IMPROVING THE POCKLINGTON AND PADRÓ-SÁEZ CUBE ROOT ALGORITHM

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Abstract. In this paper, we present a cube root algorithm using a recurrence relation. Additionally, we compare the implementations of the Pocklington and Padró-Sáez algorithm with the Adleman-Manders-Miller algorithm. With the recurrence relations, we improve the Pocklington and Padró-Sáez algorithm by using a smaller base for exponentiation. Our method can reduce the average number of $\mathbb{F}_q$ multiplications.

1. Introduction

There are two standard algorithms for computing cube roots in finite fields; the Adleman-Manders-Miller algorithm [1,2,8,9] and the Cipolla-Lehmer algorithm [3,5]. The Pocklington and Padró-Sáez algorithm [6,7] can also be used, although it is a different type of algorithm from the previously mentioned algorithms. Pocklington [7] proposed a square and cube root algorithm over $\mathbb{F}_p$ with prime $p$. Heo et al. [4] clarified and generalized the Pocklington and Padró-Sáez algorithm [6,7]. Thus, Heo et al. [4] proposed a cube root algorithm in $\mathbb{F}_q$, where $q$ is a power of a prime $p$.

In this paper, we present a cube root algorithm using a recurrence relation. We also provide the results of software implementations (using SAGE) of the Pocklington and Padró-Sáez algorithm [6,7] compared with those of the Adleman-Manders-Miller algorithm [1,2,8,9]. Our method can reduce the average number of $\mathbb{F}_q$ multiplications.

The remainder of this paper is organized as follows: In Section 2, we describe the Adleman-Manders-Miller algorithm [1,2,8,9] and refined the Pocklington and Padró-Sáez [6,7] for cube root computation. In Section 3, we implement our method with the Adleman-Manders-Miller algorithm [1,2,8,9] and compare the results. In Section 4, we conclude the paper and discuss future works.
2. Computation of cube roots in finite fields

In this section, we consider a cube root algorithm in finite fields. We describe the Adleman-Manders-Miller algorithm [1,2,8,9] over prime fields. As is known, the Adleman-Manders-Miller algorithm [1, 2, 8, 9] can be implemented over general finite fields. However, the algorithm is more complex when applied to prime fields, as well as the Pocklington and Padró-Sáez algorithm [6,7]. Subsequently, we describe the implementation of the Pocklington and Padró-Sáez algorithm [6,7] over general finite fields.

2.1. Adleman-Manders-Miller cube root algorithm

The Tonelli-Shanks [8,9] method for square root computation was extended to the general \( r \)-th roots computation by Adleman, Manders and Miller [1]. Let \( p \) be a prime such that \( p \equiv 1 \pmod{3} \). Let \( c \) be a cubic residue over \( \mathbb{F}_p \).

\begin{table}[h]
\begin{center}
\begin{tabular}{|l|}
\hline
Input: A cubic residue \( c \) in \( \mathbb{F}_p \)  \\
Output: A cube root of \( c \)  \\
\hline
Step 1:  \\
Let \( p - 1 = 3^s t \) with \( t = 3^l \pm 1 \)  \\
\hline
Step 2:  \\
Select a cubic non-residue \( b \) in \( \mathbb{F}_p \)  \\
\( a \leftarrow b^t \)  \\
\( a' \leftarrow a^{3^{t-1}} \)  \\
\hline
Step 3: (Computation of a cube root of \( (c^t)^{-1} \))  \\
\( h \leftarrow 1, r \leftarrow c^t \)  \\
\textbf{for} \( i = 1 \) to \( s - 1 \)  \\
\( d \leftarrow r^{3^{t-i-1}} \)  \\
\textbf{if} \( d = 1 \), \textbf{then} \( k \leftarrow 0 \)  \\
\textbf{else if} \( d = a' \), \textbf{then} \( k \leftarrow 2 \)  \\
\textbf{else} \( k \leftarrow 1 \)  \\
\( h \leftarrow h \cdot a^k, r \leftarrow r \cdot (a^3)^k \)  \\
\( a \leftarrow a^3 \)  \\
\textbf{end for}  \\
\hline
Step 4:  \\
\( r \leftarrow c^t \cdot h \)  \\
\textbf{if} \( t = 3^l + 1 \), \textbf{then} \( r \leftarrow r^{-1} \)  \\
Return \( r \)  \\
\hline
\end{tabular}
\end{center}
\caption{Adleman-Manders-Miller’s cube root algorithm [1]}\end{table}

We require one multiplication for each “square” and “multiply” operation over \( \mathbb{F}_p \). That is, 1.5 multiplications are required on average over \( \mathbb{F}_p \). The Adleman-Manders-Miller algorithm requires approximately \((1.5 \times 3) \log t = \)
4.5 \log t \ (\log = \log_2) \ for \ computing \ b', c', \ and \ c' \ (\log 3 \approx 1). \ Furthermore, \ it \ requires \ approximately \ 5(s-1)+\frac{(s-1)(s-2)}{2} = \frac{s^2+7s-8}{2} \ multiplications \ for \ computing \ the \ for-loop \ and \ a'. \ Therefore, \ the \ average \ number \ of \ \mathbb{F}_p \ multiplications \ is \ 4.5 \log t + \frac{s^2+7s-8}{2} \ over \ \mathbb{F}_p, \ and \ the \ complexity \ of \ the \ Adleman-Manders-Miller \ algorithm \ is \ O(\log_3 p + s^2 \log^2 p) \ [1, 2].

### 2.2. Pocklington and Padró-Sáez cube root algorithm

Assume that $q$ is a power of prime $p$ such that $q \equiv 1 \ (\text{mod } 3)$. Let $c$ be a cubic residue over $\mathbb{F}_q$. Heo et al. [4] clarified and generalized the Pocklington and Padró-Sáez algorithm [6, 7]. That is, Heo et al. presented a cube root algorithm over $\mathbb{F}_q$, which is described in Table 2.

Using the ring isomorphism $\mathbb{F}_q[X]/(X^3 - c) \cong \mathbb{F}_q \oplus \mathbb{F}_q \oplus \mathbb{F}_q$, $N(z)$ is defined as the product of all the conjugates of $z = \alpha + \beta X + \gamma X^2 \in \mathbb{F}_q[X]/(X^3 - c)$. That is, $N(z) = \overline{z}z \bar{z} \in \mathbb{F}_q$, where $\overline{z} = \alpha + \beta \epsilon X + \gamma \epsilon^2 X^2$ and a primitive cube root of unity $\epsilon$.

**Table 2.** Pocklington and Padró-Sáez cube root algorithm [4]

| Step 1: | **Input:** A cubic residue $c$ in $\mathbb{F}_q$ with $q - 1 = 3^s t$, $\gcd(3, t) = 1$ |
|         | **Output:** $x$ satisfying $x^3 = c$ in $\mathbb{F}_q$ |
|         | if $q \equiv 4 \pmod{9}$ then $x \leftarrow c^{\frac{2s+1}{3}}$ and return $x$ |
|         | if $q \equiv 7 \pmod{9}$ then $x \leftarrow c^{\frac{s+2}{3}}$ and return $x$ |
|         | Step 3: Choose random $\alpha, \beta, \gamma \in \mathbb{F}_q$ and let $z := \alpha + \beta X + \gamma X^2 \in \mathbb{F}_q[X]/(X^3 - c)$ |
|         | if $N(z) = 0$ then go to STEP 3 |
|         | $z \leftarrow z^t$ |
|         | Step 4: if $\alpha = \beta = 0$ or $\beta = \gamma = 0$ or $\gamma = \alpha = 0$ then choose new $z$ again |
|         | while $\alpha \beta \neq 0$ or $\beta \gamma \neq 0$ or $\gamma \alpha \neq 0$ do |
|         | $z_0 := \alpha_0 + \beta_0 X + \gamma_0 X^2 \leftarrow z$ |
|         | $z \leftarrow z^3$ |
|         | Step 5: if $\beta = \gamma = 0$ then $x \leftarrow \frac{z_0}{\beta_0}$ |
|         | else if $\gamma = \alpha = 0$ then $x \leftarrow -\frac{\gamma_0}{\alpha_0} \beta_0 \gamma_0 \beta_0$ |
|         | else then $x \leftarrow -\frac{\gamma_0}{\alpha_0} \beta_0 \gamma_0$ |
|         | return $x$ |

If $q \equiv 1 \pmod{9}$, then the algorithm computes $z^t$ by repeated cubing $z^3$ until $z$ has two coefficients equal to zero. That is, the complexity of the algorithm is $O(\log^3 q)$. When $s$ is large, $s \approx \log q$, the Pocklington and Padró-Sáez algorithm is more efficient than the Adleman-Manders-Miller algorithm, as it
is independent of the size of $s$. Because the while-loop (in Step 4) operates probabilistically until $z^{3^m - t}$ has exactly 2 zero coefficients with $0 \leq m \leq s - 1$, where $m$ is the number of times that the loop is executed in Step 4. The probability that a chosen invertible $z = a + X \in \mathbb{F}_q[X]/(X^3 - c)$ satisfies $z^{3^m} = \alpha + \beta X + \gamma X^2$ with 2 zero coefficients is $\frac{1}{3}$. (We can verify that the probability is identical for both random $z$ and special $z$ from [4].) Therefore the expected number of iterations of the while-loop is $s - \frac{1}{3} (1 - \frac{1}{3^m})$ [4].

We consider the average number of $F_q$ multiplications. The algorithm selected a random $\alpha, \beta, \gamma \in \mathbb{F}_q$ with $z = \alpha + \beta X + \gamma X^2 \in \mathbb{F}_q[X]/(X^3 - c)$. To compute $z^t$, we consider the classical “double and add” relations. As one can express $z^m = \alpha_m + \beta_m X + \gamma_m X^2$ by the array $[\alpha_m, \beta_m, \gamma_m]$ with respect to the basis $\{1, X, X^2\}$, we obtain the following “double” and “add” operations:

\[
\begin{align*}
\alpha_{2n} &= \alpha_n^2 + 2\beta_n \gamma_n, \\
\beta_{2n} &= c\gamma_n^2 + 2\alpha_n \beta_n, \\
\gamma_{2n} &= \beta_n^2 + 2\alpha_n \gamma_n, \\
\alpha_{m+1} &= \alpha_1 \alpha_m + c(\beta_1 \gamma_m + \gamma_1 \beta_m), \\
\beta_{m+1} &= \alpha_1 \beta_m + \alpha_m \beta_1 + c\gamma_1 \gamma_m, \\
\gamma_{m+1} &= \alpha_1 \gamma_m + \beta_m \beta_1 + \gamma_1 \alpha_m,
\end{align*}
\]

where the initial values are $\alpha_1 = \alpha, \beta_1 = \beta, \gamma_1 = \gamma$ ($z^1 = \alpha + \beta X + \gamma X^2$). The Pocklington and Padró-Sáez algorithm uses 7 multiplications for each “square” and 11 multiplications for each “add”. Therefore the algorithm can be accomplished by $12.5 \log t (= 7 + \frac{11}{2})$ multiplications on average for computing $z^t$. To extract a cube root of $c$, the algorithm requires $12.5 \log t + 18s$ multiplications (original PPS).

The number of “square” multiplications depends on the coefficients of $f(x) = x^3 - c$. Furthermore, the number of “add” multiplications depends on the initial values. For a given $f(x) = x^3 - c$, there are a fixed number of “square” multiplications. Thus, our aim is to reduce the number of “add” multiplications. We select a special $z = a + X \in \mathbb{F}_q[X]/(X^3 - c)$ with random $a$ (numerically small $a$). From the choice of the (special) initial values, we obtain

\[
\begin{align*}
\alpha_{m+1} &= \alpha \alpha_m + c \gamma_m, \\
\beta_{m+1} &= \alpha_m + a \beta_m, \\
\gamma_{m+1} &= \beta_m + a \gamma_m.
\end{align*}
\]

As $a$ is (numerically) small, we calculate $z^t$ in $7 \log t$ multiplications for “square”, and $\frac{1}{2} \log t$ multiplications for “add” on average. Subsequently, to compute a cube root of $c$, the algorithm requires $7.5 \log t + 18s (= 7 + 11)$ multiplications (Refined PPS when $a$ is small). Unfortunately, if we choose an unconstrained $a \in \mathbb{F}_q$, then the algorithm requires $9 \log t (= 7 + \frac{1}{2}) + 18s$ multiplications (Refined PPS when $a$ is random).
Table 3. Theoretical estimation (average number of $F_q$ multiplications)

<table>
<thead>
<tr>
<th></th>
<th>AMM</th>
<th>$4.5 \log t + \frac{3^2+7s-8}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>PPS</td>
<td>$12.5 \log t + 18s$</td>
<td></td>
</tr>
<tr>
<td>Refined PPS(random $a$)</td>
<td>$9 \log t + 18s$</td>
<td></td>
</tr>
<tr>
<td>Refined PPS(small $a$)</td>
<td>$7.5 \log t + 18s$</td>
<td></td>
</tr>
</tbody>
</table>

Table 4. Running time (in seconds) for cube root computation with $p \approx 2^{2000}$

<table>
<thead>
<tr>
<th>$s$</th>
<th>10</th>
<th>30</th>
<th>50</th>
<th>80</th>
<th>100</th>
<th>150</th>
<th>200</th>
<th>300</th>
<th>400</th>
<th>500</th>
<th>600</th>
</tr>
</thead>
<tbody>
<tr>
<td>AMM</td>
<td>0.013</td>
<td>0.016</td>
<td>0.021</td>
<td>0.022</td>
<td>0.024</td>
<td>0.024</td>
<td>0.026</td>
<td>0.027</td>
<td>0.029</td>
<td>0.032</td>
<td>0.041</td>
</tr>
<tr>
<td>PPS</td>
<td>0.055</td>
<td>0.056</td>
<td>0.056</td>
<td>0.056</td>
<td>0.056</td>
<td>0.057</td>
<td>0.056</td>
<td>0.056</td>
<td>0.055</td>
<td>0.057</td>
<td>0.057</td>
</tr>
<tr>
<td>Refined PPS(random $a$)</td>
<td>0.046</td>
<td>0.046</td>
<td>0.040</td>
<td>0.041</td>
<td>0.039</td>
<td>0.040</td>
<td>0.040</td>
<td>0.039</td>
<td>0.039</td>
<td>0.040</td>
<td>0.040</td>
</tr>
<tr>
<td>Refined PPS(small $a$)</td>
<td>0.028</td>
<td>0.030</td>
<td>0.031</td>
<td>0.032</td>
<td>0.032</td>
<td>0.031</td>
<td>0.030</td>
<td>0.030</td>
<td>0.030</td>
<td>0.031</td>
<td>0.030</td>
</tr>
</tbody>
</table>

3. Experimental results

In this section, we implement our proposed algorithm with the AMM (Adleman-Manders-Miller) algorithm [1, 2, 8, 9] in a finite field $F_q$. The complexity of the AMM cube root algorithm is $O(\log^4 q)$, where $q - 1 = 3^t t$ with $\gcd(3, t) = 1$ and $s \approx \log q$, and the complexity of the PPS (Pocklington and Padró-Sáez) cube root algorithm is $O(\log^3 q)$ [4, 6, 7].

We compared the average number of $F_q$ multiplications between the cube root algorithms. The AMM algorithm always required $4.5 \log t + \frac{3^2+7s-8}{2}$ multiplications. However, the PPS algorithm has a flexible number of multiplications, as shown in Table 3.

Tables 4 and 5 show the comparison of the implementation results, of the proposed methods for various cases, obtained using the software, SAGE. The implementation was performed using Intel(R) Core(TM) i7-4790K CPU @ 4.00GHz with 8GB memory.

For convenience, we used prime fields $F_p$ with two different sizes of primes $p: 2000$ and $3000$ bits. The average timings of the cube root computations for $5$ different inputs of cubic residue are computed for the cases of $s = 10, 30, 50, \ldots$, etc.

If $s$ is relatively smaller, the AMM algorithm is more efficient than the PPS type algorithms. However, if $s$ is large, the PPS type algorithms are more efficient. The tables show that our proposed methods are faster than the original PPS and AMM algorithms for large value of $s$.

4. Conclusion and future works

We proposed a refined the Pocklington and Padró-Sáez cube root algorithm over $F_q$, and successfully reduced the average number of $F_q$ multiplications...
Table 5. Running time (in seconds) for cube root computation with \( p \approx 2^{3000} \)

<table>
<thead>
<tr>
<th>s</th>
<th>10</th>
<th>30</th>
<th>50</th>
<th>80</th>
<th>100</th>
<th>150</th>
<th>200</th>
<th>300</th>
<th>400</th>
<th>500</th>
<th>600</th>
</tr>
</thead>
<tbody>
<tr>
<td>AMM</td>
<td>0.039</td>
<td>0.039</td>
<td>0.047</td>
<td>0.068</td>
<td>0.064</td>
<td>0.144</td>
<td>0.217</td>
<td>0.474</td>
<td>0.861</td>
<td>1.209</td>
<td>1.868</td>
</tr>
<tr>
<td>PPS</td>
<td>0.131</td>
<td>0.129</td>
<td>0.130</td>
<td>0.132</td>
<td>0.131</td>
<td>0.132</td>
<td>0.131</td>
<td>0.130</td>
<td>0.132</td>
<td>0.131</td>
<td>0.131</td>
</tr>
<tr>
<td>Refined PPS(random a)</td>
<td>0.091</td>
<td>0.087</td>
<td>0.089</td>
<td>0.090</td>
<td>0.091</td>
<td>0.089</td>
<td>0.088</td>
<td>0.090</td>
<td>0.091</td>
<td>0.091</td>
<td>0.091</td>
</tr>
<tr>
<td>Refined PPS(small a)</td>
<td>0.075</td>
<td>0.072</td>
<td>0.075</td>
<td>0.073</td>
<td>0.075</td>
<td>0.074</td>
<td>0.072</td>
<td>0.074</td>
<td>0.075</td>
<td>0.072</td>
<td>0.074</td>
</tr>
</tbody>
</table>

from the original Pocklington and Padró-Sáez algorithm \([4,6,7]\). Furthermore, software implementations via SAGE also indicate that the proposed methods are faster than the original Pocklington and Padró-Sáez algorithm when \( s \) is large \([4,6,7]\). However, the Adleman-Manders-Miller algorithm \([1,2,8,9]\) is more efficient than the Pocklington and Padró-Sáez algorithm for small value of \( s \). Thus, we will examine the development of an efficient algorithm for small value of \( s \) in a future work.

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