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# A NON-RECURSIVE APPROACH TO NEVANLINNA-PICK INTERPOLATION PROBLEM

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**Abstract.** A solution for Nevanlinna-Pick interpolation problem with low complexity is constructed via non-recursive method. More precisely, a stable rational function satifying the given interpolation data in the complex right half plane is found by solving a homogeneous interpolation problem related to a minial interpolation problem for the given data in the right half plane together with its mirror-image data.

## 1. Introduction

A *minimal* interpolation problem for scalar rational functions is the following: given distinct points $z_1, \cdots, z_n$ in the complex plain $\mathbb{C}$ and given complex numbers $\{w_{ij}\}_{i=1}^{\mu_j}{}_{j=1}^{n}$, find a rational function $y(z)$ in the form of

$$y(z) := \frac{n(z)}{d(z)}, \quad gcd(n, d) = 1 \tag{1.1}$$

such that

$$y^{(i-1)}(z_j) = w_{ij}, \quad i = 1, \cdots, \mu_j, \quad j = 1, \cdots, n \tag{1.2}$$

and has the *minimal complexity*, where *gcd* means the greatest common divisor of polynomials. As a measure of the *complexity* of a scalar rational function, we use the *McMillan degree.* The *McMillan degree* of $y(z)$ in (1.1) is defined by

$$\delta(y) := max\ \{deg\, n(z),\ deg\, d(z)\}.$$

Some special cases of this (minimal) interpolation problem were studied by Belevich([6]) and Donoghue([7]) and the general case was understood

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by Antoulas and Anderson([1]). In the latter approach, the *Loewner matrix* is a key notion. For simplicity, if we assume, in (1.2), $\mu_i = 1$, $i = 1, \cdots, n$ and $n = 2m+1$ for some nonnegative integer $m$, then the associated *Loewner matrix* is given by

$$L := \left[\tfrac{w_{m+1+i}-w_j}{z_{m+1+i}-z_j}\right]_{1\le i\le m,\, 1\le j\le m+1}. \tag{1.3}$$

Let $\bar{L}$ be an $(n-q-1)\times(q+1)$ Loewner matrix defined by

$$\bar{L} = [\frac{w_{q+1+i}-w_j}{z_{q+1+i}-z_j}]_{1\le i\le n-q-1,\ 1\le j\le q+1}, \tag{1.4}$$

where $q = \text{rank } L$. Due to [1], it is known that $r \times l$ Loewner matrix constructed from the same data for $L$ with $r \ge q$, $l \ge q$ has the same *rank*. Thus,

$$rank\bar{L} = q. \tag{1.5}$$

Let $c := [c_1, \cdots, c_{q+1}]^T$ be a $(q+1)$-dimensional nonzero vector satisfying

$$\bar{L}c = 0 \tag{1.6}$$

equivalently

$$\sum_{j=1}^{q+1} c_j \frac{w_k - w_j}{z_k - z_j} = 0, \quad k = q+2, q+3, \cdots, n. \tag{1.7}$$

Let

$$\bar{d}(z) := \sum_{j=1}^{q+1} c_j \prod_{i=1,\, i\ne j}^{q+1} (z - z_i) \tag{1.8}$$

$$\bar{n}(z) := \sum_{j=1}^{q+1} c_j w_j \prod_{i=1\,, i\ne j}^{q+1} (z - z_i) \tag{1.9}$$

and $\bar{f}(z)$ be a rational function satisfying

$$\bar{f}(z)\bar{d}(z) = \bar{n}(z). \tag{1.10}$$

Now we are ready to state the main result of [1] :

**Theorem 1.1.** *Let $L$ be an $m \times (m+1)$ Loewner matrix given by (1.3) and $\bar{d}(z)$ and $\bar{f}(z)$ be given as in (1.8)-(1.10).*
*(a) If $\bar{d}(z)$ has no zeros at $z_j$, $j = 1, \cdots, n$, then the minimal possible McMillan degree for the solutions of (1.1) is $rank\, L$ and $\bar{f}$ is the unique such solution.*

*(b) Otherwise, $n - rank\, L$ is the minimal possible McMillan degree and there are more than one solution of McMillan degree $n - rank\, L$.*

For the proof, readers are referred to [1] or [9]. In [9], a generalization of Theorem 1.1 to a *tangential* interpolation problem for rational matrix functions is considered. But, this classical approach to rational matrix functions seems to give very limited results. There are two main difficulties in extending [1] directly to *tangential* interpolation problems for rational matrix functions: one is, unlike the scalar case, the McMillan degree of a function does not coincide with the rank of the Loewner matrix of the interpolation data generated by that specific function. The second main difficulty for rational matrix interpolation problem is the interpolating data cannot be freely rearranged to change the size of the Loewner matrix as we did in generating $\bar{L}$ from $L$ in (1.3) and (1.4).

This paper is consisted of two sections other than this introductory one. The next section understands Theorem 1.1 from a new perspective so that the result is stated without Loewner matrix and the forementioned two difficulties in extending Theorem 1.1 to rational matrix fuctions can be avoided. The result is related to a matrix polynomial with certain *null pair* and *column reduced ( at infinity)*. The third section is about another type of interpolation problem, called Nevanlinna Pick interpolation problem where one is seeking a rational function interpolating a given array of points in the complex right half plane (*RHP*) and *stable* in the RHP. Here the stability is definded as the boundedness in the RHP. All the solutions of this problem were parametrized in [10] by using recursive algorithm. Antoulas and Anderson of [2] tried to solve this problem via a non-recursive method. But, the solution by [2] is misleading and a counter-example was presented in [5]. The main contribution of this paper is to consturct a Nevanlinna Pick solution with sufficiently small McMillan degree by a non-recursive method. More precisely, the author develops a solution of a homegeneous interpolation problem induced by a minimal interpolation problem for the given data in RHP and together with its associated so called *mirror-image array* and uses it as a main tool in the last section.

## 2. Minimal solutions and homogenous interpolation

The aim of this section is to understand Theorem 1.1 from another point of view. Suppose the interpolation data is given by (1.2) and the Loewner matrix in (1.3) has rank $q$. To construct a solution of McMillan degree $n-q$, we introduce $\hat{L}$, $(q-1)\times(n-q+1)$ Loewner matrix given by

$$\hat{L} = [\frac{\omega_{n-q+1+i} - \omega_j}{z_{n-q+1+i} - z_j}]_{1\le i\le q-1,\, 1\le j\le n-q+1}. \tag{2.1}$$

Since the dimension of the null space of $\hat{L}$ is at least 2, we can choose a $(n-q+1)$-dimensional vector $\hat{c}$ satisfying

$$\hat{L}\hat{c} = 0. \tag{2.2}$$

and

$$\hat{d}(z) := \sum_{j=1}^{n-q+1} \hat{c}_j \prod_{i=1,\, i\neq j}^{n-q+1} (z - z_i) \tag{2.3}$$

has no zeroes at $z_j$, $j = 1, \cdots, n$, where

$$\hat{c} := [\hat{c}_1, \cdots, \hat{c}_{n-q+1}]^T.$$

Let

$$\hat{n}(z) := \sum_{j=1}^{n-q+1} \hat{c}_j w_j \prod_{i=1\,, i\neq j}^{n-q+1} (z - z_i) \tag{2.4}$$

and

$$\hat{f}(z) := \frac{\hat{n}(z, \hat{c})}{\hat{d}(z, \hat{c})}.$$

Here we note that wheather $\bar{f}(z)$ in (1.10) is a solution of (1.2) or not, we can find such $\hat{n}(z)$, $\hat{d}(z)$ satisfying

$$\bar{d}(z)\hat{n}(z) - \bar{n}(z)\hat{d}(z) \neq 0. \tag{2.5}$$

By Theorem 1.1 (b) and by its construction, the McMillan degree of $\hat{f}$ has to be $n-q$. Since the choice of $\hat{c}$ is not unique, there are more than one solutions of degree $n-q$.

Let

$$A_\zeta := \begin{bmatrix} z_1 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & z_n \end{bmatrix}, \qquad B := \begin{bmatrix} 1 & -\omega_1 \\ \vdots & \vdots \\ \vdots & \vdots \\ 1 & -\omega_n \end{bmatrix} \tag{2.6}$$

Then $(A_\zeta, B)$ is a *full range pair.*

**Theorem 2.1.** *Let*

$$\hat{\Theta}(z) := \begin{bmatrix} \hat{n}(z) & \bar{n}(z) \\ \hat{d}(z) & \bar{d}(z) \end{bmatrix}, \tag{2.7}$$

*where $\bar{n}(z), \bar{d}(z), \hat{n}(z)\hat{d}(z)$ are polynomials given by (1.8), (1.9), (2.3) and (2.4). Then,*

$$\hat{\Theta}(z) \text{ has } (A_\zeta, B) \text{ as its } \sigma - \text{null pair} \tag{2.8}$$

$$\hat{\Theta}(z) \text{ is column reduced }, \tag{2.9}$$

*where $\sigma = \{z_i\}_{i=1}^n$.*

*Proof.* Remember that the polynomials of the components of $\hat{\Theta}(z)$ are constructed so that

$$(a)\, \omega_i \bar{d}(z_i) = \bar{n}(z_i), \;\; (b)\, \omega_i \hat{d}(z_i) = \hat{n}(z_i), \;\; i = 1, \cdots, n \tag{2.10}$$

and

$$(a) \text{ the index of } \begin{bmatrix} \bar{n}(z) \\ \bar{d}(z) \end{bmatrix} \le q, \;\; (b) \text{ the index of } \begin{bmatrix} \hat{n}(z) \\ \hat{d}(z) \end{bmatrix} \le n - q, \tag{2.11}$$

where the *index* of a vector polynomial means the highest polynomial degrees of the components. It is straight forward from (2.10) that $(A_\zeta, B)$ is a corestriction of a left null pair of $\hat{\Theta}(z)$.

Now, to prove (2.9), it is enough to show

$$\text{deg det } \hat{\Theta}(z) = \text{the sum of the column indices.} \tag{2.12}$$

Then, (2.12) also implies that $(A_\zeta, B)$ is in fact precisely a left null pair for $\hat{\Theta}(z)$. From (2.11), deg det$\hat{\Theta}(z) \le n$, but the condition (2.5) and (2.10) imply that the nonzero polynomial det $\hat{\Theta}(z)$ has at least $n$ zeroes. Hence,

$$\text{det } \tilde{\Theta}(z) = n \tag{2.13}$$

and, in turn,

$$\text{the index of } \begin{bmatrix} \bar{n}(z) \\ \bar{d}(z) \end{bmatrix} = q \tag{2.14}$$

$$\text{the index of } \begin{bmatrix} \hat{n}(z) \\ \hat{d}(z) \end{bmatrix} = n - q. \tag{2.15}$$

This completes the proof. □

It is known that the column indices of a column reduced matrix polynomial, $\alpha_1 \geq \alpha_2$, are the controllability indices of its (left) null pair. Hence, we have

$$\text{rank } L = q = \alpha_2, \tag{2.16}$$

For the definition of a *controllability indices*, a *full range pair* and *$\sigma$-null pair*, the readers are referred to [4]. The notion of a controllability indices of a full range pair $(A_\zeta, B)$ does not involve the related Loewner matrix.

The next theorem restates Theorem 1.1 without using the notion of Loewner matrix which is the main barrier in extending the results of [1] to rational matrix functions.

**Theorem 2.2.** *Let $n \times n$ matrix $A_\zeta$, $n \times 2$ matrix $B$ be given by (2.6) and*

$$\Theta(z) := \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$$

*be any $2\times 2$ matrix polynomial satisfying (2.8) and (2.9) with the column indices in decreasing order, $\alpha_1 > \alpha_2$.*
*(a) If $\Theta_{22}(z_i) \neq 0$ for $i = 1, \cdots, n$, then the minimal possible McMillan degree of the solutions of (1.2) is $\alpha_2$ and $f(z) := \frac{\Theta_{12}(z)}{\Theta_{22}(z)}$ is the unique solution of McMillan degree $\alpha_2$.*
*(b) Otherwise, $\alpha_1$ is the minimal McMillan degree of solutions of (1.2) and there are more than one solutions of McMillan degree $\alpha_1$.*

*Proof.* Since $\hat{\Theta}(z)$ given by (2.7) and $\Theta(z)$ have the same null pair and both are column reduced, there exists a $2 \times 2$ unimodular matrix $V(z) := \begin{bmatrix} v_{11}(z) & v_{12}(z) \\ v_{21}(z) & v_{22}(z) \end{bmatrix}$ satisfying $\Theta(z) = \hat{\Theta}(z)V(z)$. By an unimodular matrix, one means a matrix polynomial whose determinant is a nonzero constant. By the predictable degree property of a column reduced matrix polynomial (see [8]) and by (2.14)(2.15),

$$\text{the column index of } \begin{bmatrix} \Theta_{12} \\ \Theta_{22} \end{bmatrix} \geq q + deg\, v_{22}, \text{ unless } v_{22} \neq 0 \tag{2.17}$$

$$\text{the column index of} \begin{bmatrix} \Theta_{12} \\ \Theta_{22} \end{bmatrix} \geq (n-q) + deg\, v_{12}, \text{ unless } v_{12} \neq 0, \tag{2.18}$$

since $\alpha_2 = q$, $\alpha_1 = n - \alpha_2 = n - q$. Because the column index of $\begin{bmatrix} \Theta_{12} \\ \Theta_{22} \end{bmatrix} = \alpha_2 = q$ and $(n-q) > q$ in this case, (2.18) forces $v_{12} = 0$ and

(2.17) and the unimodularity of $V(z)$ forces $v_{22}$ is a nonzero constant $c$ and $v_{11}$ is a nonzero constant $\hat{c}$. By plugging $v_{22} = c,\ v_{12} = 0$ , we have

$$\bar{n}(z) = \frac{1}{c}\Theta_{12}(z), \tag{2.19}$$

$$\bar{d}(z) = \frac{1}{c}\Theta_{22}(z). \tag{2.20}$$

Upon substituting (2.19)(2.20) in Theorem 2.1, the proof is completed. □

The next Corollary follows from the proof of Theorem 2.2.

**Corollary 2.3.** *If $\Theta_1(z)$ and $\Theta_2(z)$ are $2\times 2$ matrix polynomials having the same left null pair and column reduced with decreasing column indices $i_1 > i_2$. Then the second columns of $\Theta_1(z)$ and $\Theta_2(z)$ are the same up to a nonzero constant multiplication.*

## 3. A non-recursive approach to Nevanlinna-Pick problem

A different sort of interpolation problem arsing in systems theory which is called Nevanlinna-Pick interpolation problem is the following. For given $\{z_1, z_2, ...., z_{m+1}\}$ be an $(m+1)$ distinct points in the $RHP$ and $\{\omega_1, \omega_2, ...., \omega_{m+1}\}$ be an $(m+1)$ tuple of nonzero complex numbers, one is asked to *find a rational function y satisfying*

$$y(z_i) = \omega_i, \quad \text{for } i = 1, 2, ...., m+1 \tag{3.1}$$

*and*

$$|y(z)| \leq \gamma, \quad \text{for } z \in RHP, \tag{3.2}$$

where $\gamma > 0$ is some preassigned tolerance level. Pick showed that solutions exist if and only if the so-called Pick matrix

$$\Lambda = \left[\frac{\gamma^2 - \omega_i \bar{\omega}_j}{z_i + \bar{z}_j}\right]_{1\leq i,j\leq m+1} \tag{3.3}$$

is positive definite.

Here we introduce the *Nevanlinna-Pick algorithm* which is due to Nevanlinna. It is recursive and described as follows (The unit disk in Nevanlinna's algorithm is replaced by the right half plane here). With

$$y^k(z) = \frac{n^k(z)}{d^k(z)}, \ \ k = 0, 1, \ldots, \ m, m+1, \tag{3.4}$$

consider for $k = 1, \ldots, m, m+1$ the relationship

$$\begin{bmatrix} n^{k-1}(z) \\ \gamma d^{k-1}(z) \end{bmatrix} = \Theta_k(z) \begin{bmatrix} n^k(z) \\ \gamma d^k(z) \end{bmatrix}, \tag{3.5}$$

where

$$\Theta_k(z) = \begin{bmatrix} 1 & \frac{y_k^{k-1}}{\gamma} \\ \frac{[y_k^{k-1}]^*}{\gamma} & 1 \end{bmatrix} \begin{bmatrix} z - z_k & 0 \\ 0 & z + \bar{z}_k \end{bmatrix}, \tag{3.6}$$

and where $y_k^{k-1} := y^{k-1}(z_k)$, $y^0(z) = y(z)$. When $\Lambda$ in (3.3) is positive definite, due to Nevanlinna, all the solutions of (3.1) and (3.2) can be obtained as the form of $y^0(z)$ through the recursive process given by (3.5) and (3.6) upon choosing $y^{m+1}(z)$ satisfying (3.2) with $y^{m+1}(z)$ in place of $y(z)$. Nevanlinna obtained a solution $y_{NP}$ as the result of the above recursive process upon choosing $y^{m+1}(z) = 0$ in (3.5). Then, it is easily seen

$$\delta(y_{NP}) \leq m. \tag{3.7}$$

Now, to understand the Nevanlinna's process through non-recursive approach, the author introduces the matrix $\tilde{\Theta}(z)$ defined by

$$\Theta_1(z)\Theta_2(z)\ldots\Theta_{m+1}(z) := \tilde{\Theta}(z) := \begin{bmatrix} \tilde{\Theta}_{11}(z) & \tilde{\Theta}_{12}(z) \\ \tilde{\Theta}_{21}(z) & \tilde{\Theta}_{22}(z) \end{bmatrix}. \tag{3.8}$$

The following Lemma is a restatement of the Nevanlinna's results in terms of $\tilde{\Theta}(z)$.

**Lemma 3.1.** *$y_0(z)$ defined by (3.4) is a solution of (3.1) and (3.2) if and only if*

$$\begin{bmatrix} n_0(z) \\ \gamma d_0(z) \end{bmatrix} = \tilde{\Theta}(z) \begin{bmatrix} n^{m+1}(z) \\ \gamma d^{m+1}(z) \end{bmatrix} \tag{3.9}$$

*where $y^{m+1}(z)$ given by (3.4) satiesfies the condition (3.2) with $y^{m+1}(z)$ in place of $y(z)$. Moreover,*

$$y_{NP}(z) = \gamma \frac{\tilde{\Theta}_{12}(z)}{\tilde{\Theta}_{22}(z)}. \tag{3.10}$$

The next lemma is about its *symmetric property* of $\tilde{\Theta}(z)$.

**Lemma 3.2.** *$\tilde{\Theta}(z)$ given by (3.8) satisfies the following property:*

$$\tilde{\Theta}(-\bar{z}) = (-1)^{m+1} J\, \overline{\tilde{\Theta}(z)} J, \tag{3.11}$$

*where*

$$J=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

*and* $\overline{\tilde{\Theta}(z)}$ *is the matrix obtained from* $\tilde{\Theta}(z)$ *by taking the complex conjugate of each component of* $\tilde{\Theta}(z)$.

*Proof.* It is enough to show

$$\Theta_k(-\bar{z}) = -\,J\,\overline{\Theta_k(z)}J \tag{3.12}$$

for each $k=1,2,\ldots,m+1$. Since

$$\Theta_k(z)=Y_kD_k(z) \tag{3.13}$$

with

$$Y_k := \begin{bmatrix}1 & \frac{y_k^{k-1}}{\gamma}\\ \frac{[y_k^{k-1}]^*}{\gamma} & 1\end{bmatrix} \quad\text{and}\quad D_k(z):=\begin{bmatrix}z-z_k & 0\\ 0 & z+\bar{z}_k\end{bmatrix}, \tag{3.14}$$

(3.12) is directly obtained upon noticing $D_k(-\bar{z}) = -J\overline{D_k(z)}J$ and $Y_k\,J = J\,\overline{Y_k}$, which are straightfoward from (3.14). □

**Theorem 3.3.** *Let*

$$\Theta(z):=\begin{bmatrix}\Theta_{11}(z) & \Theta_{12}(z)\\ \Theta_{21}(z) & \Theta_{22}(z)\end{bmatrix}, \tag{3.15}$$

*with* $\Theta_{i1}(z)=\tilde{\Theta}_{i1}$ *and* $\Theta_{i2}(z)=\frac{1}{z+z^*_{m+1}}\tilde{\Theta}_{i2}(z)$ *for* $i=1,2$ . *Then,*

*(a)* $\Theta(z)$ *has* $\left(\begin{bmatrix}A_\zeta & 0 & 0\\ 0 & z_{m+1} & 0\\ 0 & 0 & -A_\zeta^*\end{bmatrix}, \begin{bmatrix}\hat{B}\\ \\ \tilde{B}\end{bmatrix}\right)$ *as its (left) null pair*

*(b)* $\Theta(z)$ *is column reduced with the column indices* $\alpha_1=m+1$, $\alpha_2=m$,
*where*

$$\hat{B}=\begin{bmatrix}1 & 1 & \cdots & 1 & 1\\ -\frac{\omega_1}{\gamma} & -\frac{\omega_2}{\gamma} & \cdots & -\frac{\omega_m}{\gamma} & -\frac{\omega_{m+1}}{\gamma}\end{bmatrix}^T,$$

$$\tilde{B}:=\begin{bmatrix}1 & 1 & \cdots & 1\\ -\frac{\gamma}{\omega_1^*} & -\frac{\gamma}{\omega_2^*} & \cdots & -\frac{\gamma}{\omega_m^*}\end{bmatrix}^T$$

*and* $A_\zeta$ *is given by (2.6) with* $m$ *in place of* $n$, *and* $\alpha_j$ *is the* $j^{th}$ *column index of* $\Theta(z)$ .

*Proof.* To prove (a), first we show that for $k = 1, 2, \ldots, m+1$,

$$\begin{bmatrix} 1 & -\frac{\omega_k}{\gamma} \end{bmatrix} (\Theta_1 \Theta_2 \cdots \Theta_k)(z_k) = 0 \tag{3.16}$$

Since, by the definition of $\Theta_k(z)$ in (3.13), the first column of $(\Theta_1 \Theta_2 \cdots \Theta_k)(z_k)$ is zero and hence the first component of (3.16) is zero, and

$$\begin{aligned}
&\text{the } 2^{nd} \text{ column of } (\Theta_1 \Theta_2 \cdots \Theta_k)(z_k) \\
&= (\Theta_1 \Theta_2 \cdots \Theta_{k-1})(z_k) \begin{bmatrix} \frac{y_k^{k-1}}{\gamma} \\ 1 \end{bmatrix} (z_k + \bar{z}_k) \\
&= (\Theta_1 \Theta_2 \cdots \Theta_{k-1})(z_k) \begin{bmatrix} n^{k-1}(z_k) \\ \gamma d^{k-1}(z_k) \end{bmatrix} \frac{(z_k + \bar{z}_k)}{\gamma d^{k-1}(z_k)} \\
&= \begin{bmatrix} n^0(z_k) \\ \gamma d^0(z_k) \end{bmatrix} \frac{(z_k + \bar{z}_k)}{\gamma d^{k-1}(z_k)} \\
&= \begin{bmatrix} \frac{y^0(z_k)}{\gamma} \\ 1 \end{bmatrix} \frac{(z_k + \bar{z}_k) d^0(z_k)}{d^{k-1}(z_k)}.
\end{aligned}$$

Noting that $y^0(z_k) = \omega_k$, we find the $2^{nd}$ column of $(\Theta_1 \Theta_2 \cdots \Theta_k)(z_k)$ multiplied by $\begin{bmatrix} 1 & -\frac{\omega_k}{\gamma} \end{bmatrix}$ on its left side is simply zero. Hence, we have

$$\begin{bmatrix} 1 & -\frac{\omega_k}{\gamma} \end{bmatrix} \Theta(z_k) = 0, \ k = 1, 2, \cdots, m+1. \tag{3.17}$$

Here we note that $d^{k-1}(z_k) \neq 0$ by its construction for $1 \leq k \leq m+1$.

Now we show

$$\begin{bmatrix} 1 & -\frac{\gamma}{\omega_k^*} \end{bmatrix} \Theta(-z_k^*) = 0, \ k = 1, 2, \cdots, m. \tag{3.18}$$

For $k = 1, 2, \cdots, m$, since the second column of $\Theta_k(-z_k^*)$ is zero, so is the second column of $(\Theta_1 \Theta_2 \cdots \Theta_k)(-z_k^*)$ and hence so is the second component of (3.18). By applying (3.12), we have

$$\begin{aligned}
&\text{the } 1^{st} \text{ column of } (\Theta_1 \Theta_2 \cdots \Theta_k)(-z_k^*) \\
&= \text{the } 1^{st} \text{ column of } (-1)^k J \, \overline{(\Theta_1 \Theta_2 \cdots \Theta_k)(z_k)} \, J \\
&= \text{the } 2^{nd} \text{ column of } (-1)^k J \, \overline{(\Theta_1 \Theta_2 \cdots \Theta_k)(z_k)} \\
&= (-1)^k J \, \overline{(\Theta_1 \Theta_2 \cdots \Theta_{k-1})(z_k) \begin{bmatrix} \frac{y_k^{k-1}}{\gamma} \\ 1 \end{bmatrix}} (z_k + z_k^*).
\end{aligned} \tag{3.19}$$

In the previous page, the part under the big overline in (3.19) is reduced to a constant multiple of $\begin{bmatrix} \frac{\omega_k}{\gamma} \\ 1 \end{bmatrix}$. Upon multiplying by $J$ on the left, (3.19) is reduced to a constant multiple of $\begin{bmatrix} 1 \\ \frac{\omega_k{}^*}{\gamma} \end{bmatrix}$. Thus,

$$\begin{bmatrix} 1 & -\frac{\gamma}{\omega_k{}^*} \end{bmatrix} (\Theta_1\Theta_2 \cdots \Theta_k)(-\bar{z}_k) = 0$$

and, in turn, (3.18) is proved. By (3.17) and (3.18), $\Theta(z)$ has at least $(2m+1)$ zeros. But, the fact that $\alpha_1 \leq m$ , $\alpha_2 \leq m+1$ by the construction forces deg $(\det\Theta(z)) \leq 2m+1$ which means $\Theta(z)$ can have no more zeros. This forces (a) and $deg(det\Theta(z)) = 2m+1$ and in turn $\alpha_1 = m+1$, $\alpha_2 = m$. Thus, $\Theta(z)$ is column reduced. □

To expolit Theorem 3.3, we assume the interpolation data is given by

$$y(z_i) = \frac{\omega_i}{\gamma}, \quad 1 \leq i \leq m+1 \tag{3.20}$$

$$y(z_{m+1+i}) = \frac{\gamma}{\omega_i^*} := \omega_{m+1+i}, \quad i = 1, \cdots, m \tag{3.21}$$

with the symmetric property that $z_{m+1+i} = -z_i^*$, where $z_i \in RHP$ and $\omega_i$ is nonzero for $1 \leq i \leq m+1$. Then, the Loewner matrix $L$ in (1.3) and $\Lambda$ in (3.3) have the following relationship:

$$[I_m\ 0]\,\overline{\Lambda} = DL, \tag{3.22}$$

where $D$ is an $m \times m$ diagonal matrix having $-\gamma w_i^*$ as its $(i,i)$- component. This suggests a connection between the Nevanlinna-Pick problem and the minimal McMillan degree interpolation problem with symmetric data.

Now we suppose $\Lambda$ is positive definite. Then, by (3.22), $rankL = m$. The next theorem recovers $y_{NP}$ in a non-recursive way.

**Theorem 3.4.** *Given are the conditions (3.1) and (3.2) with positive definite $\Lambda$ in (3.3). Then,*

$$y_{NP} = \gamma \frac{\bar{n}(z)}{\bar{d}(z)}, \tag{3.23}$$

*where $\bar{n}(z), \bar{d}(z)$ are constructed as in (1.8) and (1.9) with $\bar{L} = L$, $q = m$ for the given interpolation data (3.20) and (3.21).*

*Proof.* For the given interpolation data (3.20) and (3.21), we construct $L$, $\bar{n}(z), \bar{d}(z)$ and $\hat{n}(z), \hat{d}(z)$ according to (1.3), (1.8), (1.9), (2.3) and (2.4) . Due to (3.22) $rankL = m$. Then, $\hat{\Theta}(z)$ given by (2.7) and

$\Theta(z)$ by (3.15) both are column reduced with the same left null pair $\left(\begin{bmatrix} A_\zeta & 0 & 0 \\ 0 & z_{m+1} & 0 \\ 0 & 0 & -A_\zeta^* \end{bmatrix}, \begin{bmatrix} \hat{B} \\ \\ \tilde{B} \end{bmatrix}\right)$ with decreasing column indices $m+1$ and $m$. By Corollary 2.3,

$$\begin{bmatrix} \bar{n}(z) \\ \bar{d}(z) \end{bmatrix} = \alpha \begin{bmatrix} \Theta_{12}(z) \\ \Theta_{22}(z) \end{bmatrix},$$

for some nonzero constant $\alpha$. By (3.10) and the relationship $\begin{bmatrix} \tilde{\Theta}_{12} \\ \tilde{\Theta}_{22} \end{bmatrix} = \begin{bmatrix} \Theta_{12} \\ \Theta_{22} \end{bmatrix}(z + z_{m+1}^*)$,

$$y_{NP} = \gamma \frac{\Theta_{12}}{\Theta_{22}} = \gamma \frac{\bar{n}(z)}{\bar{d}(z)}$$

□

**Corollary 3.5.** *The followings are equivalent.*
*(a)* $\delta(y_{NP}) = m$
*(b)* $\frac{1}{\gamma} y_{NP}$ *is a unique minimal solution for (3.20) and (3.21)*
*(c)* $\bar{d}(z)$ *in Theorem 3.4 has no zeroes at* $\{-z_i^* \mid i = 1, \cdots, m\}$.

In [2], it is asserted (in the equivalent context where the right half plane is replaced by the unit disk) that, if the Pick matrix $\Lambda$ is positive definite, then a minimal degree solution of (3.20) and (3.21) automatically is analytic with modulus at most $\gamma$ on $RHP$ and coincides with $\frac{1}{\gamma} y_{NP}$. But, the following example shows that it is not true.

**Example 3.6.** *Let* $z_1 = 1/2,\ z_2 = 1,\ \omega_1 = 1/2,\ \omega_2 = 1/2,\ \gamma = 1$. *Then, the Nevanlinna-Pick interpolation conditions are*

$$y(1/2) = 1/2, \quad y(1) = 1/2 \tag{3.24}$$

*while the minimal degree interpolation problem with symmetric data involves the extra interpoaltion condition*

$$y(-1/2) = 2. \tag{3.25}$$

*For this case one easily sees that* $y_{NP}$ *constructed by (3.5) is the costant function* $y_{NP} = 1/2$ *which does not meet the extra interpolation condition (3.25). Since* $L = \begin{bmatrix} -3/2 & -1 \end{bmatrix}$, *if we choose* $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ *in* $kerL$, *then by (1.8) (1.9),* $\bar{d}(z) = -(z + 1/2)$ *and* $\bar{n}(z) = -1/2(z + 1/2)$ *so that (3.23) holds. On the other hand, the minimal McMillan degree of the* *solutions satisfying (3.24) and (3.25) is* $3-$ rank$L=2$ *by Theorem 1.1, since* $\bar{d}(z)$ *has a zero at* $-\frac{1}{2}$*. So, for this problem, any minimal solution for (3.24) and (3.25) cannot be* $y_{NP}$ *whose McMilla degree is* $0$.

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