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MAPS PRESERVING JORDAN AND $*$-JORDAN TRIPLE PRODUCT ON OPERATOR $*$-ALGEBRAS

Vahid Darvish, Mojtaba Nouri, Mehran Razeghi, and Ali Taghavi

Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be two operator $*$-rings such that $\mathcal{A}$ is prime. In this paper, we show that if the map $\Phi : \mathcal{A} \to \mathcal{B}$ is bijective and preserves Jordan or $*$-Jordan triple product, then it is additive. Moreover, if Φ preserves Jordan triple product, we prove the multiplicativity or anti-multiplicativity of Φ. Finally, we show that if $\mathcal{A}$ and $\mathcal{B}$ are two prime operator $*$-algebras, $\Psi : \mathcal{A} \to \mathcal{B}$ is bijective and preserves $*$-Jordan triple product, then Ψ is a $\mathbb{C}$-linear or conjugate $\mathbb{C}$-linear $*$-isomorphism.

1. Introduction

Let $\mathcal{R}$ and $\mathcal{R}^{'}$ be rings. We say the map $\Phi : \mathcal{R} \to \mathcal{R}^{'}$ preserves product or is multiplicative if $\Phi(AB) = \Phi(A)\Phi(B)$ for all $A, B \in \mathcal{R}$. The question of when a product preserving or multiplicative map is additive was discussed by several authors, see [12] and references therein. Motivated by this, many authors pay more attention to the map on rings (and algebras) preserving the Lie product $[A, B] = AB - BA$ or the Jordan product $A \circ B = AB + BA$ (for example, see [2, 3, 8, 10, 11, 14]). These results show that, in some sense, the Jordan product or Lie product structure is enough to determine the ring or algebraic structure. Historically, many mathematicians devoted themselves to the study of additive or linear Jordan or Lie product preservers between rings or operator algebras. Such maps are always called Jordan homomorphism or Lie homomorphism. Here we only list several results [4, 12, 13].

Let $\mathcal{R}$ be a $*$-ring. For $A, B \in \mathcal{R}$, denoted by $A \bullet B = AB + BA^*$ and $[A, B]_* = AB - BA^*$, which are two different kinds of new products. This product is found playing a more and more important role in some research topics, and its study has recently attracted many author's attention (for example, see [5, 6, 9, 17, 18]).

Let Φ be a bijective map on a standard operator algebra. Molnár showed in [14] that if Φ preserves Jordan triple product (i.e., $\Phi(ABA) = \Phi(A)\Phi(B)\Phi(A)$), then Φ is additive. Later, Lu in [11] considered the case where Φ preserves

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the operation $kABA$ and proved that such a Φ is also additive. Hou and An in [1] proved that Φ preserves Jordan $*$-skew triple product maps (i.e., $\Phi(AB^*A) = \Phi(A)\Phi(B)^*\Phi(A)$) on operator algebras of indefinite inner product spaces are additive. Moreover, in [7], the authors proved that if Φ preserves $*$-Jordan triple product on $B(H)$, where H is infinite-dimensional Hilbert space, then the structure of Φ is determined.

In [16], the author asked the following question that let A and B be two unital C^*-algebras with unit I. If $\Phi : \mathcal{A}_s \to \mathcal{B}_s$, where $\mathcal{A}_s$ is the class of self-adjoint elements in C^*-algebra $\mathcal{A}$, is a surjective map which satisfies $\Phi(ABA) = \Phi(A)\Phi(B)\Phi(A)$ for all $A, B \in \mathcal{A}_s$, can we prove that Φ is Jordan homomorphism on $\mathcal{A}_s$?

The answer for the question above is negative. To show this, let $\Phi(A) = -A$, then Φ satisfies the assumptions of the question but it is easy to check that Φ is not a Jordan homomorphism map. Hence, we change the question to this, let the assumptions hold for Φ which satisfies $\Phi(ABA) = \Phi(A)\Phi(B)\Phi(A)$ for all $A, B \in \mathcal{A}_s$. Is there a Jordan homomorphism Ψ on $\mathcal{A}$ such that $\Psi(A) = \Phi(A)U$ for some $U \in \mathcal{B}$?

We note that, there is no necessity for U to be at the center of range.

In this paper, motivated by the previous results and the above question, we prove that if the map Φ from a prime $*$-ring $\mathcal{A}$ with unit I onto a $*$-ring $\mathcal{B}$ is bijective and preserves Jordan or $*$-Jordan triple product, then it is additive. In Section 3, we show that if $\Phi : \mathcal{A} \to \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are two prime rings, preserves Jordan triple product, then it is multiplicative or anti-multiplicative. Also, we show that $\Psi(A) = \Phi(A)\Phi(I)^*$, for $A \in \mathcal{A}$, is a $\mathbb{C}$-linear or conjugate $\mathbb{C}$-linear $*$-isomorphism and throughout of this paper, we use the following technique:

$$(A_{ji})^* = (P_jAP_i)^* = P_iA^*P_j = (A^*)_{ij}.$$

It is well known that a ring $\mathcal{A}$ is prime, in the sense that $A\mathcal{A}B = 0$ for $A, B \in \mathcal{A}$ implies either $A = 0$ or $B = 0$.

2. Additivity of maps preserving $*$-Jordan triple product

Our first theorem is as follows:

Theorem 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two $*$-rings such that $\mathcal{A}$ is prime and has a nontrivial projection. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a bijective map which satisfies in the following condition*

$$\Phi(AB^*A) = \Phi(A)\Phi(B)^*\Phi(A) \tag{2.1}$$

for all $A, B \in \mathcal{A}$. Then, Φ is additive.

Proof. Let P_1 be a nontrivial projection in $\mathcal{A}$ and $P_2 = I_{\mathcal{A}} - P_1$. Denote $\mathcal{A}_{ij} = P_i\mathcal{A}P_j,\ i, j = 1, 2$, then $\mathcal{A} = \sum_{i,j=1}^{2} \mathcal{A}_{ij}$. For every $A \in \mathcal{A}$ we may write $A = A_{11} + A_{12} + A_{21} + A_{22}$. In all that follow, when we write A_{ij}, it indicates that $A_{ij} \in \mathcal{A}_{ij}$.

For showing additivity of Φ on $\mathcal{A}$, we use above partition of $\mathcal{A}$ and give some claims that prove Φ is additive on each $\mathcal{A}_{ij},\ i,j=1,2$.

Claim 1. $\Phi(0)=0$.

By making use of the assumption of Theorem 2.1, we have

$$\Phi(0)=\Phi(0A^*0)=\Phi(0)\Phi(A)^*\Phi(0)$$

for every $A\in\mathcal{A}$. So, by surjectivity we can choose A such that $\Phi(A)=0$. Then we obtain $\Phi(0)=0$.

Claim 2. *For every $A_{11}\in\mathcal{A}_{11}$, $B_{12}\in\mathcal{A}_{12}$, $C_{21}\in\mathcal{A}_{21}$, $D_{22}\in\mathcal{A}_{22}$. We have*

$$\Phi(A_{11}+B_{12}+C_{21}+D_{22})=\Phi(A_{11})+\Phi(B_{12})+\Phi(C_{21})+\Phi(D_{22}).$$

Since Φ is surjective, we can find an element $T=T_{11}+T_{12}+T_{21}+T_{22}\in\mathcal{A}$ such that

$$\Phi(T)=\Phi(A_{11})+\Phi(B_{12})+\Phi(C_{21})+\Phi(D_{22}). \tag{2.2}$$

We should show $T=A_{11}+B_{12}+C_{21}+D_{22}$.

We can write (2.2) as following

$$\Phi(T)^*=\Phi(A_{11})^*+\Phi(B_{12})^*+\Phi(C_{21})^*+\Phi(D_{22})^*. \tag{2.3}$$

Multiplying the both side of (2.3) by $\Phi(P_1)$, we have

$$\begin{aligned}\Phi(P_1)\Phi(T)^*\Phi(P_1)&=\Phi(P_1)\Phi(A_{11})^*\Phi(P_1)+\Phi(P_1)\Phi(B_{12})^*\Phi(P_1)\\&\quad+\Phi(P_1)\Phi(C_{21})^*\Phi(P_1)+\Phi(P_1)\Phi(D_{22})^*\Phi(P_1).\end{aligned}$$

By (2.1), we obtain

$$\begin{aligned}\Phi(P_1T^*P_1)&=\Phi(P_1A_{11}^*P_1)+\Phi(P_1B_{12}^*P_1)\\&\quad+\Phi(P_1C_{21}^*P_1)+\Phi(P_1D_{22}^*P_1).\end{aligned}$$

It is equivalent to write $\Phi(P_1T^*P_1)=\Phi(P_1A_{11}^*P_1)$ or $\Phi(T_{11}^*)=\Phi(A_{11}^*)$. Injectivity of Φ implies that $T_{11}^*=A_{11}^*$ equivalently $T_{11}=A_{11}$.

Similar trick for $\Phi(P_2)$ leads us to $\Phi(T_{22})=\Phi(D_{22})$.

On the other hand, multiplying (2.3) by $\Phi(X_{12})$ from both sides and making use of (2.1) we have

$$\begin{aligned}\Phi(X_{12}T^*X_{12})&=\Phi(X_{12}A_{11}^*X_{12})+\Phi(X_{12}B_{12}^*X_{12})\\&\quad+\Phi(X_{12}C_{21}^*X_{12})+\Phi(X_{12}D_{22}^*X_{12}).\end{aligned}$$

So, we obtain $\Phi(X_{12}T_{12}^*X_{12})=\Phi(X_{12}B_{12}^*X_{12})$. Injectivity of Φ implies that $X_{12}T_{12}^*X_{12}=X_{12}B_{12}^*X_{12}$. By primeness, it follows that $T_{12}^*=B_{12}^*$ or $T_{12}=B_{12}$.

Similar to the above discussion, we can show $T_{21}=C_{21}$. For this purpose, we multiply $\Phi(X_{21})$ to the both sides of (2.2). Then, we have $\Phi(X_{21}TX_{21})=\Phi(X_{21}B_{12}X_{21})$. So, we obtain the desired result.

Claim 3. *For every $A_{ij},B_{ij}\in\mathcal{A}_{ij}$ such that $1\le i\ne j\le 2$, we have*

$$\Phi(A_{ij}+B_{ij})=\Phi(A_{ij})+\Phi(B_{ij}).$$

It follows from

$$((A_{ij}^*)^* + P_i + P_j)((B_{ij}^*)^* + P_j)((A_{ij}^*)^* + P_i + P_j) = (A_{ij}^*)^* + (B_{ij}^*)^* + P_j$$

and Claim 2 that

$$\begin{aligned}
&\Phi((A_{ij}^*)^* + (B_{ij}^*)^*) + \Phi(P_j) \\
&= \Phi(((A_{ij}^*)^* + P_i + P_j)((B_{ij}^*)^* + P_j)((A_{ij}^*)^* + P_i + P_j)) \\
&= \Phi(((A_{ij}^*)^* + P_i + P_j)(B_{ij}^* + P_j)^*((A_{ij}^*)^* + P_i + P_j)) \\
&= \Phi((A_{ij}^*)^* + P_i + P_j)\Phi(B_{ij}^* + P_j)^*\Phi((A_{ij}^*)^* + P_i + P_j) \\
&= \Phi((A_{ij}^*)^* + P_i + P_j)(\Phi(B_{ij}^*)^* + \Phi(P_j)^*)\Phi((A_{ij}^*)^* + P_i + P_j) \\
&= \Phi((A_{ij}^*)^* + P_i + P_j)\Phi(B_{ij}^*)^*\Phi((A_{ij}^*)^* + P_i + P_j) \\
&\quad + \Phi((A_{ij}^*)^* + P_i + P_j)\Phi(P_j)^*\Phi((A_{ij}^*)^* + P_i + P_j) \\
&= \Phi(((A_{ij}^*)^* + P_i + P_j)(B_{ij}^*)^*((A_{ij}^*)^* + P_i + P_j)) \\
&\quad + \Phi(((A_{ij}^*)^* + P_i + P_j)P_j((A_{ij}^*)^* + P_i + P_j)) \\
&= \Phi((A_{ij}^*)^*) + \Phi((B_{ij}^*)^*) + \Phi(P_j).
\end{aligned}$$

Consequently, $\Phi(A_{ij} + B_{ij}) = \Phi(A_{ij}) + \Phi(B_{ij})$.

Claim 4. *For every $A_{ii} \in \mathcal{A}_{ii}$ and $X_{ij} \in \mathcal{X}_{ij}$, we have*

$$\Phi(A_{ii}X_{ij}) = \Phi(I)\Phi(A_{ii}^*)^*\Phi(X_{ij}) + \Phi(X_{ij})\Phi(A_{ii}^*)^*\Phi(I)$$

for $1 \leq i \leq 2$.

By (2.1) and Claim 2, we can write

$$\begin{aligned}
\Phi((A_{ii}^*)^*) + \Phi((A_{ii}^*)^*X_{ij}) &= \Phi((A_{ii}^*)^* + (A_{ii}^*)^*X_{ij}) \\
&= \Phi((I + X_{ij})(A_{ii}^*)^*(I + X_{ij})) \\
&= \Phi(I + X_{ij})\Phi(A_{ii}^*)^*\Phi(I + X_{ij}) \\
&= (\Phi(I) + \Phi(X_{ij}))\Phi(A_{ii}^*)^*(\Phi(I) + \Phi(X_{ij})) \\
&= \Phi(I)\Phi(A_{ii}^*)^*\Phi(I) + \Phi(I)\Phi(A_{ii}^*)^*\Phi(X_{ij}) \\
&\quad + \Phi(X_{ij})\Phi(A_{ii}^*)^*\Phi(I) + \Phi(X_{ij})\Phi(A_{ii}^*)^*\Phi(X_{ij}) \\
&= \Phi((A_{ii}^*)^*) + \Phi(I)\Phi(A_{ii}^*)^*\Phi(X_{ij}) \\
&\quad + \Phi(X_{ij})\Phi(A_{ii}^*)^*\Phi(I).
\end{aligned}$$

So, we have

$$\Phi((A_{ii}X_{ij}) = \Phi(I)\Phi(A_{ii}^*)^*\Phi(X_{ij}) + \Phi(X_{ij})\Phi(A_{ii}^*)^*\Phi(I).$$

Claim 5. *For every $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$, $1 \leq i \leq 2$ we have*

$$\Phi(A_{ii} + B_{ii}) = \Phi(A_{ii}) + \Phi(B_{ii}).$$

Since Φ is surjective, we can find T such that

$$\Phi(T) = \Phi(A_{ii}^*) + \Phi(B_{ii}^*)$$

or equivalently

$$\Phi(T)^* = \Phi(A_{ii}^*)^* + \Phi(B_{ii}^*)^*. \tag{2.4}$$

We should show that $T = A_{ii}^* + B_{ii}^*$. By simple computation, we see that T^* is just T_{ii}^*. For this aim, it is enough to multiply both sides of equation (2.4) by P_j. Then, we have $T_{jj}^* = 0$ and similarly, from multiplying both sides of equation (2.4) by X_{ij} and X_{ji}, we obtain T_{ij}^* and T_{ji}^* are zero, respectively. Therefore, equation (2.4) converts to

$$\Phi(T_{ii})^* = \Phi(A_{ii}^*)^* + \Phi(B_{ii}^*)^*. \tag{2.5}$$

From now on, we show that $T_{ii} = A_{ii}^* + B_{ii}^*$.

By applying Claim 4 and (2.5) we have

$$\begin{aligned}
\Phi(T_{ii}^* X_{ij}) &= \Phi(I)\Phi(T_{ii})^*\Phi(X_{ij}) + \Phi(X_{ij})\Phi(T_{ii})^*\Phi(I) \\
&= \Phi(I)\Phi(A_{ii}^*)^*\Phi(X_{ij}) + \Phi(X_{ij})\Phi(A_{ii}^*)^*\Phi(I) \\
&\quad + \Phi(I)\Phi(B_{ii}^*)^*\Phi(X_{ij}) + \Phi(X_{ij})\Phi(B_{ii}^*)^*\Phi(I) \\
&= \Phi(A_{ii}X_{ij}) + \Phi(B_{ii}X_{ij}).
\end{aligned}$$

So, we obtain $\Phi(T_{ii}^* X_{ij}) = \Phi(A_{ii}X_{ij}) + \Phi(B_{ii}X_{ij})$. Now, Claim 3 gives us $T_{ii}^* X_{ij} = A_{ii}X_{ij} + B_{ii}X_{ij}$. By primeness, we have $T_{ii}^* = A_{ii} + B_{ii}$.

By Claims 2, 3 and 5 we have the additivity of Φ. □

By a similar proof, we can show that if Φ preserves Jordan triple product, then it is also additive.

Theorem 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two rings such that $\mathcal{A}$ is prime and has a nontrivial idempotent. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a bijective map which is Jordan triple multiplicative as follows*:

$$\Phi(ABA) = \Phi(A)\Phi(B)\Phi(A). \tag{2.6}$$

Then, Φ is additive.

3. Isomorphicity of maps preserving ∗-Jordan triple product

In this following theorem, we show that if $\Phi : \mathcal{A} \to \mathcal{B}$ is Jordan triple multiplicative, then it is multiplicative or anti-multiplicative.

Theorem 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two prime rings such that $\mathcal{A}$ has a nontrivial idempotent. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a bijective map which satisfies in the following condition*

$$\Phi(ABA) = \Phi(A)\Phi(B)\Phi(A). \tag{3.1}$$

Then, Φ is multiplicative or anti-multiplicative.

Proof. The proof of above theorem is presented by some steps.

Step 1. *$\Phi(I)$ commutes with each $\Phi(T)$ for $T \in \mathcal{A}$.*

We know that $\Phi(I) = \Phi(I)^3$. So,

$$\begin{aligned}\Phi(I)\Phi(T) &= \Phi(I)^2\Phi(T)\Phi(I)\\ &= \Phi(I)^3\Phi(T)\Phi(I)^2\\ &= \Phi(I)\Phi(T)\Phi(I)^2\\ &= \Phi(T)\Phi(I).\end{aligned}$$

Step 2. *$\Phi(I) = I$ or $\Phi(I) = -I$.*

From $\Phi(I) = \Phi(I)^3$ we can write $\Phi(I)(\Phi(I)^2 - I) = 0$. Multiplying the latter equation by $\Phi(T)$ from the left side and making use of Step 1, we have

$$\Phi(I)\Phi(T)(\Phi(I)^2 - I) = 0$$

for all $T \in \mathcal{A}$. Since $\mathcal{B}$ is prime, we obtain $\Phi(I)^2 - I = 0$. It follows that $(\Phi(I) - I)(\Phi(I) + I) = 0$. Again, by Step 1, we have

$$(\Phi(I) - I)\Phi(T)(\Phi(I) + I) = 0$$

for all $T \in \mathcal{A}$. Therefore, we have the result.

We prove the rest of proof by assuming that $\Phi(I) = I$. For the case $\Phi(I) = -I$, one can consider $\Psi = -\Phi$ then Ψ is Jordan triple multiplicative and $\Psi(I) = I$.

We note that it is easy to check the following relations

$$\Phi(P_i) = \Phi(P_i)^2$$

and

$$\Phi(P_i)\Phi(P_j) = 0$$

for $1 \leq i, j \leq 2$, $i \neq j$.

Step 3. *Φ is multiplicative or anti-multiplicative.*

By applying Step 2 we have Φ is a Jordan homomorphism. Since $\mathcal{B}$ is prime, Theorem A.7 from [15], results that Φ is either multiplicative or anti-multiplicative. $\square$

In the following theorem, the map $\Psi : \mathcal{A} \to \mathcal{B}$ is considered to preserve $*$-Jordan triple product. We prove that such a map is a $\mathbb{C}$-linear or conjugate $\mathbb{C}$-linear $*$-isomorphism.

Theorem 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two prime $*$-algebras such that $\mathcal{A}$ has a non-trivial projection. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a bijective map which satisfies in the following condition*

$$\Phi(AB^*A) = \Phi(A)\Phi(B)^*\Phi(A). \tag{3.2}$$

Then, there is a $\mathbb{C}$-linear or conjugate $\mathbb{C}$-linear $*$-isomorphism Ψ such that $\Psi(A) = \Phi(A)\Phi(I)^*$ for all $A \in \mathcal{A}$.

Proof. First, we show that $\Phi(I)$ is unitary. Since Φ is surjective, we can find an operator $A \in \mathcal{A}$ such that $\Phi(A) = I$. So,

$$\Phi(A^*) = \Phi(IA^*I) = \Phi(I)\Phi(A)^*\Phi(I) = \Phi(I)^2.$$

Therefore, we have

$$I = \Phi(A) = \Phi(I(A^*)^*I) = \Phi(I)\Phi(A^*)^*\Phi(I) = \Phi(I)(\Phi(I)^2)^*\Phi(I).$$

It follows that $\Phi(I)$ is invertible.

For showing that $\Phi(I)\Phi(I)^* = \Phi(I)^*\Phi(I) = I$, let $A = B = I$ in

$$\Phi(AB^*A) = \Phi(A)\Phi(B)^*\Phi(A).$$

We then get

$$\Phi(I) = \Phi(II^*I) = \Phi(I)\Phi(I)^*\Phi(I).$$

Multiplying above equation from the right and left sides by $\Phi(I)^{-1}$, respectively, we have the result.

Now, we prove that $\Psi(A^*) = \Psi(A)^*$ for all $A \in \mathcal{A}$. By (3.2), we have

$$\Psi(A^*) = \Phi(A^*)\Phi(I)^* = \Phi(I)\Phi(A)^*\Phi(I)\Phi(I)^* = (\Phi(A)\Phi(I)^*)^* = \Psi(A)^*.$$

Since Ψ preserves star. The following relation shows that Ψ is Jordan triple multiplicative

$$\Psi(ABA) = \Psi(A(B)^*)^*A) = \Psi(A)\Psi(B^*)^*\Psi(A) = \Psi(A)\Psi(B)\Psi(A)$$

for all $A, B \in \mathcal{A}$. So,

$$\Psi(ABA) = \Psi(A)\Psi(B)\Psi(A). \tag{3.3}$$

Now, by Theorem 3.1, Ψ is multiplicative.

From now on, we try to show that Ψ is $\mathbb{C}$-linear or conjugate $\mathbb{C}$-linear. It is clear that Ψ preserves positive elements, then Ψ preserves order.

For every $\lambda \in \mathbb{R}$, there exist two rational number sequences $\{r_n\}, \{s_n\}$ such that $r_n \leq \lambda \leq s_n$ and $\lim r_n = \lim s_n = \lambda$ when $n \to \infty$. So, by the additivity of Ψ we have

$$r_nI = \Psi(r_nI) \leq \Psi(\lambda I) \leq \Psi(s_nI) = s_nI.$$

Hence,

$$\Psi(\lambda I) = \lambda I \tag{3.4}$$

for $\lambda \in \mathbb{R}$. It means that Ψ is $\mathbb{R}$-linear.

On the other hand, we prove that $\Psi(iI) = iI$ or $\Psi(iI) = -iI$. For this purpose we apply the same trick as Steps 1 and 2. It is easy to check that $\Psi(iI)^3 = -\Psi(iI)$. So,

$$\begin{aligned}
\Psi(iI)\Psi(T) &= \Psi(iI)^2\Psi(-T)\Psi(iI) \\
&= \Psi(iI)^3\Psi(T)\Psi(iI)^2 \\
&= -\Psi(iI)\Psi(T)\Psi(iI)^2 \\
&= \Psi(T)\Psi(iI).
\end{aligned}$$

It implies that

$$\Psi(iI)\Psi(T) = \Psi(T)\Psi(iI) \tag{3.5}$$

for all $T \in \mathcal{A}$. By making use of $\Psi(iI)^3 = -\Psi(iI)$ again, we have

$$\Psi(iI)(\Psi(iI)^2 + I) = 0$$

by multiplying the latter equation by $\Psi(T)$ from left side and applying (3.5) we have

$$\Psi(iI)\Psi(T)(\Psi(iI)^2 + I) = 0.$$

By primeness we have $\Psi(iI)^2 + I = 0$. It follows that

$$(\Psi(iI) + iI)(\Psi(iI) - iI) = 0.$$

Multiplying the above equation by $\Psi(T)$ from left side and applying (3.5) we obtain $\Psi(iI) = iI$ or $\Psi(iI) = -iI$. These latter relations and (3.4) say that Ψ is $\mathbb{C}$-linear or conjugate $\mathbb{C}$-linear. □

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Vahid Darvish
Department of Mathematics
Faculty of Mathematical Sciences
University of Mazandaran
P. O. Box 47416-1468, Babolsar, Iran
Email address: vahid.darvish@mail.com

Mojtaba Nouri
Department of Mathematics
Faculty of Mathematical Sciences
University of Mazandaran
P. O. Box 47416-1468, Babolsar, Iran
Email address: mojtaba.nori2010@gmail.com

Mehran Razeghi
Department of Mathematics
Faculty of Mathematical Sciences
University of Mazandaran
P. O. Box 47416-1468, Babolsar, Iran
Email address: razeghi.mehran19@yahoo.com

Ali Taghavi
Department of Mathematics
Faculty of Mathematical Sciences
University of Mazandaran
P. O. Box 47416-1468, Babolsar, Iran
Email address: taghavi@umz.ac.ir