LEHMER’S GENERALIZED EULER NUMBERS IN HYPERGEOMETRIC FUNCTIONS

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Abstract. In 1935, D. H. Lehmer introduced and investigated generalized Euler numbers $W_n$, defined by

$$3 e^t + e^{\omega t} + e^{\omega^2 t} = \sum_{n=0}^{\infty} W_n \frac{t^n}{n!},$$

where $\omega$ is a complex root of $x^2 + x + 1 = 0$. In 1875, Glaisher gave several interesting determinant expressions of numbers, including Bernoulli and Euler numbers. These concepts can be generalized to the hypergeometric Bernoulli and Euler numbers by several authors, including Ohno and the second author. In this paper, we study more general numbers in terms of determinants, which involve Bernoulli, Euler and Lehmer’s generalized Euler numbers. The motivations and backgrounds of the definition are in an operator related to Graph theory. We also give several expressions and identities by Trudi’s and inversion formulae.

1. Introduction

In 1935, D. H. Lehmer [17] introduced and investigated generalized Euler numbers $W_n$, defined by the generating function

$$3 e^t + e^{\omega t} + e^{\omega^2 t} = \sum_{n=0}^{\infty} W_n \frac{t^n}{n!},$$

where $\omega = -\frac{1 + \sqrt{3}}{2}$ and $\omega^2 = \bar{\omega} = -\frac{1 - \sqrt{3}}{2}$ are the primitive cube roots of unity. Notice that $W_n = 0$ unless $n \equiv 0 \pmod{3}$. The sequence of these numbers is given by

$$\{W_{3n}\}_{n \geq 0} = 1, -1, 19, -1513, 315523, -136085041, 105261234643, -132705221399353, 254604707462013571, \ldots$$

and the sequence of these absolute values is recorded in [19, A002115]. In [14], the complementary numbers $W_n^{(j)}$ ($j = 0, 1, 2$) to Lehmer’s Euler numbers are

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defined by the generating function

\[ \sum_{n=0}^{\infty} W_n(t) t^n / n! = \left( 1 + \sum_{l=1}^{\infty} \frac{t^{3l}}{(3l+j)!} \right)^{-1}. \]

Notice that \( W_n(j) = 0 \) unless \( n \equiv 0 \pmod{3} \). When \( j = 0 \), \( W_n = W_n(0) \) are the original Lehmer’s Euler numbers. When \( j = 1 \), we also have

\[ \sum_{n=0}^{\infty} \frac{W_n(1) t^n}{n!} = \frac{3t}{e^t + \omega^2 e^{\omega t} + \omega e^{\omega^2 t}}. \]

Lehmer’s Euler numbers and their complementary numbers \( W_n(j) \) can be considered analogous of the classical Euler numbers \( E_n \) and their complementary Euler numbers \( \hat{E}_n \) ([11, 16]). For, their generating functions are given by

\[ \sum_{n=0}^{\infty} E_n(t) t^n / n! = \frac{1}{\cosh t} = \frac{2}{e^t + e^{-t}} = \left( \sum_{l=0}^{\infty} \frac{t^{2l}}{(2l)!} \right)^{-1}, \]

and

\[ \sum_{n=0}^{\infty} \hat{E}_n(t) t^n / n! = \frac{t}{\sinh t} = \frac{2t}{e^t - e^{-t}} = \left( \sum_{l=0}^{\infty} \frac{t^{2l}}{(2l + 1)!} \right)^{-1}, \]

respectively. For \( N \geq 0 \) hypergeometric Euler numbers \( E_{N,n} \) ([11, 16]) are defined by

\[ \frac{1}{_{1}F_{2}(1;N+1,(2N+1)/2;t^2/4)} \frac{t^{2N}/(2N)!}{\cosh t - \sum_{n=0}^{N-1} \frac{t^{2n}/(2n)!}{n!}}, \]

where \(_{1}F_{2}(a; b, c; z)\) is the hypergeometric function defined by

\[ _{1}F_{2}(a; b, c; z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)} z^n}{(b)^{(n)}(c)^{(n)} n!}. \]

Here \((x)^{(n)}\) denotes the rising factorial, defined by \((x)^{(n)} = x(x+1)\cdots(x+n-1)\) \((n \geq 1)\) with \((x)^{(0)} = 1\). When \( N = 0 \), \( E_n = E_{0,n} \) are the original Euler numbers. Similarly, hypergeometric Euler numbers of the second kind \( \hat{E}_{N,n} \) ([11, 16]) are defined by

\[ \frac{1}{_{1}F_{2}(1;N+1,(2N+3)/2;t^2/4)} \frac{t^{2N+1}/(2N+1)!}{\sinh t - \sum_{n=0}^{N} \frac{t^{2n+1}/(2n+1)!}{n!}}. \]
When $N = 0$, $E_n = E_{0,n}$ are the original complementary Euler numbers.
There are many kinds of generalizations of Euler numbers, but hypergeometric Euler numbers have advantages as natural extensions in terms of determinant expressions ([11,16]). For $N \geq 0$ and $n \geq 1$, we have

$$E_{N,2n} = (-1)^n (2n)! \begin{vmatrix} (2N)! & 1 & \cdots & 1 \\ (2N+2)! & (2N+4)! & \cdots & (2N+2)! \\ \vdots & \vdots & \ddots & \vdots \\ (2N+2n-2)! & (2N+2n-4)! & \cdots & (2N+2)! \\ (2N+2n)! & (2N+2n-2)! & \cdots & (2N+2)! \\ \end{vmatrix},$$

and

$$\hat{E}_{N,2n} = (-1)^n (2n)! \begin{vmatrix} (2N+1)! & 1 & \cdots & 1 \\ (2N+1)! & (2N+3)! & \cdots & (2N+1)! \\ \vdots & \vdots & \ddots & \vdots \\ (2N+2n-1)! & (2N+2n-3)! & \cdots & (2N+1)! \\ (2N+2n+1)! & (2N+2n-1)! & \cdots & (2N+1)! \\ \end{vmatrix}.$$  

When $N = 0$ in (6), we have a famous determinant expression of Euler numbers discovered by Glaisher in 1875 ([3, p. 52]).

Similar hypergeometric numbers are hypergeometric Bernoulli numbers $B_{N,n}$ ([4–9]), defined by

$$\frac{1}{1_F(1; N+1; t)} = e^t - \sum_{n=0}^{N-1} \frac{t^n}{n!} = \sum_{n=0}^{\infty} B_{N,n} \frac{t^n}{n!},$$

where $1_F(a; b; z)$ is the confluent hypergeometric function defined by

$$1_F(a; b; z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}}{(b)^{(n)}} \frac{z^n}{n!}.$$

The determinant expression of hypergeometric Bernoulli numbers are given by

$$B_{N,n} = (-1)^n n! \begin{vmatrix} N! & 1 & \cdots & N! \\ (N+1)! & (N+1)! & \cdots & (N+1)! \\ \vdots & \vdots & \ddots & \vdots \\ (N+n-1)! & (N+n-2)! & \cdots & (N+1)! \\ (N+n)! & (N+n-1)! & \cdots & (N+2)! \\ \end{vmatrix}.$$

When $N = 1$, $B_n = B_{1,n}$ are the classical Bernoulli numbers defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$
The determinant expression for the classical Bernoulli numbers was discovered by Glaisher ([3, p. 53]).

In this paper, we introduce and study the hypergeometric Euler numbers of higher grade are introduced as total generalizations of hypergeometric Euler numbers, hypergeometric Euler numbers of the second kind, hypergeometric Bernoulli numbers as well as Lehmer’s generalized Euler numbers.

2. Hypergeometric Lehmer-Euler numbers of higher grade

For \( N, n \geq 0 \), define hypergeometric Lehmer-Euler numbers \( W^{(j)}_{N,n,r} \) \( (j = 0, 1) \) of grade \( r \) by

\[
\sum_{n=0}^{\infty} W^{(j)}_{N,n,r} \frac{t^n}{n!} = \left( _1F_r \left( 1; \frac{rN + j + 1}{r}, \frac{rN + j + 2}{r}, \ldots, \frac{rN + j + r}{r}; \left( \frac{t}{r} \right)^r \right) \right)^{-1},
\]

where \(_1F_r(a; b_1, \ldots, b_r; z)\) is the hypergeometric function, defined by

\[
_1F_r(a; b_1, \ldots, b_r; z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}}{(b_1)^{(n)} \cdots (b_r)^{(n)}} \frac{z^n}{n!}.
\]

From the definition, \( W^{(j)}_{N,n,r} \equiv 0 \pmod{r} \) unless \( n \equiv 0 \pmod{r} \). When \( N = 0 \) and \( r = 3 \) in (11), \( W^{(j)}_{0,n,3} \) are the Lehmer’s generalized Euler numbers \( (j = 0) \) in (1) and their complementary numbers \( (j = 1) \) in (3). These numbers where \( r = 3 \) have been extensively studied by Ohno and the second author ([14]), including congruence properties. When \( N = 0 \) and \( r = 2 \) in (11), \( E^{(j)}_{n} = W^{(j)}_{0,n,2} \) are the classical Euler numbers \( (j = 0) \) in (4) and their complementary numbers \( (j = 1) \) in (5). When \( r = 1 \) and \( j = 0 \) in (11), \( B_{N,n} = W^{(0)}_{N,n,1} \) are the hypergeometric Bernoulli numbers. When \( N = r = 1 \) and \( j = 0 \) in (11), \( B_{n} = W^{(0)}_{1,n,1} \) are the classical Bernoulli numbers in (10).

We can write (11) as

\[
_1F_r \left( 1; \frac{rN + j + 1}{r}, \frac{rN + j + 2}{r}, \ldots, \frac{rN + j + r}{r}; \left( \frac{t}{r} \right)^r \right)
\]

\[
= \sum_{n=0}^{\infty} \frac{(rN + j + 1)(rN + j + 2) \cdots (rN + j + rn)}{(rN + j + 1)! \cdots (rN + j + rn)!} t^n
\]

\[
= \sum_{n=0}^{\infty} \frac{(rN + j)!}{(rN + j + 1)! \cdots (rN + j + rn)!} t^n = 1 + \sum_{n=1}^{\infty} \frac{(rN + j)!}{(rN + j + 1)! \cdots (rN + j + rn)!} t^n.
\]
When \( N = 0 \) in (11), \( W_{n,r}^{(j)} = W_{0,n,r}^{(j)} \) are the Lehmer-Euler numbers of grade \( r \), defined by

\[
\sum_{n=0}^{\infty} W_{n,r}^{(j)} \frac{t^n}{n!} = \binom{1}{r} \left( \frac{1}{F_r \left( \frac{j+1}{r} , \frac{j+2}{r} , \ldots , \frac{j+r}{r} ; \left( \frac{t}{r} \right)^r \right) } \right)^{-1}.
\]

If \( r = p \) is prime, the generating functions of Lehmer-Euler numbers of degree \( p \) can be expressed by

\[
\sum_{n=0}^{\infty} W_{n,p}^{(0)} \frac{t^n}{n!} = \frac{r}{\sum_{t=0}^{p-1} e^{\zeta_p t}}
\]

and

\[
\sum_{n=0}^{\infty} W_{n,p}^{(1)} \frac{t^n}{n!} = \frac{rt}{\sum_{t=0}^{p-1} \zeta_p^{p-t} e^{\zeta_p t}},
\]

where \( \zeta_p \) is the (primitive) \( p \)-th root of unity.

The definition (11) with (12) may be obvious or artificial for the readers with different backgrounds. However, there are motivations from Combinatorics, in particular, graph theory. In 1989, Cameron [2] considered the operator \( A \) defined on the set of sequences of non-negative integers as follows: for \( x = \{x_n\}_{n \geq 1} \) and \( z = \{z_n\}_{n \geq 1} \), set \( Ax = z \), where

\[
1 + \sum_{n=1}^{\infty} z_n t^n = \left( 1 - \sum_{n=1}^{\infty} x_n t^n \right)^{-1}.
\]

Suppose that \( x \) enumerates a class \( C \). Then \( Ax \) enumerates the class of disjoint unions of members of \( C \), where the order of the component members of \( C \) is significant. The operator \( A \) also plays an important role for free associative (non-commutative) algebras. More motivations and background together with many concrete examples (in particular, in the aspects of Graph theory) by this operator can be seen in [2]. In the sense of Cameron’s operator \( A \), we have the following relations:

\[
A \left\{ \frac{N!}{(N+n)!} \right\} = \left\{ \frac{B_{N,n}}{n!} \right\},
\]

\[
A \left\{ -\frac{(2N)!}{(2N+2n)!} \right\} = \left\{ \frac{E_{N,2n}}{(2n)!} \right\},
\]

\[
A \left\{ -\frac{(2N+1)!}{(2N+2n+1)!} \right\} = \left\{ \frac{\hat{E}_{N,2n}}{(2n)!} \right\},
\]

\[
A \left\{ -\frac{(3N+j)!}{(3N+3n+j)!} \right\} = \left\{ \frac{W_{N,3n,3}^{(j)}}{(3n)!} \right\}.
\]

These relations are interchangeable in the sense of determinants too. See the Section 5 about Trudi’s formula.

We have the following recurrence relation.
Proposition 2.1. For \( N \geq 0 \) and \( j = 0, 1 \), we have

\[
W^{(j)}_{N,rn,r} = -\sum_{k=0}^{n-1} \frac{(rn)!(rN+j)!}{(rN+rn-rk+j)!(rk)!} W^{(j)}_{N,rk,r} \quad (n \geq 1)
\]

with \( W^{(j)}_{N,0,r} = 1 \).

Proof. By (11), we get

\[
1 = \left( 1 + \sum_{l=1}^{\infty} \frac{(rN+j)!}{(rN+rl+j)!} tr^l \right) \left( \sum_{n=0}^{\infty} W^{(j)}_{N,rn,r} \frac{t^{rn}}{(rn)!} \right)
\]

\[
= \sum_{n=0}^{\infty} W^{(j)}_{N,rn,r} \frac{t^{rn}}{(rn)!} + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{(rN+j)!W^{(j)}_{N,rk,r}}{(rN+rn-rk+j)!(rk)!} t^{rn}.
\]

Comparing the coefficient on both sides, we obtain

\[
W^{(j)}_{N,rn,r} \frac{(rn)!}{(rn)!} + \sum_{k=0}^{n-1} \frac{(rN+j)!W^{(j)}_{N,rk,r}}{(rN+rn-rk+j)!(rk)!} = 0 \quad (n \geq 1).
\]

We have an explicit expression of \( W^{(j)}_{N,n,r} \).

Theorem 2.2. Let \( j = 0, 1 \). For \( n \geq 1 \),

\[
W^{(j)}_{N,rn,r} = (rn)! \sum_{k=1}^{n} (-1)^k \sum_{i_1+\cdots+i_k=n \atop i_1,\ldots,i_k \geq 1} \frac{(rN+j)!(rN+ri_1+j)! \cdots (rN+ri_k+j)!}{(rN+ri_1+j)! \cdots (rN+ri_k+j)!}.
\]

Proof. The proof is done by induction for \( n \). From Proposition 2.1 with \( n = 1 \),

\[
W^{(j)}_{N,r,r} = -\frac{r!(rN+j)!}{(rN+j+r)!} W^{(j)}_{N,0,r} = -\frac{r!(rN+j)!}{(rN+j+r)!}.
\]

This matches the result when \( n = 1 \). Assume that the result is valid up to \( n - 1 \). Then by Proposition 2.1

\[
\frac{W^{(j)}_{N,rn,r}}{(rn)!} = -\sum_{l=0}^{n-1} \frac{(rN+j)!}{(rN+rn-rl+j)!(rl)!} W^{(j)}_{N,rl,r} \]

\[
= -\sum_{l=1}^{n-1} \frac{(rN+j)!}{(rN+rn-rl+j)!(rl)!} \sum_{k=1}^{l} (-1)^k
\]

\[
\times \sum_{i_1+\cdots+i_k=l \atop i_1,\ldots,i_k \geq 1} \frac{(rN+j)!(rN+ri_1+j)! \cdots (rN+ri_k+j)!}{(rN+ri_1+j)! \cdots (rN+ri_k+j)!}
\]

\[
= -\frac{(rN+j)!}{(rN+rn+j)!}.
\]
\begin{align*}
&= - \sum_{k=1}^{n-1} (-1)^k \sum_{l=k}^{n-1} \frac{(rN + j)!}{(rN + rn - rl + j)!} \\
&\quad \times \sum_{i_1 + \ldots + i_k = l \atop i_1, \ldots, i_k \geq 1} \frac{((rN + j)!)^k}{(rN + ri_1 + j)! \cdots (rN + ri_k + j)!} \\
&\quad - \frac{(rN + j)!}{(rN + rn + j)!} \\
&= - \sum_{k=2}^{n} (-1)^{k-1} \sum_{l=k-1}^{n-1} \frac{(rN + j)!}{(rN + rn - rl + j)!} \\
&\quad \times \sum_{i_1 + \ldots + i_{k-1} = l \atop i_1, \ldots, i_{k-1} \geq 1} \frac{((rN + j)!)^{k-1}}{(rN + ri_1 + j)! \cdots (rN + ri_{k-1} + j)!} \\
&\quad = \frac{(rN + j)!}{(rN + rn + j)!} \\
&= \frac{n}{(rN + rn + j)!} \\
&= \frac{(rN + j)!}{(rN + rn + j)!} \\
&\quad (n - l = i_k) \\
&= \frac{n}{(rN + rn + j)!} \sum_{i_1 + \ldots + i_k = n \atop i_1, \ldots, i_k \geq 0} \frac{((rN + j)!)^k}{(rN + ri_1 + j)! \cdots (rN + ri_k + j)!}.
\end{align*}

There is an alternative form of $W^{(j)}_{N, r, n}$ by using binomial coefficients. The proof is similar to that of Theorem 2.2 and is omitted.

**Theorem 2.3.** For $n \geq 1$,

\[
W^{(j)}_{N, r, n} = (rn)! \sum_{k=1}^{n} (-1)^k \binom{n + 1}{k + 1} \sum_{i_1 + \ldots + i_k = n \atop i_1, \ldots, i_k \geq 0} \frac{((rN + j)!)^k}{(rN + ri_1 + j)! \cdots (rN + ri_k + j)!}.
\]

### 3. Determinant expressions

In this section, we shall show an expression in terms of determinants. This result is a generalization of those of Bernoulli, Euler numbers and Lehmer’s Euler numbers of grade 3.

**Theorem 3.1.** For $n \geq 1$,

\[
W^{(j)}_{N, r, n} = (-1)^n (rn)!
\]
Proof. For simplicity, put \( \bar{W}_{N,n} = (-1)^{n/r} W_{N,n,r}/n! \). Then, we shall prove that for any \( n \geq 1 \)

\[
\begin{align*}
\bar{W}_{N,n} & = (rN + j)! \\
& \quad \text{for all } r, N, 0 \leq j \leq rN - 1.
\end{align*}
\]

(15) \( \bar{W}_{N,rn} = \)

\[
\begin{pmatrix}
\frac{(rN+j)!}{(rN+j+r)!} & \frac{(rN+j)!}{(rN+j+2r)!} & \cdots & \frac{(rN+j)!}{(rN+j+r)!} \\
\frac{(rN+j)!}{(rN+j+r)!} & \frac{(rN+j)!}{(rN+j+r)!} & \cdots & \frac{(rN+j)!}{(rN+j+r)!} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{(rN+j)!}{(rN+j+r)!} & \frac{(rN+j)!}{(rN+j+r)!} & \cdots & \frac{(rN+j)!}{(rN+j+r)!}
\end{pmatrix}
\]

\( W_{N,r} = \)

\[
\begin{align*}
\frac{(rN+j)!}{(rN+j + r)!} \\
\frac{(rN+j)!}{(rN+j + 2r)!} \\
\vdots \\
\frac{(rN+j)!}{(rN+j + r)!}
\end{align*}
\]

When \( n = 1 \), (15) is valid because by Theorem 2.2

\[
\bar{W}_{N,r} = \frac{(rN + j)!}{(rN + j + r)!}.
\]

Assume that (15) is valid up to \( n - 1 \). Notice that by Proposition 2.1, we have

\[
\bar{W}_{N,rn} = \sum_{k=0}^{n-1} (-1)^{n-k-1} \frac{(rN + j)!}{(rN + rn - rk + j)!} \bar{W}_{N,rk}.
\]

Thus, by expanding the first row of the right-hand side (15), it is equal to

\[
\begin{align*}
\frac{(rN+j)!}{(rN+j+r)!} \bar{W}_{N,rn-r} & = \frac{(rN+j)!}{(rN+j+2r)!} \frac{(rN+j)!}{(rN+j+r)!} \\
& \quad \text{for all } r, N, 0 \leq j \leq rN - 1.
\end{align*}
\]

\[
\begin{align*}
& = \frac{(rN+j)!}{(rN+j+r)!} \bar{W}_{N,rn-r} - \frac{(rN+j)!}{(rN+j+2r)!} \bar{W}_{N,rn-2r} \\
& \quad + \cdots + (-1)^n \frac{(rN+j)!}{(rN+j+r)!} \frac{(rN+j)!}{(rN+j+r)!} \\
& = \sum_{k=0}^{n-1} (-1)^{n-k-1} \frac{(rN+j)!}{(rN+rN - rk + j)!} \bar{W}_{N,rk} = \bar{W}_{N,rn}.
\end{align*}
\]
Note that $\tilde{W}_{N,r} = \frac{(rN+j)!}{(rN+j+r)!}$ and $\tilde{W}_{N,0} = 1$.

Remark 3.2. When $N = 0$, $r = 3$ and $j = 0,1$, we have an determinant expression of the Lehmer’s Euler numbers and their complementary numbers $W^{(j)}_{3n} = W^{(j)}_{0,3n,3}$ ([14]):

$$W^{(j)}_{3n} = (-1)^n(3n)! \left| \begin{array}{ccc}
\frac{1}{(j+3)!} & \frac{1}{(j+6)!} & \cdots \\
\frac{1}{(3n+j-3)!} & \frac{1}{(3n+j-6)!} & \cdots \\
\frac{1}{(3n+j)!} & \frac{1}{(3n+j-3)!} & \cdots \\
\frac{1}{(j+6)!} & \frac{1}{(j+3)!} & \cdots
\end{array} \right|.$$

However, $W^{(2)}_{0,3n,3} \neq W^{(2)}_{3n}$ because these generating functions are given by

$$\sum_{n=0}^{\infty} W^{(2)}_{0,3n,3} \frac{t^{3n}}{(3n)!} = \left(1 + \sum_{l=1}^{\infty} \frac{2t^{3l}}{(3l+2)!}\right)^{-1}$$

and

$$\sum_{n=0}^{\infty} W^{(2)}_{3n} \frac{t^{3n}}{(3n)!} = \left(1 + \sum_{l=1}^{\infty} \frac{t^{3l}}{(3l+2)!}\right)^{-1},$$

respectively.

4. Incomplete Lehmer-Euler numbers

In order to generalize the hypergeometric numbers $W^{(j)}_{N,n,r}$, we shall introduce two kinds of incomplete Lehmer-Euler numbers. Similar but slightly different kinds of incomplete numbers are considered in [10,12,13,15]. For $j = 0,1$ and $n \geq m \geq 1$, define the restricted hypergeometric Lehmer-Euler numbers $W^{(j)}_{N,n,r,\leq m}$ of grade $r$ by

$$\sum_{n=0}^{\infty} W^{(j)}_{N,n,r,\leq m} \frac{t^n}{n!} = \left(1 + \sum_{l=1}^{m} \frac{(rN+j)!}{(rN+rl+j)!} t^rl\right)^{-1}$$

and the associated hypergeometric Lehmer-Euler numbers $W^{(j)}_{N,n,r,\geq m}$ of grade $r$ by

$$\sum_{n=0}^{\infty} W^{(j)}_{N,n,r,\geq m} \frac{t^n}{n!} = \left(1 + \sum_{l=m}^{\infty} \frac{(rN+j)!}{(rN+rl+j)!} t^rl\right)^{-1}.$$

When $m \to \infty$ in (16) and $m = 1$ in (17), $W^{(j)}_{N,n,r} = W^{(j)}_{N,n,r,\leq \infty} = W^{(j)}_{N,n,r,\geq 1}$ are the original hypergeometric Lehmer-Euler numbers of grade $r$, defined in (11) with (12). Hence, both incomplete numbers are reduced to the hypergeometric Lehmer-Euler numbers too.

Notice that $W^{(j)}_{N,n,r,\leq m} = W^{(j)}_{N,n,r,\geq m} = 0$ unless $n \equiv 0 \pmod{r}$. 

The restricted hypergeometric Lehmer-Euler numbers satisfy the following recurrence relation.

**Proposition 4.1.** For \( j = 0, 1 \), we have

\[
W^{(j)}_{N,rn,r,\leq m} = - \sum_{k=\max\{n-m,0\}}^{n-1} \frac{(rn)!(rN+j)!}{(rN+rn-rk+j)!(rk)!} W^{(j)}_{N,rk,r,\leq m} \quad (n \geq 1)
\]

with \( W^{(j)}_{N,0,r,\leq m} = 1 \).

The associated hypergeometric Lehmer-Euler numbers satisfy the following recurrence relation.

**Proposition 4.2.**

\[
W^{(j)}_{N,rn,r,\geq m} = - \sum_{k=0}^{n-m} \frac{(rn)!(rN+j)!}{(rN+rn-rk+j)!(rk)!} W^{(j)}_{N,rk,r,\geq m} \quad (n \geq m)
\]

with \( W^{(j)}_{N,0,r,\geq m} = 1 \) and \( W^{(j)}_{N,r,r,\geq m} = \cdots = W^{(j)}_{N,r(m-1),r,\geq m} = 0 \).

**Proof of Proposition 4.1.** By the definition (16), we get

\[
1 = \left(1 + \sum_{l=1}^{m} \frac{(rN+j)!l^r}{(rN+rl+j)!}\right) \left(\sum_{n=0}^{\infty} W^{(j)}_{N,rn,r,\leq m} \frac{t^r}{(rn)!}\right)
\]

\[
= \sum_{n=0}^{\infty} W^{(j)}_{N,rn,r,\leq m} \frac{t^r}{(rn)!} + \sum_{n=1}^{\infty} \sum_{k=\max\{n-m,0\}}^{n-1} \frac{(rN+j)!W^{(j)}_{N,rk,r,\leq m}}{(rN+rn-rk+j)!(rk)!} t^r.
\]

Comparing the coefficient on both sides, we obtain the first identity. \( \square \)

**Proof of Proposition 4.2.** By the definition (17), we get

\[
1 = \left(1 + \sum_{l=m}^{\infty} \frac{(rN+j)!l^r}{(rl+j)!}\right) \left(\sum_{n=0}^{\infty} W^{(j)}_{N,rn,r,\geq m} \frac{t^r}{(rn)!}\right)
\]

\[
= \sum_{n=0}^{\infty} W^{(j)}_{N,rn,r,\geq m} \frac{t^r}{(rn)!} + \sum_{n=m}^{\infty} \sum_{k=0}^{n-m} \frac{(rN+j)!W^{(j)}_{N,rk,r,\geq m}}{(rN+rn-rk+j)!(rk)!} t^r.
\]

Comparing the coefficient on both sides, we obtain the desired result. \( \square \)

The restricted and associated hypergeometric Lehmer-Euler numbers have the following expressions in terms of determinants. All elements in some more bands become 0, in the expression in Theorem 3.1.

**Theorem 4.3.** For integers \( n \) and \( m \) with \( n \geq m \geq 1 \), we have

\[
W^{(j)}_{N,rn,r,\leq m} = (-1)^n (rn)!
\]
Theorem 4.4. For integers $n$ and $m$ with $n \geq m \geq 1$, we have

\[
W_{N,rn,r,\leq m} = (-1)^n W_{N,rn,r,\geq m} = \begin{vmatrix}
\frac{(rN+j)!}{(rN+j+r)!} & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{(rN+j)!}{(rN+j+m)!} \\
\end{vmatrix}
\]

Proof of Theorem 4.3. For simplicity, put $\tilde{W}_{N,rn,\leq m} = (-1)^n W_{N,rn,r,\leq m}/(rn)!$ and prove that for $n \geq m \geq 1$

(18)

\[
\tilde{W}_{N,rn,\leq m} = \begin{vmatrix}
\frac{(rN+j)!}{(rN+j+r)!} & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & 0 & 1 \\
\end{vmatrix}
\]

When $n = m$, we have $\tilde{W}_{N,rm,\leq m} = \tilde{W}_{N,rm}$, which is reduced to Theorem 3.1. Assume that (18) is valid up to $n - 1$. If $n \geq 2m$, then the determinant on right-hand side of (18) is equal to

\[
\frac{\tilde{W}_{N,rn-2r,\leq m}}{(rN+j+r)!} - \frac{\tilde{W}_{N,rn-r,\leq m}}{(rN+j+2r)!} + \cdots
\]
\[ + (\pm 1)^{m-1} + (\pm 1)^{n-1} \cdots \]
If \( m \leq n < 2m \), the determinant on the right-hand side of (19) is equal to

\[
\begin{vmatrix}
0 & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & 0 & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 1 & \ddots & 0 \\
\frac{(rN+j)!}{(rN+n+m+j)!} & \cdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\frac{(rN+j)!}{(rN+n+m+j)!} & \cdots & 0 & \ddots & \ddots & 1 \\
\end{vmatrix}_{m-1}
\]

\[
= (-1)^m (-1)^{m+1} \frac{(rN+j)!}{(rN+n+m+j)!} \begin{vmatrix}
1 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \ddots & 1 \\
\end{vmatrix}
\]

\[
= (-1)^m \frac{(rN+j)!}{(rN+n+m+j)!}.
\]

Since only the term for \( k = 0 \) does not vanish in Proposition 4.2, we have

\[
W_{N,rn,m} = \frac{(rN+j)!}{(rN+n+m+j)!}.
\]

If \( n \geq 2m \), the determinant on the right-hand side of (19) is equal to

\[
\begin{vmatrix}
\frac{(rN+j)!}{(rN+n-2m+j)!} & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & 0 & \ddots & \ddots & \ddots & \vdots \\
\frac{(rN+j)!}{(rN+n-2m+m+j)!} & \cdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\frac{(rN+j)!}{(rN+n-2m+n+j)!} & \cdots & 0 & \ddots & \ddots & 1 \\
\end{vmatrix}_{m-1}
\]

\[
= (-1)^{m-1} \frac{\tilde{W}_{N,rn,m}^{(j)}}{(rN+n+m+j)!} \begin{vmatrix}
1 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \ddots & 1 \\
\end{vmatrix}
\]

\[
+ (-1)^m \frac{\tilde{W}_{N,rn,m}^{(j)}}{(rN+n+m+j)!} \begin{vmatrix}
0 & 1 & \cdots & \cdots & 0 \\
\vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \ddots & 1 \\
\end{vmatrix}
\]

\[
= (-1)^{m-1} \frac{\tilde{W}_{N,rn,m}^{(j)}}{(rN+n+m+j)!}.
\]
\[\begin{align*}
&= \cdots \\
&= - \sum_{k=m}^{n-m} \frac{(-1)^{n-k}W_{N,r,k \geq m}}{(r(N + n - k) + j)!} = \tilde{W}_{N,rn \geq m}.
\end{align*}\]

Here, we used Proposition 4.2 again. \(\square\)

There exist explicit expressions for both incomplete Lehmer-Euler numbers.

**Theorem 4.5.** For \(n, m \geq 1\),
\[
W^{(j)}_{N,rn,r \leq m} = (rn)! \sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1 + \cdots + i_k = m \\ i_1, \ldots, i_k \leq m}} \frac{(rN + j)!}{(rN + ri_1 + j)! \cdots (rN + ri_k + j)!}.
\]

**Theorem 4.6.** For \(n, m \geq 1\),
\[
W^{(j)}_{N,rn,r \geq m} = (rn)! \sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1 + \cdots + i_k = m \\ i_1, \ldots, i_k \leq m}} \frac{(rN + j)!}{(rN + ri_1 + j)! \cdots (rN + ri_k + j)!}.
\]

**Proof of Theorem 4.5.** When \(n \leq m\), the proof is similar to that of Proposition 2.1. Note that in the proof of Proposition 2.1,
\[
1 \leq n - l = i_k \leq n - k + 1 \leq n.
\]

Let \(n \geq m + 1\). By Proposition 4.1
\[
\begin{align*}
\frac{W^{(j)}_{N,rn,r \leq m}}{(rn)!} &= - \sum_{l=n-m}^{n-1} \frac{(rN + j)!W^{(j)}_{N,rn,r \leq m}}{(rN + rn - rl + j)! (rl)!} \\
&= - \sum_{l=n-m}^{n-1} \frac{(rN + j)!}{(rN + rn - rl + j)!} \sum_{k=1}^{l} (-1)^k \\
&\quad \times \sum_{\substack{i_1 + \cdots + i_k = l \\ i_1, \ldots, i_k \leq m}} \frac{(rN + j)!}{(rN + ri_1 + j)! \cdots (rN + ri_k + j)!} \\
&= - \sum_{l=1}^{n-1} \frac{(rN + j)!}{(rN + rn - rl + j)!} \sum_{k=1}^{l} (-1)^k \\
&\quad \times \sum_{\substack{i_1 + \cdots + i_k = l \\ i_1, \ldots, i_k \leq m}} \frac{(rN + j)!}{(rN + ri_1 + j)! \cdots (rN + ri_k + j)!}.
\end{align*}
\]
\[
\begin{align*}
&+ \sum_{l=1}^{n-m-1} \frac{(rN + j)!}{(rN + rn - rl + j)!} \sum_{k=1}^{l} (-1)^k \\
&\times \sum_{i_1 + \ldots + i_{k-1} = l, 1 \leq i_1, \ldots, i_{k} \leq m} \frac{((rN + j)!)^k}{(rN + ri_1 + j)! \cdots (rN + ri_{k-1} + j)!} \\
= & - \sum_{k=1}^{n-1} (-1)^k \sum_{l=k}^{n-1} \frac{(rN + j)!}{(rN + rn - rl + j)!} \\
&\times \sum_{i_1 + \ldots + i_{k-1} = l, 1 \leq i_1, \ldots, i_{k} \leq m} \frac{((rN + j)!)^k}{(rN + ri_1 + j)! \cdots (rN + ri_{k-1} + j)!} \\
&+ \sum_{k=1}^{n-m-1} (-1)^k \sum_{l=k}^{n-m-1} \frac{(rN + j)!}{(rN + rn - rl + j)!} \\
&\times \sum_{i_1 + \ldots + i_{k-1} = l, 1 \leq i_1, \ldots, i_{k} \leq m} \frac{((rN + j)!)^k}{(rN + ri_1 + j)! \cdots (rN + ri_{k-1} + j)!} \\
= & \sum_{k=2}^{n} (-1)^k \sum_{l=k-1}^{n-1} \frac{(rN + j)!}{(rN + rn - rl + j)!} \\
&\times \sum_{i_1 + \ldots + i_{k-1} = l, 1 \leq i_1, \ldots, i_{k} \leq m} \frac{((rN + j)!)^k}{(rN + ri_1 + j)! \cdots (rN + ri_{k-1} + j)!} \\
&+ \sum_{k=2}^{n-m} (-1)^k \sum_{l=k-1}^{n-m-1} \frac{(rN + j)!}{(rN + rn - rl + j)!} \\
&\times \sum_{i_1 + \ldots + i_{k-1} = l, 1 \leq i_1, \ldots, i_{k} \leq m} \frac{((rN + j)!)^k}{(rN + ri_1 + j)! \cdots (rN + ri_{k-1} + j)!} \\
&+ \sum_{k=2}^{n-m-1} (-1)^k \sum_{l=k}^{n-m} \frac{(rN + j)!}{(rN + rn - rl + j)!} \\
&\times \sum_{i_1 + \ldots + i_{k-1} = l, 1 \leq i_1, \ldots, i_{k} \leq m} \frac{((rN + j)!)^k}{(rN + ri_1 + j)! \cdots (rN + ri_{k-1} + j)!} \\
&+ \sum_{k=2}^{n} (-1)^k \sum_{l=n-m}^{n-1} \frac{(rN + j)!}{(rN + rn - rl + j)!} \\
&\times \sum_{i_1 + \ldots + i_{k-1} = l, 1 \leq i_1, \ldots, i_{k} \leq m} \frac{((rN + j)!)^k}{(rN + ri_1 + j)! \cdots (rN + ri_{k-1} + j)!}.
\end{align*}
\]
\[
\sum_{i_1 + \cdots + i_{k-1} = 1 \atop 1 \leq i_1, \ldots, i_{k-1} \leq m} \frac{((rN + j)!)^{k-1}}{(rN + ri_1 + j)! \cdots (rN + ri_{k-1} + j)!}.
\]

By putting \( i_k = n - l \), in the first term by \( n - 1 \geq l \geq k - 1 \geq n - m \), in the second term by \( n - 1 \geq l \geq n - m \), we have

\[1 \leq n - l = i_k \leq m.\]

Therefore,

\[
W_{N,rn,r,\geq m}^{(j)} = \sum_{k=n-m+1}^{n} (-1)^k \sum_{i_1 + \cdots + i_k = n \atop 1 \leq i_1, \ldots, i_k \leq m} \frac{((rN + j)!)^k}{(rN + ri_1 + j)! \cdots (rN + ri_k + j)!}.
\]

Note that the term is vanished for \( k = 1 \) as \( n > m \).

**Proof of Theorem 4.6.** Since the set

\[\{(i_1, \ldots, i_k) | i_1 + \cdots + i_k = n, \ i_1, \ldots, i_k \geq m\}\]

is empty for \( n = 1, \ldots, m - 1 \), we have \( W_{N,rn,\geq m}^{(j)} = \cdots = W_{N,rn-r,\geq m}^{(j)} = 0 \). For \( n = m \), by Theorem 4.4

\[
W_{N,rn,\geq m}^{(j)} = (-1)^m (rm)!egin{vmatrix}
0 & 1 \\
\vdots & \vdots \\
0 & 1 \\
\frac{(rN + j)!}{(rN + rm + j)!} & 0 & \cdots & 0
\end{vmatrix}
\]

which matches the result for \( n = m \). Assume that the result is valid up to \( n - 1 \geq m \). Then by Proposition 4.2

\[
W_{N,rn,\geq m}^{(j)} = - \sum_{l=0}^{n-m} \frac{(rN + j)!}{(rN + rl + j)!} W_{N,rl,\geq m}^{(j)}
\]

which gives the result for \( n = m \).
Note that \( i_k = n - l \geq m \) as \( l \leq n - m \). As \( 1 \leq m < n - 1 \), we have
\[
n < (n - m + 2)m \leq km \leq n = i_1 + \cdots + i_k ,
\]
so the set
\[
\{(i_1, \ldots, i_k) \mid i_1 + \cdots + i_k = n, \ i_1, \ldots, i_k \geq m\}
\]
is empty for \( n - m + 2 \leq k \leq n \). \( \square \)
5. Applications by the Trudi’s formula

We shall use the Trudi’s formula to obtain different explicit expressions and inversion relations for the numbers $W_{N,n,r}^{(j)}$.

**Lemma 5.1.** For a positive integer $n$, we have

$$
\begin{vmatrix}
  a_1 & a_0 & 0 & \cdots \\
  a_2 & a_1 & \ddots & \vdots \\
  \vdots & \vdots & \ddots & \ddots & 0 \\
  a_{n-1} & \cdots & a_1 & a_0 \\
  a_n & a_{n-1} & \cdots & a_2 & a_1
\end{vmatrix}
= \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} (-a_0)^{n-t_1-\cdots-t_n} a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n},
$$

where $\binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} = \frac{(t_1+t_2+\cdots+t_n)!}{t_1!\cdots t_n!}$ are the multinomial coefficients.

This relation is known as Trudi’s formula [18, Vol. 3, p. 214], [20] and the case $a_0 = 1$ of this formula is known as Brioschi’s formula [1], [18, Vol. 3, pp. 208–209].

In addition, there exists the following inversion formula (see, e.g. [15]), which is based upon the relation

$$
\sum_{k=0}^{n} (-1)^{n-k} \alpha_k D(n-k) = 0 \quad (n \geq 1)
$$

or Cameron’s operator in (14).

**Lemma 5.2.** If $\{\alpha_n\}_{n \geq 0}$ is a sequence defined by $\alpha_0 = 1$ and

$$
\alpha_n = \begin{vmatrix}
  D(1) & 1 & \cdots & \cdots & \cdots & \cdots & \cdots \\
  D(2) & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  D(n) & \cdots & D(2) & D(1) & 1 & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \alpha_1 & \alpha_2 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix},
$$

then $D(n) = \begin{vmatrix}
  \alpha_1 & 1 \\
  \alpha_2 & \alpha_1 \\
  \vdots & \ddots \\
  \alpha_n & \cdots & \alpha_2 & \alpha_1
\end{vmatrix}$.

From Trudi’s formula, it is possible to give the combinatorial expression

$$
\alpha_n = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} (-1)^{n-t_1-\cdots-t_n} D(1)^{t_1} D(2)^{t_2} \cdots D(n)^{t_n}.
$$

By applying these lemmata to Theorem 4.3 and Theorem 4.4, we obtain an explicit expression for the hypergeometric Lehmer-Euler numbers.

**Theorem 5.3.** For $n \geq m \geq 1$, we have

$$
W_{N,rn,r,\leq m}^{(j)} = (rn)! \sum_{t_1+2t_2+\cdots+mt_m=n} \binom{t_1+\cdots+t_m}{t_1,\ldots,t_m}
$$
\begin{equation*}
\times (-1)^{t_1+\cdots+t_m} \left( \frac{(rN+j)!}{(rN+j+r)!} \right)^{t_1} \cdots \left( \frac{(rN+j)!}{(rN+rm+j)!} \right)^{t_m}
\end{equation*}
and
\begin{equation*}
W_{N,rn,r}^{(j)} = (rn)! \sum_{m+t_1+\cdots+t_n=n} \binom{t_m + t_{m+1} + \cdots + t_n}{t_m, t_{m+1}, \ldots, t_n} \times (-1)^{t_1+\cdots+t_n} \left( \frac{(rN+j)!}{(rN+rm+j)!} \right)^{t_m} \left( \frac{(rN+j)!}{(rN+rm+j+r)!} \right)^{t_{m+1}} \cdots \left( \frac{(rN+j)!}{(rN+rn+j)!} \right)^{t_n}.
\end{equation*}

As a special case of Theorem 5.3, we can obtain the expressions for the original hypergeometric Lehmer-Euler numbers.

**Corollary 5.4.** For \( n \geq 1 \), we have
\begin{equation*}
W_{N,rn,r}^{(j)} = (rn)! \sum_{m+t_1+\cdots+t_n=n} \binom{t_1 + \cdots + t_n}{t_1, \ldots, t_n} \times (-1)^{t_1+\cdots+t_n} \left( \frac{(rN+j)!}{(rN+rn+j)!} \right)^{t_1} \cdots \left( \frac{(rN+j)!}{(rN+rn+j+r)!} \right)^{t_n}.
\end{equation*}

By applying the inversion relation in Lemma 5.2 to Theorem 3.1, we have the following.

**Theorem 5.5.** Let \( j = 0, 1 \). For \( n \geq 1 \), we have
\begin{equation*}
(-1)^n \frac{(rN+j)!}{(rN+rn+j)!} = \begin{vmatrix}
\frac{W_{N,r}^{(j)}}{W_{N,2r}^{(j)}} & 1 \\
\vdots & \vdots \\
\frac{W_{N,n-2r}^{(j)}}{W_{N,n-r}^{(j)}} & \frac{W_{N,n-r}^{(j)}}{W_{N,n-r}^{(j)}} & \cdots & \frac{1}{W_{N,2r}^{(j)}}
\end{vmatrix}.
\end{equation*}

In this sense, we have the inversion relation of Corollary 5.4 too.

**Corollary 5.6.** For \( n \geq 1 \), we have
\begin{equation*}
\frac{(rN+j)!}{(rN+rn+j)!} = \sum_{m+t_1+\cdots+t_n=n} \binom{t_1 + \cdots + t_n}{t_1, \ldots, t_n}.
\end{equation*}
\( \times (-1)^{t_1 + \ldots + t_n} \left( \frac{W^{(j)}_{N,n,r}}{r!} \right)^{t_1} \ldots \left( \frac{W^{(j)}_{N,n,r}}{(rn)!} \right)^{t_n} \).

6. Additional comments

The hypergeometric Lehmer-Euler numbers of higher order \( W^{(j)}_{N,n,r} \) includes hypergeometric Bernoulli and Euler numbers, and the classical Bernoulli, Euler and Lehmer’s Euler numbers as special cases. However, the numbers \( W^{(j)}_{N,n,r} \) do not include some famous generalized numbers, for example, poly-Bernoulli and Euler numbers, Apostol-Bernoulli and Euler numbers and some \( p \)-adic numbers and \( q \)-numbers, because they do not satisfy the relation (14) as definitions.

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References


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