REGULARITY OF THE SCHRÖDINGER EQUATION FOR A CAUCHY-EULER TYPE OPERATOR

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ABSTRACT. We consider the initial value problem of the Schrödinger equation for an interesting Cauchy-Euler type operator $\mathcal{A}$ on $\mathbb{C}^n$ that is an analogue of the harmonic oscillator in $\mathbb{R}^n$. We get an appropriate $L^1 - L^\infty$ dispersive estimate for the solution of the initial value problem.

1. Introduction and statement of the main result

Associated to any self-adjoint differential operator $L$ on $\mathbb{R}^n$, one can formally define an oscillatory semigroup $e^{-itL}$, using the spectral theory for $L$. Assume that $L$ has the spectral representation

$$Lf = \int_{E} \lambda dP_{\lambda}(f), \quad f \in L^2(\mathbb{R}^n),$$

where $P_{\lambda}$ is a projection valued measure supported on the spectrum $E$ of $L$. Then the operator $e^{-itL}$ can be defined by

$$e^{-itL}f = \int_{E} e^{-it\lambda} dP_{\lambda}(f), \quad f \in L^2(\mathbb{R}^n).$$

Consider the differential operator $i\partial_t - L$ and the associated initial value problem for the Schrödinger equation for $L$:

$$\begin{cases}
(i\partial_t - L)u = 0 & \text{on } \mathbb{R}^n \times \mathbb{R} \\
u(\cdot, 0) = f & \text{on } \mathbb{R}^n.
\end{cases}$$

Assuming $f \in L^2(\mathbb{R}^n)$, the solution $u$ can be represented by

$$u(x, t) = e^{-itL}f(x).$$

We thus call $e^{-itL}$, the Schrödinger oscillatory semigroup for $L$. 

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Let $H$ be the most basic Schrödinger operator in $\mathbb{R}^n$, $n \geq 1$, the Hermite operator (or the harmonic oscillator):

$$H = -\Delta + |x|^2. \tag{1}$$

Then the Schrödinger equation for $H$ can be written by

$$(i\partial_t - H)u = 0.$$ 

This is an important model in quantum mechanics (see for example [4]).

In [5], Nandakumara and Ratnakumar considered the regularity of the following initial value problem for the Schrödinger equation for $H$:

$$\begin{cases} (i\partial_t - H)u = 0 \quad \text{on} \quad \mathbb{R}^n \times \mathbb{R} \\ u(\cdot, 0) = f \quad \text{on} \quad \mathbb{R}^n. \end{cases}$$

For $f \in L^2(\mathbb{R}^n)$ the solution to the initial value problem is given by

$$u(x, t) = e^{-itH} f(x).$$

They proved the following regularity estimate

$$\int_{-\pi}^{\pi} \|u(\cdot, t)\|_{L^q(\mathbb{R}^n)}^q \, dt \leq C_n \|f\|_2^q,$$

where $1 < q < \infty$, $2 \leq p < \Lambda$, where $\Lambda = \infty$ for $n = 1$ and $\Lambda = \frac{2n}{n-2}$ for $n \geq 2$.

Let $\mathbb{C}^n$ be the complex $n$-space and $dV$ be the ordinary volume measure on $\mathbb{C}^n$. If $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ are points in $\mathbb{C}^n$, we write

$$z \cdot w = \sum_{j=1}^{n} z_j w_j, \quad |z| = (z \cdot z)^{1/2}.$$ 

For any $0 < p \leq \infty$ we let $L^p_G(\mathbb{C}^n)$ denote the space of Lebesgue measurable functions $f$ on $\mathbb{C}^n$ such that the function $f(z) e^{-\frac{1}{2}|z|^2}$ is in $L^p(\mathbb{C}^n, dV)$. When $0 < p < \infty$, it is clear that

$$L^p_G(\mathbb{C}^n) = L^p \left( \mathbb{C}^n, e^{-\frac{1}{2}|z|^2} \, dV(z) \right).$$

We define

$$\|f\|_{L^p_G} = \left( \frac{p}{2\pi} \right)^n \int_{\mathbb{C}^n} \left| f(z) e^{-\frac{1}{2}|z|^2} \right|^p \, dV(z) \right]^{\frac{1}{p}}.$$ 

For $p = \infty$ the norm in $L^\infty_G(\mathbb{C}^n)$ is defined by

$$\|f\|_{L^\infty_G} = \text{esssup} \left\{ |f(z)| e^{-\frac{1}{2}|z|^2} : z \in \mathbb{C}^n \right\}.$$ 

Let $F^p(\mathbb{C}^n)$ denote the space of entire functions in $L^p_G(\mathbb{C}^n)$. If $0 < p < q$, then $F^p \subset F^q$, and the inclusion is proper and continuous (see [7]). Note that $F^2$ is a closed subspace of the Hilbert space $L^2_G$ (see [7]) with inner product

$$\langle f, g \rangle = \frac{1}{\pi^n} \int_{\mathbb{C}^n} f(z) g(z) e^{-|z|^2} \, dV(z).$$
The Hermite operator $H$ on $\mathbb{R}^n$ has the representation

$$H = \frac{1}{2} \sum_{j=1}^{n} (a_j a_j^\dagger + a_j^\dagger a_j),$$

in terms of the creation operators $a_j = -\frac{d}{dt} + x_j$ and the annihilation operator $a_j^\dagger = \frac{d}{dx_j} + x_j$, $j = 1, 2, \ldots, n$. There is an interesting operator $\mathcal{R}$ on $\mathbb{C}^n$, given by

$$\mathcal{R} = \frac{1}{2} \sum_{j=1}^{n} (A_j A_j^* + A_j^* A_j),$$

where

$$A_j = 2 \frac{\partial}{\partial z_j}, \quad A_j^* = z_j, \quad 1 \leq j \leq n.$$

Both $A_j$ and $A_j^*$, as defined above, are densely defined linear operators on $F^p$ (unbounded though). We have

$$\mathcal{R} = 2 \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j} + n.$$

Thus $\mathcal{R}$ is a Cauchy-Euler type operator.

**Remark 1.** Let

$$f(z) = \sum_{k=0}^{\infty} \frac{z^k}{\sqrt{2^k (k + 1)!k!}}.$$

Then $f \in F^2$, but $\mathcal{R}f \notin F^2$.

The remark above tells us that $\text{Dom}(\mathcal{R}) \subsetneq F^2$. Thus $\mathcal{R}$ is an unbounded operator on $F^2$.

The Segal-Bargmann transform $B$ is defined by

$$Bf(z) = \frac{1}{\pi^{n/4}} \int_{\mathbb{R}^n} f(x)e^{x \cdot \frac{1}{2} |x|^2 - x \cdot z - \frac{1}{2} |z|^2} dV(x),$$

where $dV(x)$ is the volume measure on $\mathbb{R}^n$. It is well-known that the Segal-Bargmann transform is a unitary isomorphism between $L^2(\mathbb{R}^n)$ and $F^2(\mathbb{C}^n)$ ([1], [7]). Moreover, we know that

$$BH = \mathcal{R}B \quad \text{on} \quad L^2(\mathbb{R}^n).$$

Motivated by these relations, we consider the initial value problem:

\[
\begin{cases}
(i \partial_t - \mathcal{R})u = 0 & \text{on} \quad \mathbb{C}^n \times (0, \infty) \\
u(\cdot, 0) = f & \text{on} \quad \mathbb{C}^n.
\end{cases}
\]

We get an appropriate $L^1 - L^\infty$ dispersive estimate for the solution of the initial value problem as following.
Theorem 1.1. For \( f \in F^1(\mathbb{C}^n) \), the solution \( u(z, t) = e^{-it\mathcal{R}}f(z) \) of the initial value problem (2) satisfies the following regularity estimate
\[
\sup_{0 < t < \infty} \| u(\cdot, t) \|_{F^q} \leq \| f \|_{F^p},
\]
where \( 1 \leq p \leq 2, \ 2 \leq q \leq \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

2. Proof of Theorem 1.1

We define
\[
e_\alpha(z) = \frac{z^\alpha}{\| z^\alpha \|_{F^2}} = \frac{z^\alpha}{\sqrt{\alpha!}}.
\]
Then \( \{ e_\alpha : \alpha \in \mathbb{N}_0^n \} \) is an orthonormal basis for \( F^2 \). We know that \( \mathcal{R} \) is a positive, self-adjoint operator on \( \text{Dom}(\mathcal{R}) \) with the discrete spectrum \( \sigma(\mathcal{R}) = \{ 2|\alpha| + n : \alpha \in \mathbb{N}_0^n \} \) [2]. For \( f \in F^2 \) let
\[
f(z) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha e_\alpha(z)
\]
be the orthonormal decomposition of \( f \). Associated with the operator \( \mathcal{R} \) is a semigroup \( \{ B_t \}_{t \geq 0} \) defined by the expansion
\[
B_t f(z) = \sum_{\alpha \in \mathbb{N}_0^n} e^{-i(2|\alpha|+n)t}c_\alpha e_\alpha(z).
\]
It is easy to see that \( B_t f(z) \) converges in \( F^2 \) for every fixed \( t \geq 0 \) whenever \( f \in F^2 \). Moreover, \( B_t f(z) \rightarrow f(z) \) in \( F^2 \) as \( t \rightarrow 0^+ \) by the dominated convergence theorem since \( |e^{-i(2|\alpha|+n)t} - 1| \leq 2 \). Thus \( u(z, t) = B_t f(z) \) is the solution of the initial value problem:
\[
\begin{aligned}
(i\partial_t - \mathcal{R})u &= 0 \quad \text{on} \quad \mathbb{C}^n \times (0, \infty) \\
u(\cdot, 0) &= f \quad \text{on} \quad \mathbb{C}^n.
\end{aligned}
\]

We know that \( \{ B_t \}_{t \geq 0} \) is a strongly continuous semigroup. Moreover, \( -i\mathcal{R} \) is the infinitesimal generator of \( \{ B_t \}_{t \geq 0} \) [2]. That is,
\[
\lim_{t \rightarrow 0^+} \frac{B_t f - f}{t} = -i\mathcal{R} f.
\]
Thus, we have (see [3])
\[
B_t = e^{-it\mathcal{R}}.
\]

It is well-known ([1], [7]) that for \( f \in F^2 \) we have the reproducing formula such that
\[
f(z) = \int_{\mathbb{C}^n} f(w)K(z, w)e^{-|z|^2} dV(w),
\]
where \( K(z, w) \) is the reproducing kernel defined by
\[
K(z, w) = \sum_\alpha e_\alpha(z)e_\alpha(w).
\]
By the spectral theory,

\[ u(z, t) = e^{-it\mathcal{S}} f(z) = e^{-it\mathcal{S}} \int_{\mathbb{C}^n} f(w) \sum_{\alpha} e_{\alpha}(z) \overline{e_{\alpha}(w)} e^{-|w|^2} dV(w) \]

\[ = e^{-it\mathcal{S}} \left( \sum_{\alpha} e_{\alpha}(z) \right) \int_{\mathbb{C}^n} f(w) \overline{e_{\alpha}(w)} e^{-|w|^2} dV(w) \]

\[ = \sum_{\alpha} e^{-it(2|\alpha|+n)} e_{\alpha}(z) \int_{\mathbb{C}^n} f(w) \overline{e_{\alpha}(w)} e^{-|w|^2} dV(w) \]

\[ = \int_{\mathbb{C}^n} f(w) \sum_{\alpha} e^{-it(2|\alpha|+n)} e_{\alpha}(z) \overline{e_{\alpha}(w)} e^{-|w|^2} dV(w) \]

\[ = \int_{\mathbb{C}^n} f(w) K_t(z, w) e^{-|w|^2} dV(w). \]

Interchanging the order of summation and integration is justified by the dominated convergence theorem since

\[ \sum_{\alpha} |e_{\alpha}(z)| \int_{\mathbb{C}^n} |f(w)||e_{\alpha}(w)| e^{-|w|^2} dV(w) \leq \sum_{\alpha} \frac{|z^\alpha|}{\sqrt{\alpha!}} \|f\|_{F^2} \]

and the power series on the right side of the inequality above is convergent for every \( z \in \mathbb{C}^n \).

Note that

\[ K_t(z, w) = \sum_{\alpha} e^{-it(2|\alpha|+n)} e_{\alpha}(z) \overline{e_{\alpha}(w)} \]

\[ = e^{-int} \sum_{\alpha} e^{-2it|\alpha|} \frac{z^\alpha \overline{\alpha!}}{\alpha!} \]

\[ = e^{-int} \exp(e^{-2it} z \cdot \overline{w}). \]

Hence

\[ |K_t(z, w)| = \exp[\text{Re}(e^{-2it} z \cdot \overline{w})] \leq e^{|e^{-2it} z \cdot \overline{w}|} = e^{|z \cdot \overline{w}|}. \]

We first prove that the \( B_t = e^{-it\mathcal{S}} \) maps \( L^1 \) to \( L^\infty \) and \( L^2 \) to \( L^2 \), respectively, and combine them with Riez-Thorin interpolation to drive the desired result.
Now we can calculate that
\[
\|u(\cdot, t)\|_{F^{\infty}} = \sup_{z \in \mathbb{C}^{n}} |u(z, t)| e^{-\frac{1}{2}|z|^2}
\]
\[
\leq \sup_{z \in \mathbb{C}^{n}} \left[ \int_{\mathbb{C}^{n}} |f(w)||K_t(z, w)| e^{-\frac{1}{2}|w|^2 - \frac{1}{2}|z|^2} dV(w) \right]
\leq \sup_{z \in \mathbb{C}^{n}} \left[ \int_{\mathbb{C}^{n}} |f(w)| e^{-\frac{1}{2}|w|^2 - \frac{1}{2}|z|^2 + |z \cdot \bar{w}|} dV(w) \right]
\leq \left[ \int_{\mathbb{C}^{n}} |f(w)| e^{-\frac{1}{2}|w|^2} dV(w) \right] = \|f\|_{F^1},
\]
where we used the following relation in third inequality:
\[-|w|^2 - \frac{1}{2}|z|^2 + |z \cdot \bar{w}| \leq -|w|^2 - \frac{1}{2}|z|^2 + |z||w| \leq -\frac{1}{2}|w|^2.\]

On the other hand, for \( f \in F^2 \), we have a holomorphic expansion of \( f(z) = \sum c_{\alpha} e_{\alpha}(z) \). Then
\[
u(z, t) = e^{-it\Re f(z)} = e^{-int} \sum_{\alpha} e^{-2it|\alpha|} c_{\alpha} e_{\alpha}(z).
\]
So we have
\[
\|u(\cdot, t)\|_{F^2}^2 = \langle u(\cdot, t), u(\cdot, t) \rangle
\]
\[
= \left\langle e^{-int} \sum_{\alpha} e^{-2it|\alpha|} c_{\alpha} e_{\alpha}, e^{-int} \sum_{\beta} e^{-2it|\beta|} c_{\beta} e_{\beta} \right\rangle
\]
\[
= \sum_{\alpha, \beta} c_{\alpha} \overline{c_{\beta}} e^{-2it(|\alpha|-|\beta|)} \langle e_{\alpha}, e_{\beta} \rangle
\]
\[
= \sum_{\alpha} |c_{\alpha}|^2 = \|f\|_{F^2}^2.
\]
Hence by Riesz-Thorin interpolation theorem [6], for \( p \in [1, 2] \) we have
\[
\|u(\cdot, t)\|_{F^p} \leq \|f\|_{F^p},
\]
where \((p, q)\) is a conjugate pair.

**References**


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