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ON THE EXTREMAL TYPE I BINARY SELF-DUAL CODES WITH NEAR-MINIMAL SHADOW

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ABSTRACT. In this paper, we define near-minimal shadow and study the existence problem of extremal Type I binary self-dual codes with near-minimal shadow. We prove that there is no such codes of length $n = 24m + 2(m \geq 0)$, $n = 24m + 4(m \geq 9)$, $n = 24m + 6(m \geq 21)$, and $n = 24m + 10(m \geq 87)$.

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1. Introduction

A binary linear code C is a subspace of a vector space $GF(2)^n$ and the vectors in C are called codewords. The weight of a codeword $u = (u_1, u_2, \ldots, u_n)$ in $GF(2)^n$ is the number of nonzero u_j. The minimum weight of C is the smallest nonzero weight of any codeword in C. If the dimension of C is k and the minimum weight in C is d, we say C is an $[n, k, d]$ code.

The scalar product in $GF(2)^n$ is defined by

$$(u, v) = \sum_{j=1}^{n} u_j v_j \ , \tag{1}$$

where the sum is evaluated in $GF(2)$. The dual code of a binary linear code C is defined by

$$C^{\perp} = \{v \in GF(2)^n : (v, c) = 0 \text{ for all } c \in C\}. \tag{2}$$

If $C \subseteq C^{\perp}$, we say C is self-orthogonal and if $C = C^{\perp}$, we say C is self-dual.

A binary code is even if all its codewords have even weight. Clearly self-dual binary codes are even. In addition, some of these codes have all codewords of weight divisible by 4. A self-dual code with all codewords of weight divisible

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by 4 is called doubly-even or Type II; a self-dual code with some codeword of weight not divisible by 4 is called singly-even or Type I. Bounds on the minimum distance of binary self-dual codes were given in [10].

Theorem 1.1. *([10]) Let C be an $[n, n/2, d]$ binary self-dual code. Then $d \leq 4[n/24]+4$ if $n \not\equiv 22 (\text{mod } 24)$. If $n \equiv 22 (\text{mod } 24)$, then $d \leq 4[n/24]+6$, and if equality holds, C can be obtained by shortening a Type II code of length $n+2$. If $24|n$ and $d = 4[n/24]+4$, then C is Type II.*

A code meeting the bound of Theorem 1.1, i.e., equality holds in the bound, is called *extremal*. From Theorem 1.1, note that there is no extremal Type I code of length $n = 24m$ ($m \geq 1$). The proof of Theorem 1.1 when the code is Type I used the concept of the shadow. In [5], the shadow code of a code was introduced. The shadow code of a self-dual code C is defined as follows. Let $C^{(0)}$ be the subset of C consisting of all codewords whose weights are multiple of 4, and let $C^{(2)} = C \backslash C^{(0)}$. The shadow code of C is defined by

$$S = S(C) = \{u \in GF(2)^n : (u, v) = 0 \text{ for all } v \in C^{(0)}, (u, v) = 1 \text{ for all } v \in C^{(2)}\}. \tag{3}$$

Elkies studied the minimum weight s of the shadow code S [11]. He achieved the following things. First, $s \leq \frac{n}{2}$. Second, $s = \frac{n}{2}$ if and only if $C = \bigoplus_{i=1}^{n/2} C_2$, where C_2 is the $[2, 1, 2]$ binary code. Third, for $s = n/2 - 4$, he classified the corresponding codes and proved that $n \leq 22$.

Bachoc and Gaborit [1] studied the minimum weight d of C and the minimum weight s of S simultaneously, and they showed that $2d + s \leq \frac{n}{2} + 4$, unless $n \equiv 22$ (mod 24) and $d = 4[n/24]+6$, in which $2d + s = \frac{n}{2} + 8$. If equality holds, i.e., $2d + s = \frac{n}{2} + 4$(or $2d + s = \frac{n}{2} + 8$), then the codes are called s-extremal. Elkies' study corresponds to s-extremal codes with $d = 2$ and $d = 4$. Bachoc and Gaborit also studied s-extremal codes with $d = 6$.

Elkies, Bachoc, and Gaborit studied, in some sense, large value of minimum weight s of S. On the other hand, Bouyuklieva and Willems made a research for the smallest value s of S [4].

Definition 1.2. Let C be a Type I binary self-dual code of length $n = 24m + 8\ell + 2r$ with $\ell = 0, 1, 2$ and $r = 0, 1, 2, 3$. Then C is a code with *minimal shadow* if:

(1) $d(S) = r$ for $r > 0$; and
(2) $d(S) = 4$ for $r = 0$,

where $d(S)$ is the minimum weight of S.

They proved nonexistence of extremal self-dual codes with minimal shadow. More specific, they proved that extremal Type I binary self-dual codes of lengths $n = 24m+2, 24m+4, 24m+6, 24m+10$ and $24m+22$ with minimal shadow do not exist. They also proved that there are no extremal Type I binary self-dual codes of length n with minimal shadow if $n = 24m+8 (m \geq 53)$, $n =$

$24m + 12(m \geq 142)$, $n = 24m + 14(m \geq 146)$, $n = 24m + 16(m \geq 164)$, and $n = 24m + 18(m \geq 157)$.

Bouyuklieva, Harada, and Munemasa studied near-extremal binary self-dual codes with minimal shadow [3].

Definition 1.3. Let C be an $[n, n/2, d]$ binary self-dual code. Then C is a *near-extremal* code if:

(1) $d = 4[n/24] + 2$ for $n \not\equiv 22 \pmod{24}$; and
(2) $d = 4[n/24] + 4$ for $n \equiv 22 \pmod{24}$.

They proved that there are no near-extremal Type I binary self-dual codes of length n with minimal shadow if $n = 24m+2(m \geq 155)$, $n = 24m+4(m \geq 156)$, and $n = 24m+10(m \geq 160)$. Recently, the author [7] also proved that there are no near-extremal Type I binary self-dual codes of length n with minimal shadow if $n = 24m + 2(m \geq 323)$.

In this paper, we study near-minimal shadow. In the following, we give the definition of a code with near-minimal shadow.

Definition 1.4. Let C be a Type I binary self-dual code of length $n = 24m + 8\ell + 2r$ with $\ell = 0, 1, 2$ and $r = 0, 1, 2, 3$. Then C is a code with *near-minimal shadow* if:

(1) $d(S) = 4 + r$ for $r > 0$; and
(2) $d(S) = 8$ for $r = 0$,

where $d(S)$ is the minimum weight of S.

The main result of this paper is the following theorem.

Theorem 1.5. *There are no extremal Type I binary self-dual codes of length n with near-minimal shadow if*

(1) $n = 24m + 2$*;*
(2) $n = 24m + 4$ *and* $m \geq 9$*;*
(3) $n = 24m + 6$ *and* $m \geq 21$*;*
(4) $n = 24m + 10$ *and* $m \geq 87$.

We summarize the results so far in Table 1. In the table, we give the results of non-existence of extremal(or near-extremal) binary self-dual codes with minimal(or near-minimal) shadow of length $n = 24m + p, (0 \leq p \leq 22)$. The first row and the fifth row of the table represent the value p, and the first column of the table represents extremal(or near-extremal) w.r.t. the minimum weight d of C and minimal(or near-minimal) w.r.t. the minimum weight s of S. More specifically, the pair (ext, min) corresponds to the case d is extremal and s is minimal, the pair (n-ext, min) corresponds to the case d is near-extremal and s is minimal, and the pair (ext, n-min) corresponds to the case d is extremal and s is near-minimal. In the table, 'x' represents the non-existence of the corresponding codes. '$\geq$ number' represents the non-existence of the corresponding codes if $m \geq$ number. '*' represents that there is no Type I extremal codes of length $n = 24m$.

Table 1. Non-existence of extremal(or near-extremal) binary self-dual codes with minimal(or near-minimal) shadow of length $n = 24m + p$

$(d,s)\backslash p$	0	2	4	6	8	10
(ext, min)	*	x	x	x	≥ 53	x
(n-ext, min)	≥ 323	≥ 155	≥ 156			≥ 160
(ext, n-min)	*	x	≥ 9	≥ 21		≥ 87
$(d,s)\backslash p$	12	14	16	18	20	22
(ext, min)	≥ 142	≥ 146	≥ 164	≥ 157		x
(n-ext, min)						
(ext, n-min)						

This paper is organized by the following. In section 2, we give the proof of Theorem 1.5. In section 3, we give the summary of this paper.

2. Extremal Type I binary self-dual codes with near-minimal shadow

In this section, we give the proof of Theorem 1.5. The weight enumerator of a code is given by

$$W_C(x,y) = \sum_{i=0}^{n} A_i x^{n-i} y^i, \tag{4}$$

where there are A_i codewords of weight i in C. The following lemma is needed in this paper.

Lemma 2.1. [5] *Let C be a Type I binary self-dual code of length n and minimum weight d. Let $S(y) = \sum_{i=0}^{n} b_i y^i$ be the weight enumerator of $S(C)$. Then*

(1) $b_0 = 0$*; and*
(2) $b_i \leq 1$ *for* $i < d/2$.

Let C be a Type I binary self-dual code of length $n = 24m + 8\ell + 2r$ where $\ell = 0, 1, 2$ and $r = 0, 1, 2, 3$. By Gleason's theorem [2, 6, 9], we can write the weight enumerator of C.

$$W_C(x,y) = \sum_{i=0}^{[n/8]} c_i (x^2+y^2)^{n/2-4i} \{x^2y^2(x^2-y^2)^2\}^i, \tag{5}$$

for suitable constants c_i. Using the shadow code theory [5], we can write the weight enumerator of shadow code $S(C)$,

$$W_S(x,y) = \sum_{i=0}^{[n/8]} (-1)^i 2^{n/2-6i} c_i (xy)^{n/2-4i} (x^4-y^4)^{2i}. \tag{6}$$

We rewrite Eqn. (5) and Eqn. (6) as the following form

$$W_C(1,y) = \sum_{j=0}^{12m+4\ell+r} a_j y^{2j} = \sum_{i=0}^{3m+\ell} c_i (1+y^2)^{12m+4\ell+r-4i} \{y^2(1-y^2)^2\}^i, \quad (7)$$

$$W_S(1,y) = \sum_{j=0}^{6m+2\ell} b_j y^{4j+r} = \sum_{i=0}^{3m+\ell} (-1)^i c_i\, 2^{12m+4\ell+r-6i} y^{12m+4\ell+r-4i}(1-y^4)^{2i}. \quad (8)$$

Note that all a_j and b_j must be nonnegative integers. One can write c_i as a linear combination of the a_j for $0 \le j \le i$, and one can write c_i as a linear combination of b_j for $0 \le j \le 3m+\ell-i$ as the following form

$$c_i = \sum_{j=0}^{i} \alpha_{ij} a_j = \sum_{j=0}^{3m+\ell-i} \beta_{ij} b_j, \quad (9)$$

for suitable constants α_{ij} and β_{ij}.

In our computation, we need to calculate α_{i0} and β_{ij}. The following formula can be found in [10]. For $i > 0$,

$$\alpha_{i0} = -\frac{n}{2i}\left[\text{coeff. of } y^{i-1} \text{ in } (1+y)^{-(n/2)-1+4i}(1-y)^{-2i}\right] \quad (10)$$

and

$$\beta_{ij} = (-1)^i 2^{-\frac{n}{2}+6i} \frac{k-j}{i} \binom{k+i-j-1}{k-i-j}, \quad (11)$$

where $k = 3m+\ell$. Note that $a_0 = c_0 = \alpha_{00} = 1$.

In [3], there is a calculation formula for α_{i0}. We extend the formula in the following lemma.

Lemma 2.2. *Let $1 \le i \le 3m+\ell$. Then we have*

$$\alpha_{i,0} = \begin{cases} -\frac{n}{2i} \displaystyle\sum_{t=0,\, t+i \text{ is odd}}^{\frac{n}{2}+1-6i} (-1)^t \binom{\frac{n}{2}+1-6i}{t} \binom{\frac{n-7i-t-1}{2}}{\frac{i-t-1}{2}}, & \text{if } \frac{n}{2}+1-6i \ge 0; \\ -\frac{n}{2i} \displaystyle\sum_{t=0}^{[\frac{i-1}{2}]} \binom{\frac{n}{2}-4i+t}{t} \binom{-\frac{n}{2}+7i-2t-3}{i-2t-1}, & \text{otherwise.} \end{cases} \quad (12)$$

Proof. If $\frac{n}{2}+1-6i \ge 0$, then the proof is given in [3]. Suppose that $\frac{n}{2}+1-6i < 0$. From Eqn. (10), we have

$$\alpha_{i0} = -\frac{n}{2i}\left[\text{coeff. of } y^{i-1} \text{ in } (1+y)^{-(n/2)-1+4i}(1-y)^{-2i}\right]. \quad (13)$$

And

$$(1+y)^{-(n/2)-1+4i}(1-y)^{-2i} = (1-y^2)^{-n/2-1+4i}(1-y)^{n/2+1-6i} \quad (14)$$

$$= \left[\sum_{0\le t}\binom{\frac{n}{2}+1-4i+t-1}{t}y^{2t}\right]\times\left[\sum_{0\le j}\binom{-\frac{n}{2}-1+6i+j-1}{j}y^{j}\right] \quad (15)$$

$$= \sum_{0\le t,j}\binom{\frac{n}{2}+1-4i+t-1}{t}\binom{-\frac{n}{2}-1+6i+j-1}{j}y^{2t+j}. \quad (16)$$

Let $2t+j=i-1$. Then $j=i-1-2t$. From the above, we have the following result.

$$\alpha_{i0} = -\frac{n}{2i}\sum_{0\le t\le[\frac{i-1}{2}]}\binom{\frac{n}{2}+1-4i+t-1}{t}\binom{-\frac{n}{2}-1+6i+i-1-2t-1}{i-1-2t}$$

$$= -\frac{n}{2i}\sum_{0\le t\le[\frac{i-1}{2}]}\binom{\frac{n}{2}-4i+t}{t}\binom{-\frac{n}{2}+7i-2t-3}{i-2t-1}. \quad (17)$$

□

Let C be an extremal Type I binary self-dual code with near-minimal shadow of length $n=24m+8\ell+2r$. Since C is extremal, we have $a_0=1, a_1=a_2=\cdots=a_{2m+1}=0$.

2.1. The case $n=24m+2$. Suppose that $r=1$. By Lemma 2.1, we have $b_0=0$, $b_1=1$ if $m\ge 2$. Also we have $b_2=b_3=\cdots=b_{m-1}=0$. Otherwise, S would contain a vector v of weight less than or equal to $4m-4+1$, and if $u\in S$ is a vector of weight 5, then $u+v\in C$ with $\mathrm{wt}(u+v)\le 4m+2$, a contradiction to the minimum distance of C.

Using Eqn. (9) and the above discussion, we have the following.

$$c_i=\sum_{j=0}^{i}\alpha_{ij}a_j=\alpha_{i0}(0\le i\le 2m+1) \quad (18)$$

and

$$c_i=\sum_{j=0}^{3m+\ell-i}\beta_{ij}b_j=\beta_{i1}+\sum_{j=2}^{3m+\ell-i}\beta_{ij}b_j=\beta_{i1}(2m+\ell+1\le i\le 3m+\ell-1). \quad (19)$$

Note that $c_{3m+\ell}=0$.

Now we prove the first part of Theorem 1.5. Suppose that $\ell=0$. Then $n=24m+2$. If $m=0,1$, then the code length $n=2,26$. For this code length, there is no extremal code [8]. Now suppose that $m\ge 2$. From Eqn. (18) and Eqn. (19), we have

$$\alpha_{2m+1,0}=\beta_{2m+1,1}. \quad (20)$$

From Eqn. (10) and Eqn. (11), we have

$$\alpha_{2m+1,0}=-\frac{(12m+1)(56m+4)}{(2m+1)(m-1)}\binom{5m-1}{m-2} \quad (21)$$

and

$$\beta_{2m+1,1} = -2^5 \times \frac{3m-1}{2m+1}\binom{5m-1}{m-2}. \tag{22}$$

From Eqn. (20), (21), and (22), we have the following equation:

$$144m^2 + 58m - 7 = 0. \tag{23}$$

The equation has no integer solution. Therefore the corresponding code does not exist. This completes the first part of Theorem 1.5.

2.2. The case n = 24m+10. Now we prove the fourth part of Theorem 1.5. Suppose that $\ell = 1$. Then $n = 24m + 10$. If $m = 0, 1$, then the code length $n = 10, 34$. For this code length, there is no extremal code [8]. Now suppose that $m \geq 2$. From Eqn. (18) and Eqn. (19), we have

$$c_{2m+1} = \alpha_{2m+1,0} = \beta_{2m+1,1} + \beta_{2m+1,m} b_m. \tag{24}$$

Therefore, we get:

$$b_m = \frac{\alpha_{2m+1,0} - \beta_{2m+1,1}}{\beta_{2m+1,m}}. \tag{25}$$

From Eqn. (10) and Eqn. (11) we have

$$\alpha_{2m+1,0} = -\frac{12m+5}{2m+1}\binom{5m+1}{m} \tag{26}$$

and

$$\beta_{2m+1,m} = -2,\ \beta_{2m+1,1} = -2 \times \frac{3m}{2m+1}\binom{5m}{m-1}. \tag{27}$$

Therefore, we get:

$$b_m = \frac{27m+5}{2 \times (4m+1)}\binom{5m}{m}. \tag{28}$$

From Eqn. (18) and Eqn. (19), we have

$$c_{2m} = \alpha_{2m,0} = \beta_{2m,1} + \beta_{2m,m} b_m + \beta_{2m,m+1} b_{m+1}. \tag{29}$$

Therefore, we get:

$$b_{m+1} = \frac{\alpha_{2m,0} - \beta_{2m,1} - \beta_{2m,m} b_m}{\beta_{2m,m+1}}. \tag{30}$$

From Eqn. (10) and Eqn. (11), we have

$$\alpha_{2m,0} = \frac{4(12m+5)(5m+1)(5m+2)(32m^2+19m+3)}{(4m+1)(4m+2)(4m+3)(4m+4)(4m+5)}\binom{5m}{m} \tag{31}$$

and

$$\beta_{2m,m+1} = 2^{-5},\ \beta_{2m,1} = \frac{3}{2^6}\binom{5m-1}{m} = \frac{3}{80}\binom{5m}{m},\ \beta_{2m,m} = \frac{2m+1}{2^4}. \tag{32}$$

Therefore, we get:

$$b_{m+1} = -\frac{8(5m+1)f(m)}{5(4m+1)(4m+2)(4m+3)(4m+4)(4m+5)}\binom{5m}{m}, \tag{33}$$

where

$$f(m) = 1728m^5 - 146560m^4 - 205500m^3 - 105920m^2 - 23763m - 1935. \quad (34)$$

We can see that $f(m) > 0$ if $m \geq 87$. Therefore, if $m \geq 87$, then $b_{m+1} < 0$. This is a contradiction. This completes the fourth part of Theorem 1.5.

Remark 2.1. We made computation for $n = 24m + 10$, and found that some of coefficients of $W_S(1, y)$ are not integer if $m =$ 2, 4, 6, 8, 12, 16, 18, 24, 32, 34, 36, 38, 48, 50, 64, 66, 68, 70, 72, 76. Therefore for the corresponding code length, there is no extremal code with near-minimal shadow.

2.3. The case n = 24m+4. Now we prove the second part of Theorem 1.5. Suppose that $r = 2$. By Lemma 2.1, we have $b_0 = 0$, $b_1 = 1$ if $m \geq 3$. Also we have $b_2 = b_3 = \cdots = b_{m-2} = 0$. Otherwise, S would contain a vector v of weight less than or equal to $4m - 8 + 2$, and if $u \in S$ is a vector of weight 6, then $u + v \in C$ with $\mathrm{wt}(u + v) \leq 4m$, a contradiction to the minimum distance of C.

Using Eqn. (9) and the above discussion, we have the following.

$$c_i = \sum_{j=0}^{i} \alpha_{ij} a_j = \alpha_{i0} (0 \leq i \leq 2m + 1) \quad (35)$$

and

$$c_i = \sum_{j=0}^{3m+\ell-i} \beta_{ij} b_j = \beta_{i1} + \sum_{j=2}^{3m+\ell-i} \beta_{ij} b_j = \beta_{i1} (2m + \ell + 2 \leq i \leq 3m + \ell - 1). \quad (36)$$

Note that $c_{3m+\ell} = 0$.

Suppose that $\ell = 0$. Then $n = 24m + 4$. If $m = 0, 1, 2$, then the code length $n = 4, 28, 52$. For this code length, there is no extremal code [8]. Now suppose that $m \geq 3$. From Eqn. (35) and Eqn. (36), we have

$$c_{2m+1} = \alpha_{2m+1,0} = \beta_{2m+1,1} + \beta_{2m+1,m-1} b_{m-1}. \quad (37)$$

Therefore, we get:

$$b_{m-1} = \frac{\alpha_{2m+1,0} - \beta_{2m+1,1}}{\beta_{2m+1,m-1}}. \quad (38)$$

From Eqn. (10) and Eqn. (11), we have

$$\alpha_{2m+1,0} = -\frac{2(6m + 1)(8m + 1)}{m(2m + 1)} \binom{5m}{m - 1} \quad (39)$$

and

$$\beta_{2m+1,m-1} = -2^4, \ \beta_{2m+1,1} = -2^4 \times \frac{3m - 1}{2m + 1} \binom{5m - 1}{m - 2}. \quad (40)$$

Therefore, we get:

$$b_{m-1} = \frac{(6m + 1)(8m + 1)}{8m(2m + 1)} \binom{5m}{m - 1} - \frac{3m - 1}{2m + 1} \binom{5m - 1}{m - 2}. \quad (41)$$

From Eqn. (35) and Eqn. (36), we have

$$c_{2m} = \alpha_{2m,0} = \beta_{2m,1} + \beta_{2m,m-1}b_{m-1} + \beta_{2m,m}b_m. \tag{42}$$

Therefore, we get:

$$b_m = \frac{\alpha_{2m,0} - \beta_{2m,1} - \beta_{2m,m-1}b_{m-1}}{\beta_{2m,m}}. \tag{43}$$

From Eqn. (10) and Eqn. (11), we have

$$\alpha_{2m,0} = \frac{(6m+1)(8m+1)}{m(2m+1)}\binom{5m}{m-1} \tag{44}$$

and

$$\beta_{2m,m} = \frac{1}{4},\ \beta_{2m,1} = \frac{3m-1}{8m}\binom{5m-2}{m-1},\ \beta_{2m,m-1} = \frac{2m+1}{2}. \tag{45}$$

Therefore, we get:

$$b_m = -\frac{15(5m-2)!f(m)}{4(2m+1)(4m+1)!(m-1)!}, \tag{46}$$

where

$$f(m) = 144m^4 - 1156m^3 - 108m^2 + 39m + 5. \tag{47}$$

We can see that $f(m) > 0$ if $m \geq 9$. Therefore, if $m \geq 9$, then $b_m < 0$. This is a contradiction.

Remark 2.2. We made computation for $n = 24m + 4$, and found that some of coefficients of $W_S(1, y)$ are not integer if $m = 3, 4, 5, 6, 8$. Therefore for the corresponding code length, there is no extremal code with near-minimal shadow. The only remaining case is $m = 7$.

2.4. The case n = 24m+6. Now we prove the third part of Theorem 1.5. Suppose that $r = 3$ and $n \neq 24m + 22$. Then $b_0 = 0$, $b_1 = 1$ if $m \geq 3$. Also we have $b_2 = b_3 = \cdots = b_{m-2} = 0$. Otherwise, S would contain a vector v of weight less than or equal to $4m-8+3$, and if $u \in S$ is a vector of weight 7, then $u+v \in C$ with $\mathrm{wt}(u+v) \leq 4m+2$, a contradiction to the minimum distance of C.

Using Eqn. (9) and the above discussion, we have the following.

$$c_i = \sum_{j=0}^{i} \alpha_{ij}a_j = \alpha_{i0}(0 \leq i \leq 2m+1) \tag{48}$$

and

$$c_i = \sum_{j=0}^{3m+\ell-i} \beta_{ij}b_j = \beta_{i1} + \sum_{j=2}^{3m+\ell-i} \beta_{ij}b_j = \beta_{i1}(2m+\ell+2 \leq i \leq 3m+\ell-1). \tag{49}$$

Note that $c_{3m+\ell} = 0$.

Suppose that $\ell = 0$. Then $n = 24m + 6$. If $m = 0, 1, 2$, then the code length $n = 6, 30, 54$. For this code length, there is no extremal code [8]. Now suppose that $m \geq 3$. From Eqn. (48) and Eqn. (49), we have

$$c_{2m+1} = \alpha_{2m+1,0} = \beta_{2m+1,1} + \beta_{2m+1,m-1} b_{m-1}. \tag{50}$$

Therefore, we get:

$$b_{m-1} = \frac{\alpha_{2m+1,0} - \beta_{2m+1,1}}{\beta_{2m+1,m-1}}. \tag{51}$$

From Eqn. (10) and Eqn. (11), we have

$$\alpha_{2m+1,0} = -\frac{3(4m+1)(6m+1)}{m(2m+1)} \binom{5m}{m-1} \tag{52}$$

and

$$\beta_{2m+1,m-1} = -2^3, \ \beta_{2m+1,1} = -2^3 \times \frac{3m-1}{2m+1} \binom{5m-1}{m-2}. \tag{53}$$

Therefore, we get:

$$b_{m-1} = \frac{3(4m+1)(6m+1)}{8m(2m+1)} \binom{5m}{m-1} - \frac{3m-1}{2m+1} \binom{5m-1}{m-2}. \tag{54}$$

From Eqn. (48) and Eqn. (49), we have

$$c_{2m} = \alpha_{2m,0} = \beta_{2m,1} + \beta_{2m,m-1} b_{m-1} + \beta_{2m,m} b_m. \tag{55}$$

Therefore, we get:

$$b_m = \frac{\alpha_{2m,0} - \beta_{2m,1} - \beta_{2m,m-1} b_{m-1}}{\beta_{2m,m}}. \tag{56}$$

From Eqn. (10) and Eqn. (11), we have

$$\alpha_{2m,0} = \frac{24m+6}{m} \left\{ \binom{5m+2}{m-1} + \binom{5m+1}{m-2} \right\} \tag{57}$$

and

$$\beta_{2m,m} = \frac{1}{8}, \ \beta_{2m,1} = \frac{3m-1}{16m} \binom{5m-2}{m-1}, \ \beta_{2m,m-1} = \frac{2m+1}{4}. \tag{58}$$

Therefore, we get:

$$b_m = -\frac{5(5m-2)! f(m)}{2(4m+3)!(m-1)!}, \tag{59}$$

where

$$f(m) = 2688m^5 - 53168m^4 - 21900m^3 - 28m^2 + 837m + 87. \tag{60}$$

We can see that $f(m) > 0$ if $m \geq 21$. Therefore, if $m \geq 21$, then $b_m < 0$. This is a contradiction.

Remark 2.3. We made computation for $n = 24m + 6$, and found that some of coefficients of $W_S(1, y)$ are not integer if $3 \leq m \leq 20$ except $m = 7, 13, 14, 15$. Therefore for the corresponding code length, there is no extremal code with near-minimal shadow. The only remaining cases are $m = 7, 13, 14, 15$.

3. Summary

In this paper, we gave the definition of near-minimal shadow and proved that there is no extremal Type I binary self-dual codes with near-minimal shadow of length $n = 24m + 2 (m \geq 0)$, $n = 24m + 4 (m \geq 9)$, $n = 24m + 6 (m \geq 21)$, and $n = 24m + 10 (m \geq 87)$. We have also considered *near-extremal* Type I binary self-dual codes with *near-minimal* shadow. But we can not obtain the similar results. In the future work, it is worth while to improve Table 1.

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