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THE STABILITY OF GENERALIZED RECIPROCAL-NEGATIVE FERMAT'S EQUATIONS IN QUASI-β-NORMED SPACES

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Abstract. We introduce a reciprocal-negative Fermat's equation generalized with constants coefficients and investigate its stability in a quasi-β-normed space.

1. Introduction

In many mathematical fields we would be interested in dealing with the following question suggested first in 1940 by Ulam [32]: *Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?* In other words, we consider the conditions under which a mathematical object satisfying certain properties approximately should be close to the one satisfying the properties exactly. In 1941, Hyers [8] consider the case of linear or additive functional equation in a complete metric space, Banach space, and gave the affirmative but partial solution to Ulam's question above. This Hyers' stability result was first generalized in the

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stability involving a sum of powers of norms by T. Aoki [1], not only constants later. In 1978, Th.M. Rassias [21] provided another generalization of Hyers Theorem which allows the Cauchy difference to be unbounded. For the following sections where we show our results of stability let us define a quasi-β-normed spaces.

Let β be a real number with $0 < \beta \leq 1$ and $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. We will consider the definition and some preliminary results of a quasi-β-norm on a linear space.

Definition 1.1. Let X be a linear space over a field $\mathbb{K}$. A *quasi-β-norm* $||\cdot||$ is a real-valued function on X satisfying the followings:

(1) $||x|| \geq 0$ for all $x \in X$ and $||x|| = 0$ if and only if $x = 0$.

(2) $||\lambda x|| = |\lambda|^\beta \cdot ||x||$ for all $\lambda \in \mathbb{K}$ and all $x \in X$.

(3) There is a constant $K \geq 1$ such that $||x+y|| \leq K(||x|| + ||y||)$ for all $x,\ y \in X$.

The pair $(X, ||\cdot||)$ is called a *quasi-β-normed space* if $||\cdot||$ is a quasi-β-norm on X. The smallest possible K is called the *modulus of concavity* of $||\cdot||$. A *quasi-Banach space* is a complete quasi-β-normed space.

A quasi-β-norm $||\cdot||$ is called a $(\beta,\ p)$-norm $(0 < p \leq 1)$ if $||x+y||^p \leq ||x||^p + ||y||^p$, for all $x,\ y \in X$. In this case, a quasi-β-Banach space is called a (β, p)-Banach space; see [3] and [29].

In number theory, Fermat's Last Theorem states that no three positive integers a, b, and c satisfy the equation $c^n = a^n + b^n$ for any integer value of $n \geq 2$. Taking the reciprocal of each term in the Fermat's equation we arrive at the equation $\frac{1}{c^n} = \frac{1}{a^n} + \frac{1}{b^n}$ that is called the reciprocal-negative Fermat's equation. Solving the reciprocal equation $\frac{1}{c^n} = \frac{a^n+b^n}{a^n\, b^n}$, for c^n, we have

$$c^n = \frac{a^n\, b^n}{a^n + b^n}$$

for any integer value of $n \geq 2$. In particular, in the case of $n = 1$ the above equation should be the harmonic mean of a and b from the well-known three Pythagorean means; arithmetic mean, geometric mean, and harmonic mean in geometry.

In 2010, Ravi and Kumar [28] investigated a generalized Hyers-Ulam stability of the reciprocal functional equation $f(x+y) = \dfrac{f(x)f(y)}{f(x)+f(y)}$. Also see [11] for a fixed point approach. With the motivation of the Pythagorean means Narasimman, Ravi, and Pinelas [20] in 2015 introduced the Pythagorean mean functional equation $f(\sqrt{x^2+y^2}) =$

$\dfrac{f(x)f(y)}{f(x)+f(y)}$ for all positive numbers x and y and studied the generalized Hyers-Ulam stability of the equation providing counter-examples for singular cases. Recently Kang and Kim in [18] introduced the generalized Pythagorean mean functional equation

$$f\left(\sqrt[n]{x^n+y^n}\right)=\frac{f(x)f(y)}{f(x)+f(y)} \tag{1}$$

for a positive integer n and investigated the stabilities of the functional equation in a quasi-β-normed space.

In this paper, we consider the following weighted reciprocal-negative Fermat's functional equation:

$$f\left(\sqrt[n]{ax^n+by^n}\right)=\frac{f(x)f(y)}{bf(x)+af(y)} \tag{2}$$

for fixed positive integers n and for all $x, y \in X$ with weights a and b. We are able to see definitely that the generalized Pythagorean mean functional equation (1) given by Kang and Kim above is the special case when $a=b=1$. Due to the reciprocal-negative Fermat's equation, we still call the mapping f the reciprocal-negative Fermat's function. In Section 2 we establish the general solution of the reciprocal-negative Fermat's equation (2) in the simplest case and give the differential solution to the equation (2). In Section 3 we prove the generalized Hyers-Ulam stability of the reciprocal-negative Fermat's equation (2) in a quasi-β-normed space.

2. General Solution of the Reciprocal-negative Fermat's functional equation

In this section we establish both the general and differential solution of the weighted reciprocal-negative Fermat's equation (2) following the work by Ger [10] and Kang [18]

THEOREM 2.1 (General Solution). *Let $n \in \mathbb{N}$ be an odd integer (or even integer). The only nonzero solution $f : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ (or $f : (0, \infty) \longrightarrow \mathbb{R}$) with a finite limit of the quotient $\dfrac{f(x)}{1/x^n}$ at zero, of the equation (2) is of the form $\dfrac{c}{x^n}$ for a non-zero constant $c \in \mathbb{R}$.*

Proof. Letting $y = x$ in (2) we just have $f(\sqrt[n]{a+b}x) = \left(\frac{1}{a+b}\right) f(x)$ for all $x \in \mathbb{R} \setminus \{0\}$ (or $x \in (0, \infty)$)).
Let us define $g(x) = \frac{f(x)}{1/x}$ for all $x \in \mathbb{R} \setminus \{0\}$ (or $x \in (0, \infty)$). Then the limit

$$\lim_{x \to 0} \frac{g(x)}{\frac{1}{x^{n-1}}} = c$$

exists for some nonzero $c \in \mathbb{R}$ and using the definition of $f(x)$ we obtain

$$g\left(\sqrt[n]{a+b}x\right) = \frac{1}{\sqrt[n]{(a+b)^{n-1}}} g(x)$$

for all $x \in \mathbb{R} \setminus \{0\}$ (or $x \in (0, \infty)$). By the mathematical induction for every positive integer k, we also have

$$g\left(\frac{x}{(\sqrt[n]{a+b})^k}\right) = (\sqrt[n]{(a+b)^{n-1}})^k g(x) \tag{3}$$

for all $x \in \mathbb{R} \setminus \{0\}$ (or $x \in (0, \infty)$). Therefore we conclude from the equality (3) that

$$\frac{g(x)}{\frac{1}{x^{n-1}}} = \frac{(\sqrt[n]{(a+b)^{n-1}})^k g(x)}{(\sqrt[n]{(a+b)^{n-1}})^k \frac{1}{x^{n-1}}} = \frac{g\left(\frac{x}{(\sqrt[n]{(a+b)})^k}\right)}{\left(\frac{(\sqrt[n]{(a+b)})^k}{x}\right)^{n-1}} \longrightarrow c \tag{4}$$

as $n \to \infty$. By the definition of $g(x)$ we get the general solution

$$f(x) = \frac{1}{x} g(x) = \frac{1}{x}\left(\frac{c}{x^{n-1}}\right) = \frac{c}{x^n}$$

for all $x \in \mathbb{R} \setminus \{0\}$ (or $x \in (0, \infty)$), which completes the proof. □

Now we consider the differentiable solution of the reciprocal-negative Fermat's functional equation (2) as we suggested. For simplicity we will assume the case of an odd integer $n \in \mathbb{N}$ (we can prove the even case similarly).

THEOREM 2.2 (Differential Solution). *Let $f : (0, \infty) \longrightarrow \mathbb{R}$ be continuously differentiable function with the derivative $f'(x) \neq 0$ for all $x \in (0, \infty)$. Then f is a solution to the reciprocal-negative Fermat's* *equation (2) if and only if there exists a nonzero constant $c \in \mathbb{R}$ such that $f(x) = \dfrac{c}{x^n}$ for all $x \in (0, \infty)$.*

Proof. A simple computation of differentiation of the equation (2) with respect to x on both sides gives

$$f'(\sqrt[n]{ax^n + by^n})\left(\frac{x}{\sqrt[n]{ax^n + by^n}}\right)^{n-1} = \frac{f'(x)(f(y))^2}{(bf(x) + af(y))^2} \tag{5}$$

for all $x, y \in (0, \infty)$. Substituting $y = x$ in the equation (2) and the equation (5) above, respectively, we have

$$f(\sqrt[n]{a+b}x) = \left(\frac{1}{a+b}\right) f(x) \tag{6}$$

and

$$f'(\sqrt[n]{a+b}x) = \frac{1}{(a+b)^{\frac{n+1}{n}}} f'(x) \tag{7}$$

for all $x \in (0, \infty)$. Letting $y = \sqrt[n]{\frac{b+1}{b}}x$ in (5) again and applying (6) and (7) we can have

$$f'(\sqrt[n]{a+b+1}x) = \frac{1}{(a+b+1)^{\frac{n+1}{n}}} f'(x) \tag{8}$$

for all $x \in (0, \infty)$. Both equations (7) and (8) gives

$$f'((\sqrt[n]{a+b})^l(\sqrt[n]{a+b+1})^m x) = \frac{1}{((a+b)^{\frac{n+1}{n}})^l((a+b+1)^{\frac{n+1}{n}})^m} f'(x) \tag{9}$$

for all integers l and m. It can be easily proved that the set $\{((a+b)^{\frac{n+1}{n}})^l((a+b+1)^{\frac{n+1}{n}})^m : l, m \in \mathbb{Z}\}$ is dense in $(0, \infty)$ for fixed constants a and b. Since we assume that the function f' is continuous we derive the following first order ordinary differential equation

$$f'(\lambda) = f'(1)\frac{1}{\lambda^{n+1}} \tag{10}$$

for $\lambda \in (0, \infty)$. Therefore, the solution of the equation should be $f(x) = \dfrac{c}{x^n} + d$ for some constants c and d for $x \in (0, \infty)$. It is also obvious that the constant d should be zero since $f(\sqrt[n]{a+b}x) = \left(\dfrac{1}{a+b}\right) f(x)$ and it completes the proof.

□

3. Stability of a Reciprocal-negative Fermat's functional equation

We assume that in this entire section X is a linear space and Y a quasi-β-Banach space with a quasi-β-norm $||\cdot||_Y$. Let also K be the modulus of concavity of $||\cdot||_Y$. In this section we will investigate the generalized Hyers-Ulam stability problem for the functional equation (2) as we suggested. For a given mapping $f : X \to Y$ and a fixed positive integer n, we denote

$$D_n f(x,y) := f\Big(\sqrt[n]{ax^n + by^n}\Big) - \frac{f(x)f(y)}{bf(x) + af(y)}$$

for all $x, y \in X$ and $\mathbb{R}^+ := [0,\infty)$, i.e., the set of all nonnegative real numbers where the constants a and b are nonzero real numbers.

THEOREM 3.1. *Assume that there exists a function $\phi : X \times X \to \mathbb{R}^+$ for which a function $f : X \to Y$ satisfies*

$$||D_n f(x,y)||_Y \le \phi(x,y) \tag{11}$$

and also suppose that the series $\sum_{j=0}^{\infty}((a+b)^\beta K)^j \phi((\sqrt[n]{a+b})^j x, (\sqrt[n]{a+b})^j y)$ converges for all x, $y \in X$. Then there will be a unique reciprocal-negative Fermat's function $R : X \to Y$ which satisfies the equation (2) and the following inequality

$$||f(x) - R(x)||_Y \le \sum_{j=0}^{\infty}((a+b)^\beta K)^{j+1}\phi((\sqrt[n]{a+b})^j x, (\sqrt[n]{a+b})^j x)\,, \tag{12}$$

for all $x \in X$.

Proof. On letting $x = y$ in the equation (11), we have

$$||D_n f(x,x)||_Y = ||\frac{f(x)}{a+b} - f(\sqrt[n]{a+b}x)||_Y \le \phi(x,\, x)$$

or,

$$||f(x) - (a+b)f(\sqrt[n]{a+b}x)||_Y \le (a+b)^\beta \phi(x,\, x) \tag{13}$$

for all $x \in X$. Letting m be a fixed positive integer we note that putting $x = (\sqrt[n]{a+b})^m x$ and multiplying by $(a+b)^{m\beta}$ in the inequality (13), we can obtain

$$(14)\quad \begin{aligned} &||(a+b)^m f((\sqrt[n]{a+b})^m x) - (a+b)^{m+1} f((\sqrt[n]{a+b})^{m+1} x)||_Y \\ &\le (a+b)^{(m+1)\beta}\phi((\sqrt[n]{a+b})^m x,\, (\sqrt[n]{a+b})^m x) \end{aligned}$$

for all $x \in X$. By the mathematical induction, we conclude the following inequality:

$$(15)\quad \begin{aligned} &||f(x) - (a+b)^m f((\sqrt[n]{a+b})^m x)||_Y \\ &\le \sum_{j=0}^{m-1} ((a+b)^\beta K)^{j+1}\phi((\sqrt[n]{a+b})^j x,\, (\sqrt[n]{a+b})^j x) \end{aligned}$$

for any positive integer m and for all $x \in X$. In addition, for all positive integers s and t with $s > t$, we have

$$(16)\quad \begin{aligned} &||(a+b)^t f((\sqrt[n]{a+b})^t x) - (a+b)^s f((\sqrt[n]{a+b})^s x)||_Y \\ &\le \sum_{j=t}^{s-1} ((a+b)^\beta K)^{j+1}\phi((\sqrt[n]{a+b})^j x,\, (\sqrt[n]{a+b})^j x) \end{aligned}$$

for all $x \in X$. Since we assume that $\sum_{j=0}^{\infty}((a+b)^\beta K)^j \phi((\sqrt[n]{a+b})^j x, (\sqrt[n]{a+b})^j y)$ converges, the right-hand side of the inequality (16) tends to 0 as $t \to \infty$. Thus we just say that $\{(a+b)^m f((\sqrt[n]{a+b})^m x)\}$ is a Cauchy sequence in the quasi-β-Banach space Y . Thus we are able to let

$$R(x) = \lim_{m\to\infty} (a+b)^m f((\sqrt[n]{a+b})^m x)$$

for each $x \in X$. Now, we will show that $R(x)$ is the solution to the reciprocal-negative Fermat's equation (2). For a positive integer m letting $x = (\sqrt[n]{a+b})^m x$ and $y = (\sqrt[n]{a+b})^m y$ and multiplying by $(a+b)^{m\beta}$ in the inequality (11), we get

$$\begin{aligned} & (a+b)^{m\beta}||D_n f((\sqrt[n]{a+b})^m x,\, (\sqrt[n]{a+b})^m y)||_Y \\ = \ & (a+b)^{m\beta}||f((\sqrt[n]{a+b})^m \sqrt[n]{ax^n+by^n}) - \frac{f((\sqrt[n]{a+b})^m x) f((\sqrt[n]{a+b})^m y)}{bf((\sqrt[n]{a+b})^m x) + af((\sqrt[n]{a+b})^m y)}||_Y \\ \le \ & ((a+b)^\beta K)^m \phi((\sqrt[n]{a+b})^m x,\, (\sqrt[n]{a+b})^m y) \end{aligned}$$

for all $x\,, y \in X$. Letting m tend to the infinity, $m \to \infty$, $R(x)$ satisfies (2) for all $x,\, y \in X$, that is, $R(x)$ is the reciprocal-negative Fermat's function as the solution to it. Also, the inequality (15) implies the inequality (12).

Now, we finally have to show the uniqueness of the reciprocal-negative

Fermat's function $R(x)$. In order to do that we assume that there exists $r : X \to Y$ satisfying (2) and (12). Then we can estimate

$$\begin{aligned} ||R(x) - r(x)||_Y &= (a+b)^{m\beta}||R((\sqrt[n]{a+b})^m x) - r((\sqrt[n]{a+b})^m x)||_Y \\ &\le K(a+b)^{m\beta}\Big(||R((\sqrt[n]{a+b})^m x) - f(\sqrt[n]{a+b})^m x)||_Y \\ &\quad +||r((\sqrt[n]{a+b})^m x) - f(\sqrt[n]{a+b})^m x)||_Y\Big) \\ &\le 2K^{1-m}\sum_{j=0}^{\infty}((a+b)^\beta K)^{j+m+1}\phi((\sqrt[n]{a+b})^{j+m}x, (\sqrt[n]{a+b})^{j+m}x) \end{aligned}$$

for all $x \in X$. By letting $m \to \infty$, we just have the uniqueness of the reciprocal-negative Fermat's function $R(x)$ that completes the proof. □

Now let us present a counterpart of Theorem 3.1 by correcting the approximate $f(x)$ in (11) by scaling-down:

THEOREM 3.2. *Suppose that there exists a mapping $\phi : X \times X \to \mathbb{R}^+$ for which a mapping $f : X \to Y$ satisfies*

$$||D_n f(x,y)||_Y \le \phi(x,y) \tag{17}$$

and the series $\sum_{j=0}^{\infty}\left(\frac{K}{(a+b)^\beta}\right)^j \phi((\sqrt[n]{a+b})^{-j}x, (\sqrt[n]{a+b})^{-j}y)$ converges for all $x,\ y \in X$. Then there exists a unique reciprocal-negative Fermat's function $R : X \to Y$ which satisfies the equation (2) and the inequality

(18)

$$||f(x) - R(x)||_Y \le \sum_{j=1}^{\infty}\left(\frac{1}{a+b}\right)^{j-1} K^j \phi((\sqrt[n]{a+b})^{-j}x, (\sqrt[n]{a+b})^{-j}x)\,,$$

for all $x \in X$.

Proof. The proof can easily obtained by starting with the replacement $x = y = \dfrac{x}{\sqrt[n]{a+b}}$ in (17) as we did in Theorem 3.1. □

Now we have the following Hyers-Ulam-Rassias type stability of the functional equation (2).

COROLLARY 3.3. *Let X be a quasi-β normed space with a norm $||\cdot||$ and take a constant $p > \left(\frac{n}{\beta}\right)\left(\frac{\ln K}{\ln(a+b)} - n\right)$. Suppose that*

$f : X \to Y$ satisfies

$$||D_n f(x,y)||_Y \le c(||x||^p + ||y||^p) \tag{19}$$

for all $x, y \in X$ with a nonnegative constant c. Then there exists a unique function $R : X \to Y$ such that

$$||f(x) - R(x)||_Y \le \left(\frac{2c(a+b)^{(\beta p/n)+\beta} K}{(a+b)^{(\beta p/n)+\beta} - K} \right) ||x||^p \tag{20}$$

for each $x \in X$.

Proof. Just replacing $\phi(x,y) = c(||x||^p + ||y||^p)$ in Theorem 3.2 completes the proof. □

Remark 3.4. By the property of stability of the reciprocal-negative Fermat's equation (2) from Theorem 3.1 and 3.2 we also get the corresponding result to Corollary 3.3 as a consequence of Theorem 3.1, i.e.,

$$||f(x) - R(x)||_Y \le \left(\frac{2c(a+b)^{-(\beta p/n)-\beta} K}{(a+b)^{-(\beta p/n)-\beta} - K} \right) ||x||^p \tag{21}$$

for $p > \left(\frac{n}{\beta}\right)\left(\frac{-\ln K}{\ln 2} - n\right)$.

Remark 3.5. In physics a weighted parallel circuit with two resistors would be an application of the reciprocal-negative Fermat's equation (2). The following law is well-know from physics: The inverse of total resistance r of the circuit is sum of the inverses of the individual resistances r_1 and r_2,

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$$

or

$$r = \frac{r_1 r_2}{r_1 + r_2}$$

Take $r_1 = \frac{b}{x^n}$ and $r_2 = \frac{a}{y^n}$ for a weighted parallel circuit with weights a and b for two resistors r_1 and r_2, respectively, leads us to have

$$r = \frac{\frac{b}{x^n}\frac{a}{y^n}}{\frac{b}{x^n} + \frac{a}{y^n}}. \tag{22}$$

It is well-known that the electric conductance is reciprocal to the resistance and we, thus, have the total conductance g of the circuit as $g = \dfrac{x^n}{b} + \dfrac{y^n}{a}$. From the equation (22) we can have

$$\frac{1}{g} = \frac{\frac{b}{x^n}\frac{a}{y^n}}{\frac{b}{x^n} + \frac{a}{y^n}}, \tag{23}$$

that is,

$$1/g = \frac{1}{x^n/b + y^n/a} = \frac{\frac{b}{x^n}\frac{a}{y^n}}{\frac{b}{x^n} + \frac{a}{y^n}}, \tag{24}$$

which is exactly the reciprocal-negative Fermat's equation (2) if $f(x) = \dfrac{c}{x^n}$ for some constant c and the stability of this circuit problem can play an important role in physics as we showed earlier.

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