) = zh(z)g(z)$ is thus starlike in the disk $|z| < \rho_0^2 = 3 - 2\sqrt{2}$. \qed
The next result gives an upper estimate for arclength of all mappings $f$ in the class $T_{Ra}$.

**Theorem 3.** Let $f(z) = zh(z)g(z) \in T_{Ra}$ with respect to $a \in B_0$. Then for $|z| = r$, an upper bound for arclength $L(r)$ is given by

$$L(r) \leq \frac{2\pi r(1+r)^2}{(1-r)^2}.$$

**Proof.** Let $C_r$ denote the image of the circle $|z| = r < 1$ under the mapping $w = f(z)$. Then

$$L(r) = \int_{C_r} |df| = \int_0^{2\pi} |zf_z - \bar{z}f_{\bar{z}}|d\theta$$

$$\leq M(r) \int_0^{2\pi} \left|\frac{zf_z - \bar{z}f_{\bar{z}}}{f}\right|d\theta,$$

where $|f(z)| \leq M(r)$, $0 < r < 1$. Let $\varphi(z) = zh(z)g(z)$. Since $\varphi$ is a typically real analytic function, and $|f| = |\varphi|$, then $|z\varphi'(z)/\varphi(z)| \leq (1+r)/(1-r)$ and $M(r) \leq r/(1-r)^2$.

Further

$$\left|\frac{zf_z - \bar{z}f_{\bar{z}}}{f}\right| = \left|\frac{z\varphi'(z)}{\varphi(z)} - 2\text{Re}\left(\frac{a(z) - z\varphi'(z)}{1 + a(z) \varphi(z)}\right)\right|$$

$$\leq \frac{1+r}{1-r} + 2 \frac{r}{1-r} \frac{1+r}{1-r},$$

and thus,

$$L(r) \leq \frac{2\pi r(1+r)^2}{(1-r)^4}. \quad \square$$

### 3. Logharmonic mappings in the class $T_{Ra}$

The following result is readily established, and thus the proof is omitted. It describes the geometry of a logharmonic function in the class $T_{Ra}$ when its second dilatation has real coefficients.

**Theorem 4.** Let $f(z) = zh(z)g(z) \in T_{Ra}$ be a sense-preserving logharmonic mapping in $U$. If the second dilatation function $a$ has real coefficients, that is, $a(\overline{z}) = a(z)$, then $f(U)$ is symmetric with respect to the real axis.

The final result derives sufficient conditions for $f \in T_{Ra}$ in some subdisk of $U$.

**Theorem 5.** Let $f(z) = zh(z)g(z)$ be a univalent sense-preserving logharmonic mapping in $U$ normalized by $h(0) = 1 = g(0)$, where its second dilatation function $a$ has real coefficients. Further, suppose $f(U) = \Omega$, $\Omega \neq \mathbb{C}$, is a strictly starlike Jordan domain. If $f(U)$ is a symmetric domain with respect to the real axis, and $|a(z)| \leq k < 1$ in $U$, then $\varphi(z) = zh(z)g(z)$ is typically real in the disk $|z| < \sqrt{2} - 1$. 
Proof. The domain $\Omega$ is a strictly starlike Jordan domain provided each radial ray from 0 intersects the boundary $\partial \Omega$ of $\Omega = f(U)$ at exactly one point of $\mathbb{C}$. Further, since $|a(z)| \leq k < 1$ in $U$, it follows from [7, Lemma 2.4] that there is only one univalent logharmonic mapping from $U$ onto $\Omega$ which is a solution of (1.1) normalized by $f(0) = 0$ and $h(0) = 1 = g(0)$.

Since $a$ has real coefficients, then $a(\overline{z}) = a(z)$. On the other hand, the mapping $F(z) = f(\overline{z})$ is also univalent and logharmonic in $U$, where $F(U) = f(U)$.

If $F(z) = zH(z)G(z) = zh(z)g(z)$ with $H(z) = \overline{h(\overline{z})}$ and $G(z) = \overline{g(\overline{z})}$, then $F$ satisfies the normalization $F(0) = 0, H(0) = 1 = G(0)$, and $F$ is a solution of

$$\frac{F_{\overline{z}}(z)}{F(z)} = \overline{a(\overline{z})} \frac{F_{z}(\overline{z})}{F(z)} = a(z) \frac{F_{z}(z)}{F(z)}.$$

Thus, $F$ is a logharmonic mapping with respect to the same $a$, and consequently, $f(z) \equiv F(z)$ in $U$. This implies $f$ has real coefficients, and so $\psi(z) = zh(z)/g(z) = f(z)/|g(z)|^2$ has real coefficients.

Direct calculations yield

$$\frac{g'(z)}{g(z)} = \frac{a(z)}{1 - a(z)} \frac{\psi'(z)}{\psi(z)},$$

which upon integrating leads to

$$g(z) = \exp \int_{0}^{z} \frac{a(t)}{1 - a(t)} \frac{\psi'(t)}{\psi(t)} dt.$$

Then $g$, and so does $h$, have real coefficients, and thus $\varphi(z) = zh(z)g(z)$ has real coefficients. Furthermore, [5, Theorem 3.1] shows that $\varphi$ is starlike univalent in the disk $|z| < \rho$, where $\rho = \sqrt{2} - 1$. Thus $\varphi$ is typically real in the disk $|z| < \sqrt{2} - 1$. \qed

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