STRUCTURE OF 3-PRIME NEAR-RINGS SATISFYING SOME IDENTITIES

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Abstract. In this paper, we investigate commutativity of 3-prime near-rings \( N \) in which \((1, \alpha)\)-derivations satisfy certain algebraic identities. Some well-known results characterizing commutativity of 3-prime near-rings have been generalized. Furthermore, we give some examples show that the restriction imposed on the hypothesis is not superfluous.

1. Introduction

In the present paper, \( N \) will denote a left near-ring with center \( Z(N) \). A near-ring \( N \) is called zero-symmetric if \( 0x = 0 \) for all \( x \in N \) (recall that left distributivity yields \( x0 = 0 \)). \( N \) is 3-prime, that is, for \( a, b \in N \), \(aNb = \{0\} \) implies \( a = 0 \) or \( b = 0 \). A non empty subset \( U \) of \( N \) is said to be a semigroup left (resp. right) ideal of \( N \) if \( NU \subseteq U \) (resp. \( UN \subseteq U \)) and if \( U \) is both a semigroup left ideal and a semigroup right ideal, it is called a semigroup ideal of \( N \). As usual for all \( x,y \) in \( N \), the symbol \([x,y]\) stands for Lie product (commutator) \( xy - yx \) and \( x \circ y \) stands for Jordan product (anticommutator) \( xy + yx \). We note that for a left near-ring, \( -(x+y) = -y - x \) and \( -xy = x(-y) \).

For terminologies concerning near-rings we refer to G. Pilz [9].

An additive mapping \( d : N \to N \) is said to be a derivation if \( d(xy) = xd(y) + d(x)y \) for all \( x,y \in N \), or equivalently, as noted in [11], that \( d(xy) = d(x)y + xd(y) \) for all \( x,y \in N \). An additive mapping \( d : N \to N \) is called a semiderivation if there exists a map \( g : N \to N \) such that \( d(xy) = d(x)g(y) + xd(y) = d(x)y + g(x)d(y) \) and \( d(g(x)) = g(d(x)) \) hold for all \( x,y \in N \). An additive mapping \( d : N \to N \) is called a two sided \( \alpha \)-derivation if there exists a map \( \alpha : N \to N \) such that \( d(xy) = d(x)y + \alpha(x)d(y) \) and \( d(xy) = d(x)\alpha(y) + xd(y) \) hold for all \( x,y \in N \). An additive mapping \( d : N \to N \) is called \((1, \alpha)\)-derivation if there exists a map \( \alpha : N \to N \) such that \( d(xy) = d(x)y + \alpha(x)d(y) \) holds for all \( x,y \in N \). An additive mapping \( d : N \to N \) is called \((\alpha, 1)\)-derivation if there exists a map \( \alpha : N \to N \) such that \( d(xy) = d(x)\alpha(y) + xd(y) \) holds for all \( x,y \in N \).

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$x, y \in \mathcal{N}$. Obviously, a two sided $\alpha$-derivation is both a $(1, \alpha)$-derivation as well as an $(\alpha, 1)$-derivation. Also, any derivation on $\mathcal{N}$ is a $(1, \alpha)$-derivation, but the converse is not true in general (see [6]). The study of commutativity of 3-prime near-rings by using derivations was initiated by H. E. Bell and G. Mason in 1987 (see [5]). In [8] A. A. M. Kamal generalizes some results of Bell and Mason by studying the commutativity of 3-prime near-rings using $\sigma$-derivations instead of the usual derivations, where $\sigma$ is an automorphism on the near-ring. M. Ashraf, A. Ali and Shakir Ali in [1] and N. Aydin and Ö. Gölbaşı in [7] generalize Kamal’s work by using a $(s, t)$-derivation instead of a $s$-derivation, where $s$ and $t$ are automorphisms. Recently many authors (see [2], [4], [5]) for reference where further references can be found) have studied commutativity of 3-prime near-rings satisfying certain identities involving derivations, semiderivations, two sided $\alpha$-derivations. Now our aim in this paper is to study the commutativity behavior of 3-prime near-ring which admits $(1, \alpha)$-derivations satisfying certain properties. In fact, our results generalize, extend and complement several results obtained earlier in [2], [6], [10] on derivations, semiderivations and two sided $\alpha$-derivations for 3-prime near-rings.

2. Some preliminaries

In this section, we give some well-known results and we add some new lemmas which will be used throughout the next sections of the paper.

**Lemma 2.1** ([4, Theorem 2.9]). Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{U}$ is a nonzero semigroup ideal of $\mathcal{N}$, then the following assertions are equivalent:

(i) $[x, y] \in Z(\mathcal{N})$ for all $x, y \in \mathcal{U}$;

(ii) $\mathcal{N}$ is a commutative ring.

**Lemma 2.2** ([4, Theorem 2.10]). Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and $\mathcal{U}$ be a nonzero semigroup ideal. If $u \circ v \in Z(\mathcal{N})$ for all $u, v \in \mathcal{U}$, then $\mathcal{N}$ is a commutative ring.

**Lemma 2.3.** Let $\mathcal{N}$ be a 3-prime near-ring and $\mathcal{U}$ be a nonzero semigroup ideal of $\mathcal{N}$.

(i) [3, Lemma 1.5] If $\mathcal{U} \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.

(ii) [3, Lemma 1.4(i)] If $x, y \in \mathcal{N}$ and $x \circty = \{0\}$, then $x = 0$ or $y = 0$.

(iii) [3, Lemma 1.3(i)] If $x$ is an element of $\mathcal{N}$ such that $\mathcal{U}x = \{0\}$ (resp. $x \circty = \{0\}$), then $x = 0$.

(iv) If $z$ centralizes a nonzero semigroup ideal $\mathcal{U}$, then $z \in Z(\mathcal{N})$.

**Lemma 2.4.** Let $\mathcal{N}$ be a near-ring and $\mathcal{d}$ an additive map.

(i) If $\mathcal{d}$ is a $(1, \alpha)$-derivation associated with a map $\alpha$, then $\mathcal{N}$ satisfies the following property:

$$
(d(x)y + \alpha(x)d(y))z = d(x)yz + \alpha(x)d(y)z + \alpha(x)\alpha(y)d(z) - \alpha(xy)d(z)
$$

for all $x, y, z \in \mathcal{N}$.
(ii) If $d$ is an $(\alpha, 1)$-derivation, then $N$ satisfies the following relation:
\[
(d(x)\alpha(y) + xd(y))\alpha(t) = d(x)\alpha(yt) + xd(y)\alpha(t)
\]
for all $t, x, y \in N$.

Proof. (i) We have
\[
d((xy)z) = d(xy)z + \alpha(xy)d(z)
\]
\[
= (d(x)y + \alpha(x)d(y))z + \alpha(xy)d(z)
\]
for all $x, y, z \in N$.

Also
\[
d(xyz) = d(xy)z + \alpha(x)d(yz)
\]
\[
= d(xyz) + \alpha(x)d(yz) + \alpha(x)\alpha(y)d(z)
\]
for all $x, y, z \in N$.

Combining the above two equalities, we find that
\[
(d(x)y + \alpha(x)d(y))z + \alpha(xy)d(z) = d(xyz) + \alpha(x)d(yz) + \alpha(x)\alpha(y)d(z)
\]
for all $x, y, z \in N$, which gives the required result.

(ii) Using the same proof of [10, Lemma 2.1], we find the required result. \hfill \Box

Lemma 2.5. Let $N$ be a 3-prime near-ring and $U$ be a nonzero semigroup ideal of $N$. If $N$ admits a nonzero $(1, \alpha)$-derivation $d$ associated with a map $\alpha$, the following properties are satisfied:

(i) If $d(U) = \{0\}$, then $d = 0$.

(ii) If $ad(U) = \{0\}$, $a \in N$ and $\alpha(U) = U$, then $a = 0$.

(iii) If $d(U)a = \{0\}$, $a \in N$ and $\alpha(xy) = \alpha(x)\alpha(y)$ for all $x, y \in U$, then $a = 0$.

Proof. (i) Suppose that $d(U) = \{0\}$. Then
\[
0 = d(mnu)
\]
\[
= d(m)nu + \alpha(m)d(nu)
\]
\[
= d(m)nu \quad \text{for all } u \in U, m \in N,
\]
which implies that $d(m)Nu = \{0\}$ for all $u \in U, m \in N$. But $N$ is 3-prime and $U \neq \{0\}$, then $d = 0$.

(ii) If $ad(U) = \{0\}$ and $a \in N$, then $ad(xy) = 0$ for all $x, y \in U$. This implies that $ad(x)\alpha(xy)d(y) = 0$ for all $x, y \in U$, and hence $a\alpha(x)d(y) = 0$ for all $x, y \in U$. But $\alpha(U) = U$, then $a\alpha(Ud)(y) = \{0\}$ for all $y \in U$. Using (i) and Lemma 2.3(ii), we obtain $a = 0$.

(iii) If $d(U)a = \{0\}$, then $d(xy)a = 0$ for all $x, y \in U$. By Lemma 2.4, we get $d(xy)a + \alpha(x)d(y)a + \alpha(x)\alpha(y)d(a) - \alpha(xy)d(a) = 0$ for all $x, y \in U$. Using the given hypothesis, we find that $d(xy)a = 0$ for all $x, y \in U$, i.e., $d(U)a = \{0\}$ for all $x \in U$. Since $d \neq 0$, we arrive at $a = 0$. \hfill \Box
3. Main results

In [5], H. E. Bell and G. Mason proved that a 3-prime near-ring \( N \) must be commutative if it admits a derivation \( d \) such that \( d(N) \subseteq Z(N) \). This result was generalized by the authors in [6], [10]. They replaced the derivation by a semiderivation or a two sided \( \alpha \)-derivation where \( \alpha \) is an homomorphism. Our objective in the following theorem is to generalize this result by treating the cases of \( (1, \alpha) \)-derivations, \( (\alpha, 1) \)-derivations and two sided \( \alpha \)-derivations where \( \alpha \) is an additive map.

**Theorem 3.1.** Let \( N \) be a 3-prime near-ring. If \( N \) admits a nonzero map \( d \) such that \( d(N) \subseteq Z(N) \), then \( N \) is a commutative ring if \( d \) has one of the following properties:

(i) \( d \) is a \((1, \alpha)\)-derivation associated with an additive map \( \alpha \).

(ii) \( d \) is a \((\alpha, 1)\)-derivation associated with an additive map \( \alpha \).

(iii) \( d \) is a two sided \( \alpha \)-derivation associated with an additive map \( \alpha \).

**Proof.** (i) Using our assumptions, we have \( zd(xy) = d(xy)z \) and \( d(z)d(xy) = d(xy)d(z) \) for all \( x, y, z \in N \). By Lemma 2.4, we obtain

\[
zd(x)y + z\alpha(x)d(y) = d(xy)z + \alpha(x)d(y)z + \alpha(x)\alpha(y)d(z) - \alpha(xy)d(z)
\]

and

\[
d(z)d(xy) + d(z)\alpha(x)d(y) = d(xy)d(z) + \alpha(x)d(y)d(z) + \alpha(x)\alpha(y)d^2(z) - \alpha(xy)d^2(z).
\]

Since \( d^2(z) = d(d(z)) \in Z(N) \), (3.2) becomes

\[
d^2(z)N(\alpha(xy) - \alpha(x)\alpha(y)) = \{0\} \quad \text{for all } x, y, z \in N.
\]

Since \( N \) is 3-prime, we have

\[
d^2(z) = 0 \quad \text{or} \quad \alpha(xy) = \alpha(x)\alpha(y) \quad \text{for all } x, y, z \in N.
\]

Assume that \( \alpha(xy) = \alpha(x)\alpha(y) \) for all \( x, y \in N \). For \( z = y \), (3.1) implies \( d(y)N[\alpha(x), y] = \{0\} \) for all \( x, y \in N \). Using \( N \) is 3-prime, we obtain \( d(y) = 0 \) or \( y\alpha(x) = \alpha(x)y \) for all \( x, y \in N \). The last two cases give the following equation

\[
d(x)N[y, z] = \{0\} \quad \text{for all } x, y, z \in N.
\]

Since \( N \) is 3-prime, for each \( y \in N \), either \( y \) centralizes \( N \) or \( d(N) = \{0\} \); and by Lemma 2.3(iv) together with Lemma 2.5(i), we conclude that \( N \) is a commutative ring.

Suppose that \( d^2(z) = 0 \) for all \( z \in N \). We have

\[
0 = d^2(xy) = d(d(x)y + \alpha(x)d(y)) = \alpha(d(x)) + d(\alpha(x))d(y) \quad \text{for all } x, y \in N,
\]
this implies that \((\alpha(d(x)) + d(\alpha(x)))d(y) = 0\) for all \(x, y \in \mathcal{N}\). Replacing \(y\) by \(yt\) in the preceding equation and using it again, we arrive at
\[
d(y)\mathcal{N}(\alpha(d(x)) + d(\alpha(x))) = \{0\} \quad \text{for all } x, y \in \mathcal{N}.
\]
By 3-primeness of \(\mathcal{N}\) and \(d \neq 0\), we obtain
\[
\alpha(d(x)) + d(\alpha(x)) = 0 \quad \text{for all } x \in \mathcal{N}.
\] (3.5)

Using our hypothesis, we have \(d(xd(y)) = d(d(y)x)\) for all \(x, y \in \mathcal{N}\). By the definition of \(d\), we get
\[
d(x)d(y) + \alpha(x)d^2(y) = d^2(y)x + \alpha(d(y))d(x) \quad \text{for all } x, y \in \mathcal{N},
\]
which can be rewritten as
\[
(\alpha(d(y)) - d(y))d(x) = 0 \quad \text{for all } x, y \in \mathcal{N}.
\]
From the above, one can easily see that
\[
\alpha(d(y)) = d(y) \quad \text{for all } y \in \mathcal{N}.
\] (3.6)

Using (3.5) and (3.6), we obtain
\[
d(x) + d(\alpha(x)) = 0 \quad \text{for all } x \in \mathcal{N}.
\] (3.7)

Taking \(d(u)\) instead of \(y\) in (3.1), we find that
\[
(\alpha(xd(u)) - \alpha(x)d(u))d(z) = 0 \quad \text{for all } x, u, z \in \mathcal{N}.
\]
Which implies that
\[
\alpha(xd(u)) = \alpha(x)d(u) \quad \text{for all } x, u \in \mathcal{N}.
\] (3.8)

Using (3.6) with the definition of \(d\), we get
\[
\alpha(d(x)y + d(x)d(y)) = d(x)y + \alpha(x)d(y) \quad \text{for all } x, y \in \mathcal{N}.
\] (3.9)

Since \(\alpha\) is an additive map, using the fact that \(d(\mathcal{N}) \subseteq Z(\mathcal{N})\) and (3.8), we arrive at
\[
d(x)\alpha(y) + \alpha^2(x)d(y) = d(x)y + \alpha(x)d(y) \quad \text{for all } x, y \in \mathcal{N}.
\] (3.10)

Setting \(x = y\) in (3.10) and using our hypothesis, we obviously get
\[
(\alpha^2(x) - x)\mathcal{N}d(x) = \{0\} \quad \text{for all } x \in \mathcal{N}.
\] (3.11)

Since \(\mathcal{N}\) is 3-prime, by (3.11) we can easily find that
\[
\alpha^2 = id_\mathcal{N} \text{ or } d(x) = 0 \quad \text{for all } x \in \mathcal{N}.
\] (3.12)

Suppose there exists an element \(u \in \mathcal{N}\) such that \(d(u) = 0\) and writing \(u\) instead of \(y\) in (3.10), we find that \(d(x)\alpha(u) = d(x)u\) for all \(x \in \mathcal{N}\) which forces that
\[
d(\mathcal{N})\mathcal{N}(\alpha(u) - u) = \{0\}.
\] (3.13)
Since \( d \neq 0 \) and \( \mathcal{N} \) is 3-prime, we conclude that \( \alpha(u) = u \) so that \( \alpha^2(u) = \alpha(u) = u \). In this case, (3.12) forces that \( \alpha^2 = id_{\mathcal{N}} \). Using Lemma 2.4, we get

\[
(3.14) \quad \alpha(z)d(x)\alpha(y)+\alpha(z)xd(y) = d(x)\alpha(yz)+xd(y)\alpha(z) \quad \text{for all } x, y, z \in \mathcal{N}.
\]

For \( z = \alpha(x) \), (3.14) becomes

\[
(3.15) \quad d(x)\mathcal{N}(\alpha(y\alpha(x)) - x\alpha(y)) = \{0\} \quad \text{for all } x, y \in \mathcal{N}.
\]

Since \( \mathcal{N} \) is 3-prime, either \( d(x) = 0 \) or \( \alpha(y\alpha(x)) = x\alpha(y) \) for all \( x, y \in \mathcal{N} \).

Suppose there exists \( x_0 \in \mathcal{N} \) such that \( d(x_0) = 0 \). By (3.14), we arrive at \( d(y)\mathcal{N}[x_0, \alpha(z)] = \{0\} \) for all \( y, z \in \mathcal{N} \). Replacing \( z \) by \( \alpha(u) \) and using the 3-primeness of \( \mathcal{N} \) and \( d \neq 0 \), we get \( x_0u = ux_0 \) for all \( u \in \mathcal{N} \).

If there exists \( x_0 \in \mathcal{N} \) such that \( \alpha(y\alpha(x_0)) = x_0\alpha(y) \) for all \( y \in \mathcal{N} \). Replace \( z \) by \( \alpha(x_0) \) in (3.14), we get \( d(y)\mathcal{N}[x_0x - xx_0] = \{0\} \) for all \( x, y \in \mathcal{N} \). Since \( \mathcal{N} \) is 3-prime and \( d \neq 0 \), we obtain \( x_0x = xx_0 \) for all \( x \in \mathcal{N} \). In both cases, we conclude that \( x \) centralizes \( \mathcal{N} \) which forces that \( \mathcal{N} \) is a commutative ring by Lemma 2.3(iv).

(ii) Assume that \( d(xy) = d(x)\alpha(y) + xd(y) \) for all \( x, y \in \mathcal{N} \). By hypothesis, we have \( d(xy) = \alpha(y)d(x) + d(y)x \) for all \( x, y \in \mathcal{N} \). Calculating \( d(x(y + y)) \) in two different ways, we obtain

\[
d(x)\alpha(y) + xd(y) = xd(y) + d(x)\alpha(y)
\]

\[
= d(y)x + \alpha(y)d(x) \quad \text{for all } x, y \in \mathcal{N}.
\]

From the last expression, we remark that \( d \) plays a role of a \((1, \alpha)\)-derivation, in this case, using the same proof of (i), we find that \( \mathcal{N} \) is a commutative ring.

(iii) It is clear that if \( d \) is a two sided \( \alpha \)-derivation, then \( d \) is both a \((1, \alpha)\)-derivation and an \((\alpha, 1)\)-derivation, which proves that \( \mathcal{N} \) is a commutative ring by (i) and (ii).

\( \square \)

In the following, we study the commutativity of a near-ring \( \mathcal{N} \) admitting nonzero two sided \( \alpha \)-derivations \((1, \alpha)\)-derivations \( d \) satisfying the condition \( d(xy) = d(xy) \) \((d(xy) = -d(yx)) \) for all \( x, y \in \mathcal{N} \). These results have been demonstrated by several authors in cases the derivations, semiderivations and two sided \( \alpha \)-derivations on 3-prime near-rings for more details see the following references [2], [3], [5], [6] and [10]. Our goal in the next part is to generalize these results in the case of \((1, \alpha)\)-derivations and two sided \( \alpha \)-derivations where \( \alpha \) is an additive map instead of a homomorphism.

**Theorem 3.2.** Let \( \mathcal{N} \) be a 3-prime near-ring and \( \mathcal{U} \) be a nonzero semigroup of \( \mathcal{N} \). If \( \mathcal{N} \) admits a nonzero map \( d \) such that \( d([x, y]) = 0 \) for all \( x, y \in \mathcal{U} \), then \( \mathcal{N} \) is a commutative ring if \( d \) has one of the following properties:

(i) \( d \) is a \((1, \alpha)\)-derivation associated with a map \( \alpha \).

(ii) \( d \) is a two sided \( \alpha \)-derivation associated with a map \( \alpha \).
Proof. (i) By our assumptions, we have \( d([x, y]) = 0 \) for all \( x, y \in \mathcal{U} \). Replacing \( y \) by \( xy \), then
\[
0 = d([x, xy]) = d(x)[x, y] + \alpha(x)d([x, y]) = d(x)[x, y] \quad \text{for all } x, y \in \mathcal{U}
\]
which implies that \( d(x)xy = d(x)yx \) for all \( x, y \in \mathcal{U} \). Taking \( yz \) instead of \( y \) where \( z \in \mathcal{N} \), we obtain \( d(x)\mathcal{U}[x, z] = \{0\} \) for all \( x \in \mathcal{U} \), \( z \in \mathcal{N} \). Invoking Lemma 2.3(ii), we get
\[
(3.16) \quad d(x) = 0 \quad \text{or} \quad x \in Z(\mathcal{N}) \quad \text{for all } x \in \mathcal{U}.
\]
Suppose there is an element \( x_0 \in \mathcal{U} \) such that \( d(x_0) = 0 \). Using the fact that \( d(x_0y) = d(yx_0) \) for all \( y \in \mathcal{U} \), we obtain \( \alpha(x_0)d(y) = d(y)x_0 \) for all \( y \in \mathcal{U} \).

Putting \( yt \) instead of \( y \) and using Lemma 2.4, we get
\[
\alpha(x_0)d(y)t + \alpha(x_0)\alpha(y)d(t) = d(y)t x_0
\]
which can be rewritten as
\[
d(y)x_0t + \alpha(x_0)\alpha(y)d(t) = d(y)t x_0 + \alpha(y)d(t)x_0 \quad \text{for all } y \in \mathcal{U}, t \in \mathcal{N}.
\]
Taking \( t = [u, v] \) in last equation, we obviously get \( d(y)(x_0[u, v] - [u, v]x_0) = 0 \) for all \( y, u, v \in \mathcal{U} \). Calculating the expression \( d(y)(x_0[u, v] - [u, v]x_0) \), one can easily find that \( d(y)(x_0[u, v] - [u, v]x_0) = d(y)(x_0[u, v] - [u, v]x_0) = 0 \) for all \( y, u, v \in \mathcal{U} \). Substituting \( yt \) for \( y \), where \( t \in \mathcal{U} \) in the above equation, we arrive at \( d(y)\mathcal{U}([u, v] - [u, v]x_0) = \{0\} \) for all \( y, u, v \in \mathcal{U} \). Lemma 2.5(ii) and Lemma 2.3(ii) for \( x_0[u, v] = [u, v]x_0 \) for all \( u, v \in \mathcal{U} \), in this case, \( (3.16) \) forces that \( x[u, v] = [u, v]x \) for all \( x, u, v \in \mathcal{U} \). Replacing \( x \) by \( xt \) where \( t \in \mathcal{N} \) in the preceding equation and using it again, we arrive at \( \mathcal{U}([u, v], t) = \{0\} \) for all \( u, v \in \mathcal{U}, t \in \mathcal{N} \). By virtue of Lemma 2.3(iii), we obtain \( [u, v] \in Z(\mathcal{N}) \) for all \( u, v \in \mathcal{U} \) which together with Lemma 2.1, yields that \( \mathcal{N} \) is a commutative ring.

(ii) It is clear that if \( d \) is a two sided \( \alpha \)-derivation, then \( d \) is a \((1, \alpha)\)-derivation, which proves that \( \mathcal{N} \) is a commutative ring by (i).

As an application of Theorem 3.1, we obtain the following corollaries.

**Corollary 3.1.** Let \( \mathcal{N} \) be a 2-torsion free 3-prime near-ring and \( d \) a nonzero derivation.

(i) [5, Theorem 2] If \( d(\mathcal{N}) \subseteq Z(\mathcal{N}) \), then \( \mathcal{N} \) is a commutative ring.

(ii) [2, Theorem 4.1] If \( d([x, y]) = 0 \) for all \( x, y \in \mathcal{N} \), then \( \mathcal{N} \) is a commutative ring.

**Corollary 3.2.** Let \( \mathcal{N} \) be a 2-torsion free 3-prime near-ring and \( d \) a nonzero semi-derivation.

(i) [6, Theorem 1] If \( d(\mathcal{N}) \subseteq Z(\mathcal{N}) \), then \( \mathcal{N} \) is a commutative ring.

(ii) [6, Theorem 2] If \( d([x, y]) = 0 \) for all \( x, y \in \mathcal{N} \), then \( \mathcal{N} \) is a commutative ring.
Corollary 3.3. Let $N$ be a 2-torsion free 3-prime near-ring and $d$ a nonzero two sided $\alpha$-derivation.

(i) [10, Theorem 1] If $d(N) \subseteq Z(N)$, then $N$ is a commutative ring.

(ii) [10, Theorem 2] If $d([x, y]) = 0$ for all $x, y \in N$, then $N$ is a commutative ring.

The following example shows the necessity of the 3-primeness of $N$ in the previous theorems.

Example 3.1. Let $S$ be a 2-torsion free near-ring. Let us define $N$ and $d, \alpha : N \to N$ by:

$$N = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \middle| x, y \in S \right\}.$$  

$$d \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \alpha \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}.$$  

It is clear that $N$ is not a 3-prime near-ring and $d$ is a nonzero two sided $\alpha$-derivation associated with an additive map $\alpha$ satisfying the following properties:

(i) $d(N) \subseteq Z(N)$, (ii) $d([A, B]) = 0$ for all $A, B \in N$, but, since the addition in $N$ is not commutative, $N$ cannot be a commutative ring.

The conclusion of Theorem 3.2 no remains valid if we replace the product $[x, y]$ by $x \circ y$, provided that $N$ is 2-torsion free. In fact, we obtain the following result.

Theorem 3.3. Let $N$ be a 2-torsion free 3-prime near-ring and $U$ be a nonzero semigroup ideal of $N$. Then there exists no nonzero map $d$ on $N$ such that $d(x \circ y) = 0$ for all $x, y \in U$ in the following cases:

(i) $d(N) \subseteq Z(N)$, (ii) $d([A, B]) = 0$ for all $A, B \in N$.

Proof. (i) Suppose that $d$ is a $(1, \alpha)$-derivation associated with an additive map $\alpha$ such that $d(x \circ y) = 0$ for all $x, y \in U$. Replacing $y$ by $xy$, then

$$0 = d(x \circ xy) = d(x)(x \circ y) + \alpha(x)d(x \circ y) = d(x)(x \circ y) \quad \text{for all}\ x, y \in U$$

which implies that $d(x)xy = -d(x)yx$ for all $x, y \in U$. Taking $yz$ instead of $y$ where $z \in N$, we obtain $d(x)U(-z(-x) + (-x)z) = \{0\}$ for all $x \in U, z \in N$.

Using Lemma 2.3(ii), we get

$$d(x) = 0 \quad \text{or} -x \in Z(N) \quad \text{for all}\ x \in U.$$  

(3.17)
Suppose there exists an element $x_0 \in U$ such that $-x_0 \in Z(N)$. We have

\[
0 = -d(x_0 \circ x_0) \\
= d(-x_0 \circ x_0) \\
= d(2(-x_0)x_0).
\]

By the 2-torsion freeness of $N$, we get $d((-x_0)x_0) = d(x_0(-x_0)) = 0$. On the other hand, we have

\[
0 = d((x_0 \circ x_0)(-x_0)) \\
= 2d((-x_0)x_0^2) \\
= d((x_0x_0)(-x_0)) \\
= d(x_0x_0(-x_0) + \alpha(x_0)d(x_0(-x_0)) \\
= d(x_0)x_0(-x_0)
\]

which implies that $d(x_0)x_0N(-x_0) = \{0\}$. In light of 3-primeness of $N$, we conclude that $d(x_0)x_0 = 0 = d(x_0)(-x_0)$ and $d(x_0)N(-x_0) = \{0\}$. By the 3-primeness of $N$, we obtain $d(x_0) = 0$. In all cases (3.17) becomes $d(x) = 0$ for all $x \in U$ which is a contradiction with Lemma 2.5(i).

(ii) It is clear that if $d$ is a two sided $\alpha$-derivation, then $d$ is a $(1, \alpha)$-derivation, which proves that $N$ is a commutative ring by (i). \hfill $\square$

The following example shows the necessity of the 3-primeness of $N$ in the previous theorems.

**Example 3.2.** Let $S$ be a 2-torsion free near-ring. Let us define $N$, $d$ and $\alpha : N \rightarrow N$ by:

\[
N = \left\{ \left( \begin{array}{ccc} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{array} \right) \mid x, y \in S \right\}
\]

\[
d \left( \begin{array}{ccc} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{array} \right) = \left( \begin{array}{ccc} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \text{and} \quad \alpha \left( \begin{array}{ccc} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{array} \right) = \left( \begin{array}{ccc} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right).
\]

It is clear that $N$ is a non 3-prime near-ring and $d$ is a nonzero two sided $\alpha$-derivation such that $d(A \circ B) = 0$ for all $A, B \in N$, but $N$ is not a commutative ring because the addition is not commutative.

**Example 3.3.** Let $N = M_2(\mathbb{Z}_3)$ the noncommutative prime ring and $d$ the nonzero map on $N$ such that $d \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} 0 & d-a \\ 0 & -d \end{array} \right)$. Taking $x = (1 \, 1)$ and $y = (0 \, 1)$. Then $d(x \circ y) = (0 \, 2) \neq 0$, which shows that the condition “$d(x \circ y) = 0$ for all $x, y \in N$” is necessary.
Example 3.4. Let $\mathcal{N} = \mathbb{Z}_2[x]$. Then $\mathcal{N}$ is an integral domain which means that $\mathcal{N}$ is a commutative prime ring. Also, we have $2\mathcal{N} = \{0\}$. If we take $d$ to be the identical application on $\mathcal{N}$ and $\alpha = 0$. Then $d$ is a nonzero $(1, \alpha)$-derivation and also is a nonzero two sided $\alpha$-derivation on $\mathcal{N}$ and $d(p \circ q) = 2d(pq) = 0$ for all $p, q \in \mathcal{N}$. But $\mathcal{N}$ is not 2-torsion free.

References


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