ON CLEAN AND NIL CLEAN ELEMENTS IN SKEW T.U.P. MONOID RINGS

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Abstract. Let \( R \) be an associative ring with identity, \( M \) a t.u.p. monoid with only one unit and \( \omega : M \to \text{End}(R) \) a monoid homomorphism. Let \( R \) be a reversible, \( M \)-compatible ring and \( \alpha = a_1g_1 + \cdots + a_ng_n \) a non-zero element in skew monoid ring \( R \ast M \). It is proved that if there exists a non-zero element \( \beta = b_1h_1 + \cdots + b_mh_m \) in \( R \ast M \) with \( \alpha \beta = c \) is a constant, then there exist \( 1 \leq i_0 \leq n, 1 \leq j_0 \leq m \) such that \( g_{i_0} = e = h_{j_0} \) and \( a_{i_0}b_{j_0} = c \) and there exist elements \( a, 0 \neq r \) in \( R \) with \( \alpha r = ca \). As a consequence, it is proved that \( \alpha \in R \ast M \) is unit if and only if there exist \( 1 \leq i_0 \leq n \) such that \( g_{i_0} = e, a_{i_0} \) is unit and \( a_j \) is nilpotent for each \( j \neq i_0 \), where \( R \) is a reversible or right duo ring. Furthermore, we determine the relation between clean and nil clean elements of \( R \) and those elements in skew monoid ring \( R \ast M \), where \( R \) is a reversible or right duo ring.

1. Introduction and preliminaries

Throughout the paper, unless mentioned otherwise, all rings are associated. We use the notation \( U(R) \), \( \text{Idem}(R) \), \( \text{nil}(R) \), \( \text{cln}(R) \), \( \text{nil}^*(R) \), \( \text{L-rad}(R) \) and \( J(R) \) to denote the set of unit elements, idempotent elements, nilpotent elements, clean elements, nil clean elements, the prime radical, the upper nil radical, the Levitzki radical and the Jacobson radical of a ring \( R \), respectively.

Recall that a ring \( R \) is reduced if it has no non-zero nilpotent element. According to Krempa [16] an endomorphism \( \sigma \) of a ring \( R \) is called rigid if \( a\sigma(a) = 0 \) implies that \( a = 0 \) for \( a \in R \). A ring \( R \) is said to be \( \sigma \)-rigid if there exists a rigid endomorphism \( \sigma \) of \( R \). Concept of \( \sigma \)-compatible rings introduced in [11, 13]. A ring \( R \) is called \( \sigma \)-compatible if for each \( a, b \in R, ab = 0 \) if and only if \( a\sigma(b) = 0 \). We now make a survey of several kinds of generalizations of reduced rings. A ring \( R \) is reversible if \( ab = 0 \) implies \( ba = 0 \) for \( a, b \in R \). A ring \( R \) is semicommutative if \( ab = 0 \) implies \( aRb = 0 \) for \( a, b \in R \). On
the other hand, a ring $R$ is called 2-primal if $\operatorname{nil}_2(R) = \operatorname{nil}(R)$ (see [3]). Shin in [25, Proposition 1.11] showed that a ring $R$ is 2-primal if and only if every minimal prime ideal $P$ of $R$ is completely prime (i.e., $R/P$ is a domain). A ring $R$ is weakly 2-primal if $\operatorname{nil}(R) = \operatorname{L-rad}(R)$. A ring $R$ is NI if $\operatorname{nil}(R) = \operatorname{nil}^*(R)$. It is known that reduced $\Rightarrow$ reversible $\Rightarrow$ semicommutative $\Rightarrow$ 2-primal $\Rightarrow$ weakly 2-primal $\Rightarrow$ NI, but the converse does not hold (see [7, 14]). Moreover, a ring is right (resp., left) duo if every right (resp., left) ideal is an ideal. Reversible as well as (one-sided) duo rings are semicommutative. A ring $R$ is called abelian if each idempotent element of $R$ is central. It is known that reversible rings and also semicommutative rings are abelian. But these implications are irreversible.

Let $M$ be a monoid. In the following, we denote the identity element of $M$ by $e$. A monoid $M$ with a partial order $\leq$ is called ordered monoid if for any $a_1, a_2, b \in M, a_1 \leq a_2$ implies $a_1 b \leq a_2 b$ and $b a_1 \leq b a_2$. A strictly ordered monoid $(M, \leq)$ is an ordered monoid such that for any $a_1, a_2, b \in S, a_1 < a_2$ implies $a_1 b < a_2 b$ and $b a_1 < b a_2$. A strictly ordered monoid $M$ is said to be positively strictly ordered if $m \geq e$ for all $m \in M$.

We use the following terminology. If $A$ and $B$ are non-empty subsets of a monoid $M$, then an element $s_0 \in AB = \{ab : a \in A, b \in B\}$ is said to be a unique product element (u.p. element for short) in the product of $AB$ if it is uniquely presented in the form of $s = ab$ where $a \in A$ and $b \in B$.

Recall that a monoid $M$ is called unique product monoid (u.p. monoid for short) if for any two non-empty finite subsets $A, B \subseteq M$ there exist $a \in A$ and $b \in B$ such that $ab$ is u.p. element in the product of $AB$. Also, a monoid $M$ is said to be two unique product monoid (or simply t.u.p. monoid) if for any two non-empty finite subsets $A, B \subseteq M$ such that $|A| + |B| > 2$ there exist two u.p. elements in the product of $AB$. Strojnowski in [26, Theorem 1] proved that a group is t.u.p. if and only if it is u.p. Clearly, each strictly totally ordered monoid is t.u.p. monoid.

Assume that $R$ is a ring, $M$ a monoid and $\omega : M \to \operatorname{End}(R)$ a monoid homomorphism. For each $g \in M$ we denote the image of $g$ by $\omega_g$ (i.e., $\omega(g) = \omega_g$). Then all finite formal combinations $\sum_{i=1}^{n} a_i g_i$, with point-wise addition and multiplication induced by $(ag)(bh) = (a \omega_g(b)) gh$ form a ring that is called skew monoid ring and it is denoted by $M \ast R$. Let $R$ be a ring, $M$ a monoid and $\omega : M \to \operatorname{End}(R)$ a monoid homomorphism. We say that $R$ is $M$-compatible (resp. $M$-rigid) if $\omega_g$ is compatible (resp. rigid) for any $g \in M$. The construction of skew monoid ring generalizes some classical ring construction such as polynomial rings, skew polynomial rings, Laurent polynomial rings, skew Laurent polynomial rings and monoid rings. Hence any result on skew monoid ring has its counterpart in each of the subclasses.

Unit, idempotent and nilpotent elements play an important role in noncommutative ring theory. An element $a \in R$ is said to be clean if it present as the sum of a unit and an idempotent. Rings in which every element is sum of unit and idempotent are called clean, have been extensively studied. The concept of clean ring originally defined by Nicholson [21]. In recent decades
many authors studied this class of ring, many generalization and variation of it. Anderson and Camilo in [2] defined uniquely clean ring in commutative rings. An element \( a \in R \) is said to be uniquely clean if it uniquely present as the sum of a unit and an idempotent. A ring \( R \) is called uniquely clean if every element is uniquely clean. Later, Nicholson and Zhou [22] extend the notation of uniquely clean in noncommutative ring.

Recently, Diesl [9] modified the definition of clean ring and obtained an interesting new concept he called nil clean. Following [9], an element \( a \in R \) is said to be (strongly) nil-clean if there exists an idempotent \( e \) and a nilpotent \( b \) in \( R \) such that \( a = e + b \) (and \( eb = be \)). A ring \( R \) is called (strongly) nil clean if every element of \( R \) is (strongly) nil clean. Nil clean and strongly nil clean rings are naturally connected to clean and strongly clean ring. Kosan et al. [15] proved that an element \( a \) is strongly nil clean if and only if \( a \) is clean and \( a - a^2 \) is nilpotent, and that a ring \( R \) is strongly nil clean if and only if \( R/J(R) \) is boolean and \( J(R) \) is nil.

The aim of this paper is to determine some type of elements such as idempotents, units, cleans, and nil cleans in a skew monoid ring \( R \ast M \).

In Section 2, we show that if \( 0 \neq \alpha = a_1g_1 + \cdots + a_ng_n \) is an idempotent in \( R \ast M \), then there exists \( 1 \leq i \leq n \) such that \( g_i = e \) and \( a_i - f \in \text{nil}(R) \) for some idempotent \( f \in R \) and \( a_j \in \text{nil}(R) \) for all \( j \neq i \), where \( R \) is a 2-primal and \( M \)-compatible ring and \( M \) is a u.p. monoid. Also, we prove that the set of idempotent elements in \( R \ast M \) are coincide to the set of idempotents in \( R \) when the base ring \( R \) is a semicommutative ring and \( M \) is a u.p. monoid.

In Section 3, we recall the definition of two unique product monoid (t.u.p. monoid) and bring some examples about t.u.p. monoids.

In Section 4, first we prove some results which concern the constant products of elements in skew monoid ring \( R \ast M \), then we study unit elements of skew monoid ring \( R \ast M \). It is shown that a non-zero element \( \alpha = a_1g_1 + \cdots + a_ng_n \) of \( R \ast M \) is unit, if there exists \( 1 \leq i \leq n \) such that \( g_i = e \) and \( a_i \) is unit in \( R \) and \( a_j \) is nilpotent for all \( j \neq i \), where \( M \) is a t.u.p. monoid with only one unit and \( R \) is a \( M \)-compatible reversible or right duo ring. Therefore we can determine (strongly) clean and (strongly) nil clean elements in skew monoid ring \( R \ast M \).

2. Idempotent elements in skew monoid ring over semicommutative ring

Idempotent elements play an important role in the study of noncommutative rings. In this section, we determine construction of idempotents in skew monoid ring \( R \ast M \) and relation between idempotent elements in \( R \) and skew monoid ring \( R \ast M \) over semicommutative ring \( R \) when \( M \) is a u.p. monoid.

Lemma 2.1 ([10, Lemma 2.8]). Let \( R \) be a ring, \( M \) a monoid and \( \omega : M \rightarrow \text{End}(R) \) a monoid homomorphism such that \( R \) is \( M \)-compatible. Then for each
elements $a_1, a_2, \ldots, a_n \in R$ we have $a_1a_2 \cdots a_n \in \text{nil}(R)$ if and only if
$$\omega_m(a_1)\omega_m(a_2) \cdots \omega_m(a_n) \in \text{nil}(R)$$
for all elements $m_1, m_2, \ldots, m_n \in M$.

Let $R$ be a ring, $M$ a monoid and $\omega : M \to \text{End}(R)$ a monoid homomorphism. According to [10, Definition 3.1], $R$ is said to be skew Armendariz ring relative to $M$ (or simply skew $M$-Armendariz) if whenever non-zero elements
$$\alpha = \sum_{i=1}^{n} a_i g_i, \beta = \sum_{j=1}^{m} b_j h_j$$
in $R \ast M$ satisfy $\alpha\beta = 0$, then $a_i \omega(g_i)(b_j) = 0$ for each $1 \leq i \leq n$ and $1 \leq j \leq m$.

**Proposition 2.2** ([10, Proposition 3.3]). Let $R$ be a ring, $M$ a monoid and $\omega : M \to \text{End}(R)$ a monoid homomorphism. If $R$ is $M$-rigid, then $R$ is skew $M$-Armendariz.

**Theorem 2.3** ([10, Theorem 4.4]). Let $R$ be a 2-primal ring, $M$ a u.p. monoid and $\omega : M \to \text{End}(R)$ a monoid homomorphism. If $R$ is $M$-compatible, then $\text{nil}(R \ast M) = \text{nil}(R) \ast M$.

Let $I$ be a nil ideal in $R$ and $a \in R$ be such that $\bar{a} \in \bar{R} := R/I$ is an idempotent. Then by [17, Theorem 21.28] there exists an idempotent $f \in aR$ such that $\bar{f} = \bar{a} \in \bar{R}$. Also, let $\sigma$ be an endomorphism of a ring $R$. According to [13, Lemma 2.2], $\sigma$ is rigid if and only if $R$ is reduced and $\sigma$ is compatible. Thus for a monoid $M$ and monoid homomorphism $\omega : M \to \text{End}(R)$, $R$ is $M$-rigid if and only if $R$ is reduced and $M$-compatible.

**Proposition 2.4.** Let $R$ be a 2-primal ring, $M$ a u.p. monoid and $\omega : M \to \text{End}(R)$ a monoid homomorphism. Let $R$ be a $M$-compatible ring and $0 \neq \alpha = a_1 g_1 + \cdots + a_n g_n$ an idempotent element of $R \ast M$. Then there exists $1 \leq i \leq n$ such that $g_i = e, \bar{\sigma} = \bar{f} \in R/\text{nil}(R)$ for some idempotent $f \in R$ and $a_j \in \text{nil}(R)$ for all $j \neq i$.

**Proof.** Since $R$ is NI, $\bar{R} = R/\text{nil}(R)$ is reduced. We define $\varpi : M \to \text{End}(\bar{R})$ such that $\varpi(m) = \varpi_m; \varpi_m(\bar{a}) = \omega_m(a)$. Assume that $a \varpi_m(b) = 0$. Then $a \omega_m(b) \in \text{nil}(R)$ and so by Lemma 2.1, $ab \in \text{nil}(R)$. Thus $\bar{R} = R/\text{nil}(R)$ is $M$-compatible (i.e., $\varpi_m$ is compatible for each $m \in M$), which implies that $\bar{R}$ is $M$-rigid and so it is skew $M$-Armendariz, by Proposition 2.2. The element $\bar{\sigma} = \bar{\sigma}_1 g_1 + \cdots + \bar{\sigma}_n g_n$ is non-zero. Otherwise, if $\bar{\sigma} = 0$, then $a_j \in \text{nil}(R)$ for each $1 \leq i \leq n$, which implies that $\alpha \in \text{nil}(R) \ast M$. Therefore $\alpha \in \text{nil}(R \ast M)$, by Theorem 2.3, which is a contradiction. From $\bar{\sigma}^2 = \bar{\sigma}$ we have $(\bar{\sigma} - 1)e \bar{\sigma} = (\bar{\sigma}_1 g_1 + \cdots + \bar{\sigma}_n g_n - 1)e \bar{\sigma} = 0$. Therefore $\bar{\sigma}_i = \bar{\sigma}_i 1 = 0$, if $g_i \neq e$ for each $1 \leq i \leq n$, since $R$ is skew $M$-Armendariz. This means $\bar{\sigma} = 0$, which is a contradiction. Thus we can assume that there exists $1 \leq i \leq n$ such that $g_i = e$. Without loss of generality assume that $i = 1$. Thus $0 = \bar{\pi}(\bar{\sigma} - 1) = \overline{(a_1 - 1)e + a_2 g_2 + \cdots + a_n g_n}$ which implies that $\overline{a_1} = \overline{\sigma}_1$ and $a_i \in \text{nil}(R)$ for all $i \neq 1$, since $\bar{R}$ is skew $M$-Armendariz. Then by [17, Theorem 21.28], there exists an idempotent $f \in R$ such that $\overline{\sigma}_1 = \overline{f} \in \bar{R}$, as desired. \hfill \Box
Now we can determine the relation between idempotent elements of $R$ and $R*M$ when $M$ is u.p. monoid and $R$ is a semicommutative and $M$-compatible ring.

**Theorem 2.5.** Let $R$ be a semicommutative and $\omega : M \to \text{End}(R)$ a monoid homomorphism, where $M$ is a u.p. monoid. If $R$ is $M$-compatible, then $\text{Idem}(R*M) = \text{Idem}(R)$.

**Proof.** Let $\alpha = a_1g_1 + \cdots + a_ng_n \in \text{Idem}(R*M)$. By Proposition 2.4 there exists $1 \leq i \leq n$ such that $g_i = e$, $\overline{f} = f \in R/\text{nil}(R)$ for some idempotent $f \in R$ and $a_j \in \text{nil}(R)$ for all $j \neq i$. Hence $\alpha_i = f + t$ where $t$ is a nilpotent element of $R$.

Let $i = 1$ and $\alpha_1 = te + a_2g_2 + \cdots + a_ng_n$. Then $\alpha = f + \alpha_1$. Since $R$ is $2$-primal, $\alpha_1 \in \text{nil}(R)*M = \text{nil}(R*M)$, by Theorem 2.3. If $\alpha_1 \neq 0$, then there exists non-negative integer $k$ such that $\alpha_1^k = 0 \neq \alpha_1^{k-1}$. By the proof of [1, Theorem 2.41] any compatible homomorphism is idempotent-stabilizing, thus $\omega_i(f) = f$ and so $\alpha_1f = f\alpha_1$. Since $\alpha^2 = \alpha$, we have $0 = (f + \alpha_1)(1 - \alpha - \alpha_1) = \alpha_1 - 2f\alpha_1 - \alpha_1^2$. Then $\alpha_1^2 = (1 - 2f)\alpha_1$. By multiplying $\alpha_1^{k-2}$ from right hand side to $\alpha_1^2 = (1 - 2f)\alpha_1$ we have $0 = \alpha_1^k = (1 - 2f)\alpha_1^{k-1}$. Since $1 - 2f$ is invertible, $\alpha_1^{k-1} = 0$, which is a contradiction. Hence $\alpha_1 = 0$ and so $\alpha = f$. \hfill $\Box$

If $R$ is a semicommutative ring, $M$ a u.p. monoid and $\omega : M \to \text{End}(R)$ identity homomorphism (i.e., $\omega_m$ is identity endomorphism of $R$ for each $m \in M$), then the skew monoid ring $R*M$ is isomorphic to monoid ring $R[M]$. As a result of Theorem 2.5 we have $\text{Idem}(R[M]) = \text{Idem}(R)$.

**Corollary 2.6.** Let $R$ be a semicommutative and $\omega : M \to \text{End}(R)$ a monoid homomorphism, where $M$ is a u.p. monoid. If $R$ is $M$-compatible, then $R*M$ is abelian.

Let $M$ be a monoid generated by \{x\} and $\sigma$ be an endomorphism of $R$. Let $\omega : M \to \text{End}(R)$ a monoid homomorphism such that $\omega_{x^i} = \sigma^i$ for each $x^i \in M$. Then the skew monoid ring $R*M$ is isomorphic to skew polynomial ring $R[x; \sigma]$.

**Corollary 2.7.** Let $R$ be a semicommutative ring and $\sigma$ an endomorphism of $R$. If $R$ is a $\sigma$-compatible ring, then $\text{Idem}(R[x; \sigma]) = \text{Idem}(R)$ and $R[x; \sigma]$ is abelian.

Note that if $M,N$ are two u.p. monoids, then $M \times N$ is also u.p. monoid, by [18, Lemma 1.13]. On the other hand, for a ring $R$ and indeterminates $x$ and $y$ we have $R[x,y] \cong R[x][y]$. Thus, we have the following result.

**Corollary 2.8.** Let $R$ be a semicommutative ring. Then $\text{Idem}(R[x_1,x_2,\ldots,x_n]) = \text{Idem}(R)$ and $R[x_1,x_2,\ldots,x_n]$ is abelian.

3. Two unique product monoids

As mentioned in the introduction, a monoid $M$ is said to be t.u.p. if for any two non-empty finite subsets $A,B \subseteq M$ with $|A| + |B| > 2$ there exist two
A partially ordered set \((M, \leq)\) is called 
\textit{artinian} if every strictly decreasing sequence of elements of \(M\) is finite, and \((M, \leq)\) is called 
\textit{narrow} if every subset of a pairwise order-incomparable elements of \(M\) is finite. Thus \((M, \leq)\) is artinian and narrow if and only if every non-empty subset of \(M\) has at least one but only a finite number of minimal elements. Let \((M, \leq)\) be an 
ordered monoid. We say that \((M, \leq)\) is an 
\textit{artinian narrow unique product monoid} (or \textit{a.n.u.p. monoid}) if for every two artinian and narrow subsets \(A\) and \(B\) of \(M\) there exists a \textit{u.p.} element in the product of \(AB\). An ordered 
monoid \((M, \leq)\) is called 
\textit{artinian narrow unique product monoid} (or \textit{a.n.u.p. monoid}) if for every two artinian and narrow subsets \(A\) and \(B\) of \(M\) there exists a \textit{u.p.} element in the product of \(AB\). An ordered monoid \((M, \leq)\) is called 
\textit{artinian narrow unique product monoid} (or \textit{a.n.u.p. monoid}) if for every two artinian and narrow subsets \(A\) and \(B\) of \(M\) there exists a \textit{u.p.} element in the product of \(AB\). A monoid \(M\) is said to be 
\textit{totally orderable} if \((M, \leq)\) is an ordered monoid 
for some total order \(\preceq\) (any two different elements of \(M\) are comparable). An ordered 
monoid \((M, \leq)\) is said to be 
\textit{quasi-totally ordered} (and \(\leq\) is called a 
\textit{quasi-total order} on \(M\)) if \(\leq\) can be refined to an order 
\(\preceq\) with respect to which \(M\) is a strictly totally ordered monoid.

One could define an \textit{a.n.t.u.p. monoid} in the obvious way:

\textbf{Definition 3.1.} An ordered monoid \(M\) is an 
\textit{a.n.t.u.p. monoid} if for every 
two artinian narrow subsets \(A, B\) of \(M\) with 
\(|A| + |B| > 2\) there exist at least 
two \textit{u.p.} elements in the product \(AB\). An ordered 
monoid \((M, \leq)\) is called 
\textit{minimal artinian narrow unique product monoid} (or \textit{m.a.n.u.p. monoid}) if for every two artinian and narrow subsets \(A\) and \(B\) of \(M\) with 
\(|A| + |B| > 2\) there exist at least two \textit{u.p.} elements \(a_1b_1, a_2b_2\) in the product 
\(AB\) such that \(a_1, a_2 \in \min(A)\) and \(b_1, b_2 \in \min(B)\).

If \((M, \leq)\) is an ordered monoid the following implications hold:

\[
\begin{align*}
(M, \leq) \text{ is quasi-totally ordered monoid} & \downarrow \\
(M, \leq) \text{ is a m.a.n.u.p. monoid} & \leftarrow (M, \leq) \text{ is a m.a.n.t.u.p. monoid} \\
(M, \leq) \text{ is an a.n.u.p. monoid} & \downarrow \\
(M, \leq) \text{ is an a.n.t.u.p. monoid} & \downarrow \\
M \text{ is a u.p. monoid} & \leftarrow M \text{ is a t.u.p. monoid}
\end{align*}
\]

Marks, Mazurek and Ziembowski \cite{19} showed that left implications in the 
above diagram are irreversible. To prove quasi-totally ordered monoids are 
\textit{m.a.n.t.u.p.} note that by assumption \(\leq\) can be refined to a total order \(\preceq\) such 
that \((M, \preceq)\) is a strictly ordered monoid. If \(A\) and \(B\) are artinian and narrow 
subsets of \((M, \preceq)\), then the sets \(\min(A), \min(B)\) are finite, and thus there exist 
\(a_1 \in \min(A)(b_1 \in \min(B))\) which is smallest under the total order \(\preceq\) and 
\(a_2 \in \min(A)(b_2 \in \min(B))\) which is greatest under \(\preceq\). Now, clearly \(a_1b_1\) and 
\(a_2b_2\) are two \textit{u.p.} elements of \(AB\). The remaining implications are obvious.
Clearly, any t.u.p. monoid is u.p. The following example shows that the converse is not true in general.

**Example 3.2** ([24, Example 13 of Chap. 10]). Let $M$ be the monoid generated by $x_1, x_2, x_3, X_1, X_2, X_3$ subject to the following relations:

$$x_1 X_1 = x_2 X_3, \quad x_1 X_2 = x_3 X_1, \quad x_1 X_3 = x_2 X_2, \quad x_3 X_2 = x_2 X_1.$$ 

As shown in [24], $M$ is a u.p. monoid. Let $A=\{x_1, x_2, x_3\}$ and $B=\{X_1, X_2, X_3\}$.

Clearly, $x_3 X_3$ is the only u.p. element in the product $AB$. Hence $M$ is not t.u.p.

Marks et al. in [19, Example 2.7] showed that the monoid mentioned in Example 3.2 is m.a.n.u.p. Therefore m.a.n.u.p. monoids are not t.u.p. in general. Our next example shows that t.u.p. does not imply a.n.u.p.

**Example 3.3.** Let $M$ be the monoid generated by $\{x_i \mid i \in \mathbb{N}\} \cup \{X_j \mid j \in \mathbb{N}\}$ with the following relations:

$$x_i X_j = \begin{cases} x_{i-2} X_{i-2} & \text{if } i \geq 3 \text{ and } j = i + (-1)^{i+1} \\ x_j X_i & \text{otherwise.} \end{cases}$$

Hence $x_i X_j = x_j X_i$ for any $i \neq j$ except for the following products:

$$x_3 X_4 = x_1 X_1, \quad x_4 X_3 = x_2 X_2, \quad x_5 X_6 = x_3 X_3, \quad x_6 X_5 = x_4 X_4,$$

and so on. Marks et al. [19, Example 2.6], showed that $M$ admits a strict total ordering (and hence is t.u.p. monoid) but $(M, \leq)$ is not a.n.u.p.

4. Unit elements in skew monoid ring over reversible or right duo ring

It is well known that a polynomial over commutative ring $R$ is unit if and only if its constant term is a unit in $R$ and other coefficients are nilpotent. Chen [5, Example 2.8] showed that the conclusion is not true for noncommutative ring in general. Also, in [6] he generalized the constant-product theorem for a commutative polynomial ring (which is proved in [14]) to skew polynomial ring $R[x; \sigma]$, where $R$ is a reversible ring which is $\sigma$-compatible for an endomorphism $\sigma$ of $R$. Also he proved a skew polynomial $f(x)$ in $R[x; \sigma]$ is a unit if and only if its constant term is a unit in $R$ and other coefficients are all nilpotent, when $R$ is a weakly 2-primal ring which is $\sigma$-compatible. In this section, we generalize these results to skew monoid ring $R * M$.

For an element $\alpha = a_1 g_1 + \cdots + a_n g_n \in R * M$ with $a_i \neq 0$ for each $i$, we say that length $\alpha = n$ and denote it by $\ell(\alpha)$.

**Lemma 4.1.** Let $R$ be a ring and $M$ a t.u.p. monoid with only one unit and $\omega : M \to \text{End}(R)$ a monoid homomorphism. Let $R$ be $M$-rigid and $\alpha = a_1 g_1 + \cdots + a_n g_n$, $\beta = b_1 h_1 + \cdots + b_m h_m$ non-zero elements of $R * M$ with $\alpha \beta = e \in R$. Then there exist $i_0, j_0$ such that $g_{i_0} = e = h_{j_0}, a_{i_0} b_{j_0} = e$ and $a_i b_j = 0$ for all $i + j \neq i_0 + j_0$. 

\[ \text{(Example 3.2)} \]
Proof. Let \( \alpha = a_1g_1 + \cdots + a_ng_n, \beta = b_1h_1 + \cdots + b_nh_m \) be non-zero elements of \( R\ast M \) with \( \ell(\alpha) = n \) and \( \ell(\beta) = m \). If \( c = 0 \), then \( a_i\omega_{g_i}(b_{j}) = 0 \) for each \( i, j \), since \( R \) is skew \( M\)-Armendariz by [10, Proposition 3.3]. Thus \( a_ib_j = 0 \) for each \( i, j \), as desired. Now, suppose that \( c \neq 0 \). We proceed by induction on \( m + n \). For \( m = 1 \) or \( n = 1 \) the result is clear. Now, let \( m, n > 1 \) and suppose that the result is true for all the smaller values than \( m + n \). Since \( M \) is a t.u.p. monoid, there exist \( 1 \leq i_0, i_1 \leq n, 1 \leq j_0, j_1 \leq m \) such that \( g_{i_0}b_{j_0} \) and \( g_{i_1}h_{j_1} \) are t.u.p. elements in the product of two subsets \( \{g_1, \ldots, g_n\} \) and \( \{h_1, \ldots, h_m\} \) of \( M \). Since \( M \) has only one unit, from \( \alpha\beta = c \) we have \( g_{i_0} = e = h_{j_0}, a_i\omega_{g_i}(b_{j_0}) = 0 \), which implies that \( a_{i_0}b_{j_0} = c \) and \( a_{i_1}b_{j_1} = 0 \), since \( \omega_{g_{i_0}} = id_R \) and \( R \) is \( M \)-compatible. Without loss of generality we can assume that \( i_0 = j_0 = 1, i_1 = n \) and \( j_1 = m \). Thus \( a_{1}b_{n} = 0 = b_{m}a_{n} \) since \( R \) is reduced. Then \( b_{m}c = b_{m}\alpha\beta = (b_{m}a_{1}e + b_{m}a_{2}g_{2} + \cdots + b_{m}a_{n-1}g_{n-1})\beta \). Let \( \alpha_{1} = b_{m}\alpha \). Then \( b_{m}c = \alpha_{1}\beta \). Therefore by induction hypothesis we have \( b_{m}a_{1}b_{m} = b_{n}a_{2}b_{m} = \cdots = b_{m}a_{n-1}b_{m} = 0 \). Thus \( b_{m}a_{1} = 0 \) and so \( a_{1}b_{m} = a_{2}b_{m} = \cdots = a_{n}b_{m} = 0 \), since \( R \) is reduced. Then \( c = \alpha\beta = a(\beta - b_{m}g_{m}) = \alpha\beta_{1} \). Since \( \ell(\alpha) + \ell(\beta_{1}) \leq m + n - 1 \), then by induction hypothesis we get \( a_{1}b_{1} = c \) and \( a_{i}b_{j} = 0 \) for each \( i + j > 2 \). \( \square \)

The following example shows that the assumption “\( M \) has only one unit” in Theorem 4.1 is not superfluous. Note that if \( M \) is a monoid generated by \( \{x, x^{-1}\} \) and \( \omega \) be the identity homomorphism, then \( R\ast M \) is isomorphic to Laurent polynomial ring \( R[x, x^{-1}] \).

**Example 4.2.** Let \( R \) be a ring with \( |\text{Idem}(R)| \geq 3 \). Let \( a \) be a nontrivial idempotent element of \( R \) and \( \alpha = ax^{-1} + (1 - a)x, \beta = (1 - a)x^{-1} + ax \in R[x, x^{-1}] \). Then \( \alpha\beta = 1 \) but any product of coefficients \( \alpha \) and \( \beta \) is not identity.

**Lemma 4.3** ([6, Lemma 2.3]). Let \( R \) be a 2-primal ring which is \( \sigma \)-compatible for an endomorphism \( \sigma \) of \( R \). If \( P \) is any minimal prime ideal of \( R \), then both \( \sigma(P) \) and \( \sigma^{-1}(P) \) are contained in \( P \).

Chen [6, Theorem 2.4] proved that for a reversible ring \( R \) which is \( \sigma \)-compatible, if \( f = a_0 + a_1x + \cdots + a_mx^m, g = b_0 + b_1x + \cdots + b_mx^m \) are non-zero elements of \( R[x; \sigma] \) such that \( gf = c \) is a constant element of \( R \), then \( b_0a_0 = c \) and there exist non-zero elements \( a \) and \( r \) in \( R \) such that \( rf(x) = ac \) with \( r = b_0a \) for some \( p, 0 \leq p \leq m \), and \( a \) is either one or a product of at most \( m \) coefficients from \( f(x) \). In the next theorem we extend this result to skew monoid ring \( R \ast M \), where \( R \) is a reversible ring and \( M \) is a t.u.p. monoid.

**Theorem 4.4.** Let \( R \) be a reversible ring, \( M \) a t.u.p. monoid with only one unit and \( \omega : M \to \text{End}(R) \) a monoid homomorphism and \( \alpha = a_1g_1 + \cdots + a_ng_n \) a non-zero element of \( R \ast M \). Let \( R \) be a \( M \)-compatible ring. If there is a non-zero element \( \beta = b_1h_1 + \cdots + b_nh_m \in R \ast M \) and \( c \in R \) with \( \alpha\beta = c \), then there exist \( 1 \leq i_0 \leq n, 1 \leq j_0 \leq m \) such that \( g_{i_0} = c = h_{j_0}, a_{i_0}b_{j_0} = c \) and there exist elements \( a \) and \( 0 \neq r \) in \( R \) such that \( r\beta = ac \). Furthermore, if \( a_{i_0} \) is a unit in \( R \), then \( b_{j_0} \) is nilpotent for all \( j \neq j_0 \). 
First we prove that the conclusion is true for any element $\beta \in R \ast M$ with $\ell(\beta) = 1$. Let $0 \neq \beta = b_1h_1$. Then $c = \alpha \beta = \alpha b_1h_1$. Therefore, there exist $1 \leq i \leq n$ such that $g_i = e = h_1$ and $a_i\omega_g(b_1) = e$, since any u.p. monoid is cancelative by [4, Lemma 1.1]. Therefore $a_i b_1 = e$, since $g_i = e$ and $\omega_g = id_R$. Hence $a_i \alpha \beta = c$. In this case $r = a_i$ and $a = 1$.

Now, assume that $\ell(\beta) = m \geq 1$. We proceed by induction on $\ell(\alpha) = n$. If $\ell(\alpha) = 1$, then $r = a_1g_1$. Since $\alpha \beta = a_1g_1(b_1h_1 + \cdots + b_nh_m) = c$, there exist $1 \leq j_1, j_2 \leq m$ such that $g_1 = e = h_{j_1}, a_i\omega_g(b_{j_1}) = c$ and $a_1\omega_g(b_{j_2}) = 0$. Without loss of generality we can assume that $j_1 = 1$ and $j_2 = m$. Thus $a_1b_1 = c$ and $a_1b_m = 0$. Hence $c = \alpha \beta = a_1c(b_1e + b_2h_2 + \cdots + b_{m-1}h_{m-1})$ and so there exist $a, r \neq 0$ in $R$ such that $r \beta = ac$, as desired. Thus, we assume that it is true for any element has length less than $\ell(\alpha) = n$ with $n \geq 2$. From $\alpha \beta = c$, there exist non-zero integers $1 \leq i_0, i_1 \leq n$ and $1 \leq j_0, j_1 \leq m$ such that $g_{i_0}h_{j_0}$ and $g_{i_1}h_{j_1}$ are two u.p. elements in the product of two subsets $\{g_1, \ldots, g_n\}$ and $\{h_1, \ldots, h_m\}$ of $M$. Since $M$ has only one unit $g_{i_0} = e = h_{j_0}, a_0\omega_{g_{i_0}}(b_{j_0}) = c$ and $a_1\omega_{g_{i_1}}(b_{j_1}) = 0$. Therefore $a_0b_{j_0} = e$ and $a_1b_{j_1} = 0$, since $\omega_{g_{i_0}} = id_R$ and $R$ is $M$-compatible. Without loss of generality, we can assume that $i_0 = j_0 = 1$ and $i_1 = n, j_1 = m$. Thus $g_1 = e = h_1, a_1b_1 = c$ and $a_n b_m = 0$. We consider the following two cases.

Case 1: Let $c = 0$. We can conclude it with a similar argument as used in the proof of [12, Proposition 1.2].

Case 2: Let $c \neq 0$. If $a_0b_k = 0$ for each $k \neq 1$, then $a_1 b_k = 0$ for each $k \neq 1$, since $R$ is $M$-compatible. Hence $a_1 \beta = c$. In this case $r = a_1$ and $a = 1$. Now, assume that there exists positive integer $k \neq 1$ such that $a_0 b_k \neq 0$. We consider the following two sub-cases.

Sub-case 1: Suppose that $k = m$. Thus $a_0 b_m \neq 0 \neq b_m a_1$, since $R$ is reversible and $M$-compatible. Therefore, $b_m a_1$ is a non-zero element in $R \ast M$ which has it less length than $\ell(\alpha)$ satisfying $b_m a_1 \beta = b_m c$. By induction hypothesis, there is $a, 0 \neq r \in R$ such that $r \beta = a b_m c$, as desired.

Sub-case 2: Suppose that $k \neq m$ and $a_0 b_k \neq 0$. Then there exists $1 \leq t \leq m$ such that $a_0 b_r \neq 0$ for each $1 \leq r \leq t$ and $a_0 b_s = 0$ for each $t < s \leq m$ (we can rewrite $\beta$ if it is necessary). This implies that $c = \alpha \beta = \alpha (b_1h_1 + \cdots + b_t h_t)$. Thus, there exist $1 \leq i \leq n$ and $1 \leq r \leq t$ such that $a_i b_r$ is u.p. element in the product of $\{g_1 = e, g_2, \ldots, g_n\}$ and $\{h_1 = e, h_2, \ldots, h_t\}$ and $a_i b_{j_0} = 0$, since $R$ is t.u.p. with only one unit and $g_1 = e = h_1$. Therefore $a_i b_r = 0$, since $R$ is $M$-compatible. Without loss of generality we can assume that $i = n$ and $r = t$. Therefore $a_n b_t = b_t a_n = 0$. This implies that $b_t a_0 \beta$ has length less than $\ell(\alpha)$ satisfying $b_t a_0 \beta = b_c$ hence, there exist elements $a', 0 \neq r \in R$ such that $r \beta = a' b_t c$. In this case, $a = a' b_t$.

Now assume that $a_1$ is unit in $R$. We prove $b_j$ is nilpotent for all $j > 1$. Let $P$ be a minimal prime ideal of $R$ and $\overline{R} = R/P$. We define $\overline{\omega} : M \to End(\overline{R})$ with $\overline{\omega}(m) = \overline{\omega}_m; \overline{\omega}_m(\overline{a}) = \overline{\omega_m(a)}$. Thus $\overline{R} \ast M$ is a skew monoid ring with monoid homomorphism $\overline{\omega}$. Since reversible rings are 2-primal, $P$ is a completely
prime ideal of \( R \) and so \( \overline{R} = R/P \) is a domain. We prove that \( \overline{R} \) is \( M \)-compatible. If \( \overline{a\overline{b}} = 0 \) for \( a, b \in R \), then \( ab \in P \). Since \( P \) is completely prime, \( a \in P \) or \( b \in P \). This implies that \( a\omega_{m}(b) \in P \), by Lemma 4.3, and so \( a\overline{\omega_{m}(b)} = 0 \). Conversely, assume that \( a\overline{\omega_{m}(b)} = 0 \). This means that \( a\omega_{m}(b) \in P \) and so \( a \in P \) or \( \omega_{m}(b) \in P \). Hence, by Lemma 4.3 we have \( ab \in P \) and so \( \overline{a\overline{b}} = 0 \).

If \( \overline{b} = 0 \), then \( b_j \in P \) for all \( 1 \leq j \leq m \), as desired. Now assume that \( \overline{b} \neq 0 \). Since \( \alpha_1 \) is unit in \( R \), \( \overline{\alpha_1} \neq 0 \). Since \( \overline{\alpha_1}\overline{\beta} = \overline{\alpha_1\beta} \) in \( \overline{R} \ast M \), then there exist elements \( \overline{\alpha_1}, 0 \neq \overline{\alpha} \in R \) such that \( \overline{\alpha_1}\overline{\beta} = \overline{\alpha_1\beta} \). This means \( r\overline{b_j} = 0 \) for all \( j \neq 1 \). Therefore \( rb_j \neq P \) and since \( r \notin P \), we have \( b_j \notin P \) for all \( j \neq 1 \). Since \( P \) is an arbitrary minimal prime ideal of \( R \), so \( b_j \in \text{nil}(R) \) for all \( j \neq 1 \). This implies that \( b_j \in \text{nil}(R) \) for all \( j \neq 1 \), since \( \text{nil}(R) = \text{nil}(R) \).

\[ \square \]

**Theorem 4.5.** Let \( R \) be a reversible ring, \( M \) a t.u.p. monoid with only one unit, \( \omega : M \rightarrow \text{End}(R) \) a monoid homomorphism and \( \alpha = a_1g_1 + \cdots + a_ng_n \) a non-zero element of \( R \ast M \). If there is non-zero element \( \beta = b_1h_1 + \cdots + b_nh_m \in R \ast M \) and \( c \in R \) with \( c\alpha = c \), then there exist elements \( a, 0 \neq r \in R \) such that \( g_m = e = b_{j_0}, a_{i_0}b_{j_0} = c \) and there exist elements \( a \neq 0 \neq r \in R \) with \( ar = ca \). Furthermore, if \( b_{j_0} \) is a unit in \( R \), then \( a_1 \) is nilpotent for each \( i \neq i_0 \).

**Proof.** By a similar argument as used in the proof of Theorem 4.4 one can prove it. \[ \square \]

Let \( M \) be the monoid generated by \( \{ x \} \) and \( \sigma \) be an endomorphism of \( R \). Let \( \omega : M \rightarrow \text{End}(R) \) a monoid homomorphism such that \( \omega_{\omega_{i}} = \sigma^{i} \) for each \( i \geq 0 \). Then it is clear that the skew monoid ring \( R \ast M \) is isomorphic to skew polynomial ring \( R[x; \sigma] \).

**Corollary 4.6** ([6, Theorem 2.4]). Let \( R \) be a reversible ring which is \( \sigma \)-compatible for an endomorphism \( \sigma \) of \( R \) and \( f = a_0 + a_1x + \cdots + a_nx^n \) a non-zero element in \( R[x; \sigma] \). If there is a non-zero skew polynomial \( g = b_0 + b_1x + \cdots + b_mx^m \) in \( R[x; \sigma] \) with \( xf = c \) a constant, then \( b_{0}a_0 = c \) and there exist elements \( a \) and \( 0 \neq r \in R \) such that \( rf(x) = ac \). Furthermore, if \( b_{0} \) is unit in \( R \), then \( a_1, a_2, \ldots, a_n \) are all nilpotent.

Nielsen in [23] called a ring \( R \) is right (resp. left) McCoy if for each pair of non-zero polynomials \( f, g \in R[x] \) providing \( fg = 0 \), then there exists a non-zero element \( r \) in \( R \) such that \( fr = 0 \) (resp. \( rg = 0 \)). The author in [12] extend the concept of McCoy ring to monoid ring \( R[M] \). Let \( R \) be a ring, \( M \) a monoid and \( \alpha, \beta \) be non-zero elements in \( R[M] \). Then \( R \) is said to be right \( M \)-McCoy if \( \alpha\beta = 0 \), then there exists a non-zero element \( r \) in \( R \) such that \( \alpha r = 0 \). Left \( M \)-McCoy is defined similarly. Finally, in [20] the McCoy condition was extended to skew monoid ring \( R \ast M \). According to [20, Definition 2.16] \( R \) is called right skew \( M \)-McCoy, if for each pair of non-zero elements \( \alpha = a_1g_1 + \cdots + a_ng_n \) and \( \beta = b_1h_1 + \cdots + b_nh_m \) in \( R \ast M \), \( \alpha\beta = 0 \) implies that \( \alpha r = 0 \) for some
Lemma 4.10. Let $\omega$ be a skew-McCoy ring, then we say that $R$ is skew-McCoy.

As a consequence of Theorems 4.4 and 4.5 we have the following results.

**Corollary 4.7.** Let $R$ be a reversible, $M$ a t.u.p. monoid with only one unit and $\omega: M \to \text{End}(R)$ a monoid homomorphism. If $R$ is a $M$-compatible ring, then $R$ is skew-McCoy.

**Corollary 4.8.** Let $R$ be a reversible ring, $M$ a quasi-totally ordered monoid with only one unit and $\omega: M \to \text{End}(R)$ a monoid homomorphism. If $R$ is a $M$-compatible ring, then $R$ is skew-McCoy.

**Corollary 4.9** ([23, Theorem 2]). Let $R$ be a reversible ring. Then $R$ is McCoy.

**Lemma 4.10.** Let $R$ be a semicommutative ring, $M$ a t.u.p. monoid with only one unit and $\omega: M \to \text{End}(R)$ a monoid homomorphism. Let $R$ be a $M$-compatible ring and $0 \neq \alpha = a_1 g_1 + \cdots + a_n g_n, 0 \neq \beta = b_1 h_1 + \cdots + b_m h_m \in R+M$ with $\alpha \beta = c \in R$. Then $a_i \cdots a_i \beta = 0$ or $a_i \cdots a_i a_{i+1} \beta = a_i a_{i+1} \cdots a_i c$ for some $\{i_1, \ldots, i_t, i_{t+1}\} \subseteq \{1, \ldots, n\}$.

**Proof.** First assume that $c = 0$. Then $a_i \cdots a_i \beta = 0$ for some $\{i_1, \ldots, i_t\} \subseteq \{1, \ldots, n\}$ and $1 \leq t \leq m$, by [12, Lemma 1.19].

Now, assume that $a_i a_{i+1} \cdots a_t c \neq 0$ for any $\{i_1, \ldots, i_t\} \subseteq \{1, \ldots, n\}$. There exist $1 \leq i_0, i_1 \leq n$ and $1 \leq j_0, j_1 \leq m$ such that $g_{i_0} h_{j_0}$ and $g_{i_1} h_{j_1}$ are two u.p. elements in the product of two subsets $\{g_1, \ldots, g_n\}$ and $\{h_1, \ldots, h_m\}$ of $M$. Since $\alpha \beta = c$ and $M$ has only one unit, $g_{i_0} = c = h_{j_0}, a_{i_0} \omega_{g_{i_0}} (b_{j_0}) = c$ and $a_i \omega_{g_{i_0}} (b_{j_1}) = 0$, which implies that $a_i b_{j_0} = 0$ and $a_i b_{j_1} = 0$, since $R$ is $M$-compatible and $\omega_{g_{i_0}} = id_R$. Without loss of generality, we can assume that $i_0 = j_0 = 1$ and $j_1 = m$. Since $R$ is semicommutative and $M$-compatible, we have $a_{i_1} c = a_{i_1} \alpha \beta = (a_{i_1} a_{i_1} e + a_{i_1} a_{i_1} g_2 + \cdots + a_{i_1} a_{i_1} g_n) (b_{1 e} + b_{2 h_2} + \cdots + b_{m-1} h_{m-1})$. Thus there exist $1 \leq j_2 \leq n$ and $1 \leq j_2 \leq m$ such that $g_{i_0} h_{j_2}$ is u.p. in the product of $\{g_1 = e, g_2, \ldots, g_n\}$ and $\{h_1 = e, h_2, \ldots, h_{m-1}\}$ and $a_i a_{i_1} \omega_{g_{i_1}} (b_{j_2}) = 0 = a_i a_{i_1} b_{j_2}$. Without loss of generality, we can assume that $j_2 = m - 1$. Again, since $R$ is semicommutative we have $a_{i_1} a_{i_1} c = a_{i_1} a_{i_1} \alpha (b_{1 e} + b_{2 h_2} + \cdots + b_{m-2} h_{m-2})$. Continuing this process we can prove $a_i a_{i_1} \cdots a_{i_t} c = a_i a_{i_1} \cdots a_{i_t} a_{i_1} \omega_{g_{i_1}} (b_{j_2})$ for some $\{i_1, \ldots, i_t\} \subseteq \{1, \ldots, n\}$. Thus $a_i a_{i_1} \cdots a_{i_t} \alpha (b_{1 e} + b_{2 h_2} + \cdots + b_{m-2} h_{m-2})$. But $\omega_{g_{i_1}} (b_{j_2}) = 0$. Then $\omega_{g_{i_1}} (b_{j_2}) = 0$.

As a consequence of Lemma 4.10 we have the following result.

**Corollary 4.11.** Let $R$ be a semicommutative ring which is $\sigma$-compatible for an endomorphism $\sigma$ of $R$ and $f = a_0 + a_1 x + \cdots + a_n x^n$, $g = b_0 + b_1 x + \cdots + b_m x^m$ be non-zero elements of $R[x; \sigma]$ with $fg = c \in R$. Then $a_0 b_0 = c$ and $a_i a_{i+1} \cdots a_{i+1} g = 0$ or $a_i a_{i+1} \cdots a_{i+1} a_{i+1} g = a_i a_{i+1} \cdots a_{i+1} c$ for some $\{i_1, \ldots, i_t\} \subseteq \{1, \ldots, n\}$.

For a ring $R$ and element $a \in R$ we denote the set of right (resp. left) annihilators of $a$ over $R$ by $ann_R^R(a)$ (resp. $ann_R^L(a)$). Let $c = 0$ in the Corollary.
4.1. Then by a similar argument as used in the proof of [23, Theorem 4] we can conclude the following result.

**Corollary 4.12.** Let $R$ be a semicommutative ring and $\sigma$ an endomorphism of $R$. Given $f(x)g(x) = c \in R$ with non-zero skew polynomials $f(x), g(x)$ in $R[x; \sigma]$, then (at least) on of $\text{ann}_R R[x; \sigma] f(x) \cap R \neq 0$ or $\text{ann}_R R[x; \sigma] g(x) \cap R \neq 0$.

Let $R$ be a ring and $M$ a u.p. monoid. Cheon and Kim in [8, Theorem 3] proved that $\text{nil}_R(R[M]) = \text{nil}_R(R)[M]$. Also, they proved that for a ring $R$ and a u.p. monoid $M$, $R$ is 2-primal if and only if monoid ring $R[M]$ is 2-primal [8, Theorem 5]. By a similar argument as used in the [8, Theorem 5] with small changes one can prove the following lemma.

**Lemma 4.13.** Let $R$ be a ring, $M$ a u.p. monoid and $\omega : M \to \text{End}(R)$ a monoid homomorphism. Let $R$ be a $M$-compatible ring. Then $R$ is 2-primal if and only if the skew monoid ring $R \ast M$ is 2-primal.

Now we determine all of the unit elements of $R \ast M$, when $R$ is a reversible or right duo ring and $M$ a t.u.p. monoid with only one unit.

**Proposition 4.14.** Let $R$ be a reversible or right duo ring, $M$ a t.u.p. monoid with only one unit and $\omega : M \to \text{End}(R)$ a monoid homomorphism. Let $R$ be a $M$-compatible ring. Then $0 \neq \alpha = a_1g_1 + \cdots + a_ng_n \in R \ast M$ is unit if and only if there exists $1 \leq i_0 \leq n$ such that $g_{i_0} = e, a_{i_0} \in U(R)$ and $a_i$ is nilpotent for each $i \neq i_0$.

**Proof.** Assume that $\alpha$ is unit. Then there exists $0 \neq \beta = b_1h_1 + \cdots + b_nh_m$ such that $\beta\alpha = 1 = \alpha\beta$. Since $M$ is u.p. monoid with only one unit, there exist $i_0,j_0$ such that $g_{i_0}h_{j_0}$ is a u.p. element in the product of two subsets $\{g_1,\ldots, g_n\}$ and $\{h_1,\ldots, h_m\}$ of $M$, $g_{i_0} = e = h_{j_0}$ and $a_{i_0}\omega g_{i_0}(b_{j_0}) = 1$. Therefore $a_{i_0}b_{j_0} = 1$ since $\omega g_{i_0} = \text{id}_R$. Without loss of generality, we can assume that $i_0 = j_0 = 1$. Thus $a_1b_1 = 1$ and so $b_1$ is unit in $R$. Hence, if $R$ is a reversible ring, then $a_i$ is nilpotent for all $i \neq 1$, by Theorem 4.5.

Now, assume that $R$ is a right duo ring. Since $R$ is NI, $R/\text{nil}(R)$ is reduced. For each $m \in M$, $\omega_m(\text{nil}(R)) \subseteq \text{nil}(R)$. Thus we can assume that $\omega_m \in \text{End}(R/\text{nil}(R))$. Hence we define $\overline{\omega} : M \to \text{End}(R/\text{nil}(R))$ such that $\overline{\omega}(m) = \omega_m$. Thus $\overline{R} = R/\text{nil}(R)$ is $M$-compatible and so is $M$-rigid. Since $\overline{M} = M = \text{End}(\overline{R}/\text{nil}(R))$ and $\overline{R} \ast M$ is t.u.p. monoid with only one unit, there exist $1 \leq i_1 \leq n, 1 \leq j_1 \leq m$ such that $g_{i_1} = e = h_{j_1}, \overline{\pi}_i \overline{\beta}_j = \overline{\pi}_j \overline{\beta}_i = 0$ for each $i+j \neq i_1+j_1$, by Lemma 4.1. In the other hand, $M$ has only one unit and $g_1 = e = h_1$, so $i_1 = j_1 = 1$. Thus $\overline{\pi}_1 \overline{\beta}_1 = \overline{\pi}_1 \overline{\beta}_1 = 0$ for each $i+j > 2$. Hence, $\overline{\pi}_i \overline{\beta}_j = 0$ for all $i \geq 2$, which implies that $a_i$ is nilpotent in $R$ for all $i \geq 2$, since $b_1$ is unit in $R$.

Conversely, assume that $a_1$ is unit and $a_2,\ldots, a_n$ are nilpotent. Thus $a_2g_2 + \cdots + a_ng_n \in \text{nil}(R) \ast M = \text{nil}(R \ast M)$, by Theorem 2.3. Since reversible and right duo rings are 2-primal, by Lemma 4.13 the skew monoid ring $R \ast M$ is 2-primal. On the other hand, every 2-primal ring is weakly 2-primal, thus $\text{nil}(R \ast M) =$
L-rad($R * M$). Also, we have L-rad($R * M$) ⊆ $J(R * M)$ by [17, Lemma 10.32], so $a_2g_2 + \cdots + a_ng_n \in J(R * M)$. Thus $\alpha \in U(R * M)$. \hfill \Box

**Corollary 4.15.** Let $R$ be a reversible or right duo ring, $M$ a t.u.p. monoid with only one unit and $\omega : R \to \text{End}(R)$ a monoid homomorphism. Let $R$ be a $M$-compatible ring. Then $U(R * M) = \{a_1g_1 + a_2g_2 + \cdots + a_ng_n \mid n \geq 1, g_1 = e, a_1 \in U(R) \text{ and } a_i \text{ is nilpotent for all } i \neq 1\}.

As a consequence we obtain a generalization of [6, Corollary 2.9], which is just the first part of the following corollary. Not that, if $M,N$ are two t.u.p. monoids, then by a similar argument as used in the proof of [18, Lemma 1.13], one can show that $M \times N$ is also t.u.p. Thus we have the second part of the following result.

**Corollary 4.16.** Let $R$ be a reversible or right duo ring and $\sigma$ an endomorphism of $R$. If $R$ is $\sigma$-compatible, then

1. $U(R[x;\sigma]) = U(R) + \text{nil}(R[x])x$.
2. A non-zero element $f \in R[x_1,\ldots,x_n]$ is unit if and only if the constant term of $f$ is unit and any other coefficients are nilpotent.

Now, we are in the position to determine clean elements of skew monoid ring $R * M$.

**Proposition 4.17.** Let $R$ be a reversible or right duo ring, $M$ a t.u.p. monoid with only one unit and $\omega : R \to \text{End}(R)$ a monoid homomorphism. Let $R$ be a $M$-compatible ring. Then $\text{chn}(R * M) = \{a_1g_1 + a_2g_2 + \cdots + a_ng_n \mid n \geq 1, g_1 = e, a_1 \in \text{chn}(R) \text{ and } a_i \text{ is nilpotent for each } i \neq 1\}.$

Proof. It follows from Theorem 2.5 and Corollary 4.15. \hfill \Box

Let $R$ be a reversible (or right duo) $M$-compatible ring, $M$ a t.u.p. monoid with only one unit and $\omega : M \to \text{End}(R)$ a monoid homomorphism. Then by Corollary 2.6, the skew monoid ring $R * M$ is abelian. Hence every clean element of $R * M$ is also strongly clean.

**Corollary 4.18.** Let $R$ be a reversible or right duo ring and $\sigma$ an endomorphism of $R$. If $R$ is $\sigma$-compatible, then

1. $\text{chn}(R[x;\sigma]) = \text{chn}(R) + \text{nil}(R[x;\sigma])x = \text{chn}(R) + (\text{nil}(R[x;\sigma])x).
2. A non-zero element $f \in R[x_1,\ldots,x_n]$ is clean if and only if the constant term of $f$ is clean and any other coefficients are nilpotent.

Let $R$ be a reversible or right duo ring, $M$ a positively strictly totally ordered monoid or a t.u.p. monoid with only one unit, and $\omega : M \to \text{End}(R)$ a monoid homomorphism. Let $\alpha = 1g \in R * M$. Since $1 \notin \text{nil}(R)$ then $\alpha = 1g \notin \text{chn}(R * M)$. Hence, the skew monoid ring $R * M$ is never clean. As a consequence, for a $\sigma$-compatible ring $R$, the skew polynomial ring $R[x;\sigma]$ is never clean and so the polynomial ring $R[x]$ is never clean [22, Proposition 13].

In the following, we determine nil clean elements of skew monoid ring $R * M$. 

Corollary 4.19. Let $R$ be a semicommutative, $M$ a u.p. monoid and $\omega : M \to \text{End}(R)$ a monoid homomorphism. Let $R$ be a $M$-compatible ring. Then \( \alpha \in R \ast M \) is nil clean if and only if \( \alpha = a_1g_1 + \cdots + a_ng_n \) such that \( g_1 = e, a_1 \in \text{nil} - \text{cln}(R) \) and \( a_i \in \text{nil}(R) \) for all \( i \geq 2 \).

Proof. It follows from Theorem 2.5 and Corollary 2.3. \(\square\)

Let $R$ be a semicommutative and $M$-compatible ring, $M$ a u.p. monoid and $\omega : M \to \text{End}(R)$ a monoid homomorphism. By Corollary 2.6, the skew monoid ring $R \ast M$ is abelian. Hence every nil clean element of $R \ast M$ is also strongly nil clean.

Corollary 4.20. Let $R$ be a semicommutative ring and $\sigma$ an endomorphism of $R$. If $R$ is $\sigma$-compatible, then

1. \( \text{nil-cln}(R[x;\sigma]) = \{a_0 + a_1x + \cdots + a_nx^n \mid n \geq 0, a_0 \in \text{nil-cln}(R) \) and \( a_i \in \text{nil}(R) \) for \( 1 \leq i \leq n \}\).

2. A non-zero element \( f \in R[x_1,\ldots,x_n] \) is nil clean if and only if the constant term of $f$ is nil clean and any other coefficients are nilpotent.

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References

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