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CERTAIN NEW EXTENSION OF HURWITZ-LERCH ZETA FUNCTION

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Abstract. In the present research paper, we introduce a further extension of Hurwitz-Lerch zeta function by using the generalized extended Beta function defined by Parmar et al. [9]. We investigate its integral representations, Mellin transform, generating functions and differential formula. In view of diverse applications of the Hurwitz-Lerch Zeta functions, the results presented here may be potentially useful in some related research areas.

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1. Introduction

The well known Hurwitz-Lerch zeta function is defined by (see [2], [10], [11]):

$$\Phi(z,s,a)=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s}, \tag{1.1}$$

$(a\in\mathbb{C}\setminus\mathbb{Z}_0; s\in\mathbb{C}, \text{ when } \mid z\mid<1; \Re(s)>1, \text{ when } \mid z\mid=1)$.

Goyal and Laddha [4] and Garg et al. [3] introduced to investigate certain interesting extensions of the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ in (1.1) which are defined respectively, by

$$\Phi^*_\mu(z,s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_n z^n}{n!(n+a)^s}, \tag{1.2}$$

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$(\mu \in \mathbb{C}; a \in \mathbb{C} \setminus \mathbb{Z}_0; s \in \mathbb{C}$, when $| z |< 1; \Re(s-\mu) > 1$, when $| z |= 1)$
and

$$\Phi_{\lambda,\mu,\nu}(z,s,a) = \sum_{n=0}^{\infty} \frac{(\lambda)_n(\mu)_n z^n}{(\nu)_n n!(n+a)^s}, \tag{1.3}$$

$(\lambda, \mu \in \mathbb{C}; a \in \mathbb{C} \setminus \mathbb{Z}_0; s \in \mathbb{C}$, when $| z |< 1; \Re(s+\nu-\lambda-\mu) > 1$, when $| z |= 1)$.

The following known integral representations of (1.2) and (1.3) are given, respectively, by

$$\Phi^*_{\mu}(z,s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}e^{-at}}{(1-zt)^{\mu}} dt = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}e^{(a-1)t}}{(1-zt)^{\mu}} dt, \tag{1.4}$$

$(\Re(a) > 0; \Re(s) > 0$ when $| z |\leq 1 (z \neq 1); \Re(s) > 1$ when $z = 1)$
and

$$\Phi_{\lambda,\mu,\nu}(z,s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1}e^{-at} 2F1(\lambda, \mu; \nu; ze^t)dt, \tag{1.5}$$

$(\Re(a) > 0; \Re(s) > 0$ when $| z |\leq 1 (z \neq 1); \Re(s) > 1$ when $z = 1)$.

Very recently, Parmar et al. [9] introduced and investigated the following extended Hurwitz-Lerch zeta function:

$$\Phi^{(\rho,\sigma)}_{\lambda,\mu;\gamma}(z,s,a;p) = \sum_{n=0}^{\infty} \frac{(\lambda)_n B^{(\rho,\sigma)}_p(\mu+n, \gamma-\mu)}{n! B(\mu, \gamma-\mu)} \frac{z^n}{(n+a)^s}, \tag{1.6}$$

$(p \geq 0, \Re(\rho) > 0, \Re(\sigma) > 0; \lambda, \mu \in \mathbb{C}; \gamma, a \in \mathbb{C} \setminus \mathbb{Z}_0; s \in \mathbb{C}$, when $| z |< 1; \Re(s+\gamma-\lambda-\mu) > 1$, when $| z |= 1)$.

where $B^{(\rho,\sigma)}_p(x,y)$ is the extended Beta function defined as follows (see [6]):

$$B^{(\rho,\sigma)}_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} {}_1F_1(\rho;\sigma; \frac{-p}{t(1-t)})dt. \tag{1.7}$$

They also defined the integral representation of (1.2) by

$$\Phi^{(\rho,\sigma)}_{\lambda,\mu;\gamma}(z,s,a;p) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1}e^{-at} F^{(\rho,\sigma)}_p(\lambda, \mu; \gamma; ze^{-t})dt. \tag{1.8}$$

For $\rho = \sigma$, (1.6) reduces to the Hurwitz-Lerch zeta function defined by Parmar and Raina [8], which further for $p = 0$, gives the known extension of (1.1) gives by Garg et al. [3].

Further, Srivastava et al. [13] introduced the following generalizations of the extended Beta and hypergeometric functions which are defined, respectively, by

$$B^{(\rho,\sigma;m,n)}_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} {}_1F_1(\rho;\sigma; \frac{-p}{t^m(1-t)^n})dt, \tag{1.9}$$

$$(\Re(\sigma) \geq 0; \min\{\Re(\rho), \Re(\sigma), \Re(x), \Re(y)\} > 0; \min\{\Re(m), \Re(n)\} > 0)$$

and

$$F_p^{(\rho,\sigma;m,n)}(a,b;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k B_p^{(\rho,\sigma;m,n)}(b+n,c-b)}{B(b,c-b)}\frac{z^k}{k!}, \tag{1.10}$$

$(|\, z \,|< 1; \min\{\Re(\rho),\Re(\sigma),\Re(m),\Re(n)\}>0; \Re(c)>\Re(b)>0; \Re(p)\geq 0)$.

Due to diverse applications of Hurwitz-Lerch zeta functions, several extensions of $\Phi(z,s,a)$ have been introduced and investigated by a number of authors (see, for example [1], [5], [7], [12], [14] etc).

In a sequel of such type of works mentioned above in this paper, we introduce a further extension of $\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma)}(z,s,a;p)$ by using the generalized Beta function defined by Srivastava et al. [13].

2. A new extension of Hurwitz-Lerch Zeta function

In this section, we establish the following new extension of Hurwitz-Lerch Zeta function:

$$\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m,k)}(z,s,a;p)=\sum_{n=0}^{\infty}\frac{(\lambda)_n B_p^{(\rho,\sigma;m,k)}(\mu+n,\gamma-\mu)}{n!B(\mu,\gamma-\mu)}\frac{z^n}{(n+a)^s}, \tag{2.1}$$

$(p\geq 0, \Re(\rho)>0, \Re(\sigma)>0, \Re(m)>0, \Re(k)>0; \lambda,\mu\in\mathbb{C}; \gamma,a\in\mathbb{C}\setminus\mathbb{Z}_0; s\in\mathbb{C}$, when $|\, z\,|<1; \Re(s+\gamma-\lambda-\mu)>1$, when $|\, z \,|=1)$.

Where $B_p^{(\rho,\sigma;m,k)}(x,y)$ is the generalized Beta function, which is defined by (see [13]):

$$B_p^{(\rho,\sigma;m,k)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1(\rho;\sigma;\frac{-p}{t^m(1-t)^k})dt. \tag{2.2}$$

He also given the following extension of Gauass hypergeometric function:

$$F_p^{(\rho,\sigma;m,k)}(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n B_p^{(\rho,\sigma;m,k)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}. \tag{2.3}$$

On substituting $m=k=1$ in (2.1), we get the extended Hurwitz-Lerch Zeta function given by (1.6).

Remark 2.1. The generalized new extension of Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m,k)}(z,s,a;p)$ has the following limiting case.

$$\Phi_{\lambda,\mu;\gamma}^{*(\rho,\sigma;m,k)}(z,s,a;p)=\lim_{|\lambda|\longrightarrow\infty}\left\{\Phi_{\mu;\gamma}^{(\rho,\sigma;m,k)}\left(\frac{z}{\lambda},s,a;p\right)\right\}$$

$$=\sum_{n=0}^{\infty}\frac{B_p^{(\rho,\sigma;m,k)}(\mu+n,\gamma-\mu)}{n!B(\mu,\gamma-\mu)}\frac{z^n}{(n+a)^s}, \tag{2.4}$$

$(p \geq 0, \Re(\rho) > 0, \Re(\sigma) > 0, \Re(m) > 0, \Re(k) > 0; \mu \in \mathbb{C}; \gamma, a \in \mathbb{C} \setminus \mathbb{Z}_0; s \in \mathbb{C}$, when $\mid z \mid < 1; \Re(s + \gamma - \mu) > 1$, when $\mid z \mid = 1)$.

Remark 2.2. On setting $m = k = 1$, $\lambda = \gamma = 1$ in (2.1), we get another known result of Hurwitz-Lerch Zeta function, which is defined by (see [7]):

$$\Phi_{1,\mu;1}^{(\rho,\sigma;1,1)}(z, s, a; p) = \Phi_{\mu}^{*(\rho,\sigma)}(z, s, a; p) = \sum_{n=0}^{\infty} \frac{B_p^{(\rho,\sigma)}(\mu + n, 1 - \mu)}{n! B(\mu, 1 - \mu)} \frac{z^n}{(n + a)^s}, \tag{2.5}$$

$(p \geq 0, \Re(\rho) > 0, \Re(\sigma) > 0; \mu \in \mathbb{C}; \gamma, a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C}$, when $\mid z \mid < 1; \Re(s + 1 - \mu) > 1$, when $\mid z \mid = 1)$.

3. Integral representations of $\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m,k)}(z, s, a; p)$

In this section, we derive the following integral representations of our new generalized Hurwitz-Lerch Zeta function.

Theorem 3.1. *The following integral representation of* $\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m,k)}(z, s, a; p)$ *holds true:*

$$\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m,k)}(z, s, a; p) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-at} F_p^{(\rho,\sigma;m,k)}(\lambda, \mu; \gamma; ze^{-t}) dt. \tag{3.1}$$

($\Re(p) \geq 0, \Re(\rho) > 0, \Re(\sigma) > 0, \Re(m) > 0, \Re(k) > 0; p = 0, \Re(a) > 0; \Re(s) > 0$, *when* $\mid z \mid \leq 1; \Re(s) > 1$, *when* $z = 1$*).*

Proof. We have

$$\frac{1}{(n + a)^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-(n+a)t} dt.$$

By using the above result in (2.1) and then interchanging the order of summation and integration (which is valid under the given condition), we get

$$\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m,k)}(z, s, a; p)$$
$$= \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-at} \left(\sum_{n=0}^{\infty} \frac{(\lambda)_n B_p^{(\rho,\sigma;m,k)}(\mu + \gamma, \gamma - \mu)}{B(\mu, \gamma - \mu)} \frac{(ze^{-t})^n}{n!} \right) dt.$$

In view of definition (2.3), we arrive at the desired result (3.1). □

Theorem 3.2. *The following integral representation of* $\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m,k)}(z, s, a; p)$ *holds true:*

$$\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m,k)}(z, s, a; p) = \frac{1}{\Gamma(\lambda)} \int_0^{\infty} t^{\lambda-1} e^{-t} \Phi_{\mu,\gamma}^{*(\rho,\sigma;m,k)}(zt, s, a; p) dt. \tag{3.2}$$

($\Re(p) \geq 0, \Re(\rho) > 0, \Re(\sigma) > 0, \Re(m) > 0, \Re(k) > 0; p = 0, \Re(\nu) > 0; \Re(a) > 0; \Re(s) > 0$, when $\mid z \mid \leq 1; \Re(s) > 1$, when $z = 1$).

Proof. We have

$$(\lambda)_n = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda+n-1} e^{-t} dt.$$

By using the above result in (2.1) and then interchanging the order of summation and integration (which is valid under the given condition), we get

$$\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m,k)}(z,s,a;p)$$
$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-t} \left(\sum_{n=0}^{\infty} \frac{B_p^{(\rho,\sigma;m,k)}(\mu+n,\gamma-\mu)}{B(\mu,\gamma-\mu)} \frac{(zt)^n}{n!(n+a)^s} \right) dt.$$

Which further on using the definition of (2.4), gives the required result (3.2). □

4. Mellin Transform

The Mellin transform of the function $f(x)$ is given by

$$M\{f(x)\} = \phi(r) = \int_0^\infty x^{r-1} f(x) dx. \tag{4.1}$$

Theorem 4.1. *For the new extended Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)$, we have the following Mellin transform representation:*

$$M\left\{\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m,k)}(z,s,a;p)\right\}$$
$$= \frac{\Gamma^{(\rho,\sigma)}(s)}{B(\mu,\gamma-\mu)} B(m\alpha+\mu; k\alpha+\gamma-\mu) \Phi_{\lambda,m\alpha+\mu;m\alpha+k\alpha+\gamma}(z,s,a;p). \tag{4.2}$$

($\Re(s) > 0, \Re(m\alpha+\mu) > 0, \Re(m\alpha+k\alpha+\gamma) > 0, 0 < \Re(\mu) < \Re(\gamma)$),

where $\Gamma^{(\rho,\sigma)}(s)$ and $\Phi_{\lambda,\mu;\gamma}(z,s,a)$ are extended Gamma and Hurwitz-Lerch Zeta function defined by Parmar [7] and Garg et al. [3,p.313], respectively.

Proof. Using the definition of Mellin transform (4.1) on the L.H.S of (4.2) and then expanding $\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m)}(z,s,a;p)$ with the help of (2.1), we get

$$M\left\{\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m,k)}(z,s,a;p)\right\}$$
$$= \int_0^\infty p^{\alpha-1} \left(\sum_{n=0}^{\infty} \frac{(\lambda)_n B_p^{(\rho,\sigma;m,k)}(\mu+n,\gamma-\mu)}{n!B(\mu,\gamma-\mu)} \frac{z^n}{(n+a)^s} \right) dp.$$

Now changing the order of summation and integration, we get

$$M\left\{\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m,k)}(z,s,a;p)\right\}$$
$$=\sum_{n=0}^{\infty}\frac{(\lambda)_n z^n}{n!(n+a)^s B(\mu,\gamma-\mu)}\int_0^{\infty} p^{\alpha-1}B_p^{(\rho,\sigma;m,k)}(\mu+n,\gamma-\mu)dp$$
$$=\sum_{n=0}^{\infty}\frac{(\lambda)_n z^n}{n!(n+a)^s}\frac{B(m\alpha+\mu+n,\gamma-\mu+\alpha k)}{B(\mu,\gamma-\mu)}\Gamma^{(\rho,\sigma)}(s).$$

Now expanding $B(m\alpha+\mu+n,\gamma-\mu+\alpha k)$ in terms of Gamma function and then by using the result $\Gamma(\lambda+n)=\Gamma(\lambda)(\lambda)_n$,

$$M\left\{\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m,k)}(z,s,a;p)\right\}$$
$$=\frac{\Gamma^{(\rho,\sigma)}(s)\Gamma(m\alpha+\mu)\Gamma(\alpha k+\gamma-\mu)}{B(\mu,\gamma-\mu)\Gamma((m+k)\alpha+\gamma)}\sum_{n=0}^{\infty}\frac{(\lambda)_n(m\alpha+\mu)_n}{n!(m\alpha+k\alpha+\gamma)_n}\frac{z^n}{(n+a)^s}.$$

Finally using the definition of Hurwitz-Lerch Zeta function given in [6,p.313], we are led to the desired result.

□

5. Generating relations

Theorem 5.1. *For $p\geq 0$, $\lambda\in\mathbb{C}$ and $\mid t\mid<1$, the following generating function holds true:*

$$\sum_{n=0}^{\infty}(\lambda)_n\Phi_{\lambda+n,\mu;\gamma}^{(\rho,\sigma;m,k)}(z,s,a;p)\frac{t^n}{n!}=(1-t)^{-\lambda}\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m,k)}\left(\frac{z}{1-t},s,a;p\right). \quad (5.1)$$

Proof. For convenience, let the left hand side of assertion (5.1) of Theorem 5.1 be denoted by L_1. Then by substituting the series expression from (2.1) into L_1, we find that

$$L_1=\sum_{n=0}^{\infty}(\lambda)_n\left\{\sum_{r=0}^{\infty}\frac{(\lambda+n)_r B_p^{(\rho,\sigma;m,k)}(\mu+k,\gamma-\mu)}{B(\mu,\gamma-\mu)}\frac{z^r}{r!(r+a)^s}\right\}\frac{t^n}{n!}, \quad (5.2)$$

which upon changing the order of summation and after a little simplification, gives

$$L_1=\sum_{r=0}^{\infty}\frac{(\lambda)_r B_p^{(\rho,\sigma;m,k)}(\mu+k,\gamma-\mu)}{B(\mu,\gamma-\mu)}\left\{\sum_{n=0}^{\infty}(\lambda+k)_n\frac{t^n}{n!}\right\}. \quad (5.3)$$

Now applying the following binomial expansion

$$(1-\lambda)^{-(\lambda+k)}=\sum_{n=0}^{\infty}(\lambda+k)_n\frac{t^n}{n!},(\mid t\mid<1),$$

for evaluating the inner sum in (5.3) and then by using (2.1), we get the desired assertion (5.1) of Theorem 5.1.

□

Theorem 5.2. *For* $p \geq 0$, $\lambda \in \mathbb{C}$ *and* $\mid t \mid < \mid a \mid ; s \neq 1$, *the following generating function holds true:*

$$\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m,k)}(z,s,a;p) = \sum_{n=0}^{\infty} \frac{(s)_n}{n!} \Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m,k)}(z,s+n,a;p). \tag{5.4}$$

Proof. Let the left hand side of (5.4) denoted by L_2. Then by using (2.1) into L_2, we get

$$\begin{aligned} L_2 &= \sum_{l=0}^{\infty} \frac{(\lambda)_l B_p^{(\rho,\sigma;m,k)}(\mu+l,\gamma-\mu)}{B(\mu,\gamma-\mu)} \frac{z^l}{l!(l+a-t)^s} \\ &= \sum_{l=0}^{\infty} \frac{(\lambda)_l B_p^{(\rho,\sigma;m,k)}(\mu+l,\gamma-\mu)}{B(\mu,\gamma-\mu)} \frac{z^l}{l!(l+a)^s}(1-\frac{t}{l+a})^{-s} \\ &= \sum_{l=0}^{\infty} \frac{(\lambda)_l B_p^{(\rho,\sigma;m,k)}(\mu+l,\gamma-\mu)}{B(\mu,\gamma-\mu)} \frac{z^l}{l!(l+a)^s} \left\{ \sum_{n=0}^{\infty} \frac{(s)_n}{n!} \left(\frac{t}{l+a} \right)^n \right\} \\ &= \sum_{n=0}^{\infty} \frac{(s)_n}{n!} \left(\sum_{l=0}^{\infty} \frac{(\lambda)_l B_p^{(\rho,\sigma;m,k)}(\mu+l,\gamma-\mu)}{B(\mu,\gamma-\mu)} \frac{z^l}{l!(l+a)^{s+n}} \right) t^n. \end{aligned}$$

Finally, by making use of (2.1), we get the desired assertion (5.4) of Theorem 5.2.

□

Remark 5.1. On setting $k = m = 1$, the generating function (5.1) and (5.4) asserted by Theorem 5.1 and Theorem 5.2, respectively, were derived earlier by Parmar et al. [9].

6. Derivation of $\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m,k)}(z,s,a;p)$

For the extended Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m,k)}(z,s,a;p)$, we have a differential formula given in Theorem 6.1.

Theorem 6.1. *The following differential formula holds true:*

$$\frac{d^r}{dz^r}\left[\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m,k)}(z,s,a;p)\right] = \frac{(\mu)_r(\lambda)_r}{(\gamma)_r} \Phi_{\lambda+r,\mu+r;\gamma+r}^{(\rho,\sigma;m,k)}(z,s,a+r;p), \tag{6.1}$$

where $r \in \mathbb{N} = \{1,2,3,\cdots\}$.

Proof. Taking the derivative of $\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m,k)}(z,s,a;p)$ with respect to z, we get

$$\begin{aligned}
&\frac{d}{dz}\left[\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m,k)}(z,s,a;p)\right]\\
&=\frac{d}{dz}\left[\sum_{n=0}^{\infty}\frac{(\lambda)_n B_p^{(\rho,\sigma;m,k)}(\mu+n,\gamma-\mu)}{n!B(\mu,\gamma-\mu)}\frac{z^n}{(n+a)^s}\right]\\
&=\sum_{n=1}^{\infty}\frac{(\lambda)_n B_p^{(\rho,\sigma;m,k)}(\mu+n,\gamma-\mu)}{(n-1)!B(\mu,\gamma-\mu)}\frac{z^{n-1}}{(n+a)^s}.
\end{aligned}$$

Replacing n by $n+1$, we get

$$\begin{aligned}
&\frac{d}{dz}\left[\Phi_{\lambda,\mu;\gamma}^{(\rho,\sigma;m,k)}(z,s,a;p)\right]\\
&=\frac{\mu\lambda}{\gamma}\sum_{n=0}^{\infty}\frac{(\lambda+1)_n B_p^{(\rho,\sigma;m,k)}(\mu+n+1,\gamma-\mu)}{n!B(\mu+1,\gamma-\mu)}\frac{z^n}{(n+1+a)^s}\\
&=\frac{\mu\lambda}{\gamma}\Phi_{\lambda+1,\mu+1;\gamma+1}^{(\rho,\sigma;m,k)}(z,s,a+1;p).
\end{aligned}$$

Recursive application of this procedure yields us the desired result (6.1). □

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