Almost Cohen-Macaulayness of Koszul Homology

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Abstract. Let \((R, m)\) be a commutative Noetherian ring, \(I\) an ideal of \(R\) and \(M\) a non-zero finitely generated \(R\)-module. We show that if \(M\) and \(H_0(I, M)\) are aCM \(R\)-modules and \(I = (x_1, \ldots, x_{n+1})\) such that \(x_1, \ldots, x_n\) is an \(M\)-regular sequence, then \(H_i(I, M)\) is an aCM \(R\)-module for all \(i\). Moreover, we prove that if \(R\) and \(H_i(I, R)\) are aCM for all \(i\), then \(R/(0 : I)\) is aCM. In addition, we prove that if \(R\) is aCM and \(x_1, \ldots, x_n\) is an aCM \(d\)-sequence, then \(\text{depth} H_i(x_1, \ldots, x_n; R) \geq i - 1\) for all \(i\).

Introduction

Throughout this paper, we assume that \(R\) is a commutative Noetherian ring with non-zero identity, \(I\) an ideal of \(R\) and \(M\) a non-zero finitely generated \(R\)-module. Let \(H_i(I, M)\) denote the \(i\)th Koszul homology of the ideal \(I\) with respect to some fixed system of generators for \(I\).

The \(R\)-module \(M\) is called almost Cohen-Macaulay (i.e., aCM) if for every \(p \in \text{Supp}_R(M)\) \(\text{grade}(p, M) = \text{grade}(p_R, M_p)\), and \(R\) is called an aCM ring if it is an aCM \(R\)-module. It is clear that all CM \(R\)-modules are aCM. Several fundamental properties and some characterizations of aCM modules have been proved in [9]. In particular, Kang in [9] proved that if \((R, m)\) is a local ring, then \(M\) is an aCM \(R\)-module if and only if \(\text{dim} M \leq 1 + \text{depth} M\). Moreover, several interesting examples have been given in [10]. After that, several authors studied aCM modules (see for example [2], [8], [12], [13] and [14]).

Huneke in [6] and [7] studied the Cohen-Macaulayness of Koszul homology of \(H_i(I, R)\). The main aim of this paper is to prove the following:

Theorem 0.1. Let \(R\) be a Noetherian ring and \(I\) be an ideal of \(R\).

(i) If \(I = (x_1, \ldots, x_{n+1})\) such that \(x_1, \ldots, x_n\) is an \(M\)-regular sequence and \(H_0(I, M)\) is an aCM \(R\)-module, then \(H_i(I, M)\) is an aCM \(R\)-module for all \(i \geq 0\).

(ii) If \((R, m)\) is an aCM local ring and \(H_i(I, R)\) is aCM for all \(i\), then \(R/(0 : I)\) is aCM.
(iii) If $(R, m)$ is an aCM local ring and $x_1, \ldots, x_n$ is an aCM $d$-sequence, then \( \text{depth} H_i(x_1, \ldots, x_n; R) \geq i - 1 \) for all \( i \geq 0 \) whenever \( H_i(x_1, \ldots, x_n; R) \neq 0 \).

For basic definitions and unexplained terminologies, we refer the reader to [1] or [15].

1. The results

We begin this section by the following lemma which is a generalization of [7, Remark 1.5].

**Lemma 1.1.** Let \( M \) be a CM \( R \)-module and let \( I = (x_1, \ldots, x_n) \) be an ideal of \( R \) with \( H_i(I, M) \neq 0 \). Then \( \dim H_i(I, M) = \dim M/IM \).

**Proof.** It is known that \( I + \text{Ann}(M) \subseteq \text{Ann}(H_i(I, M)) \). Hence \( \dim H_i(I, M) \leq \dim M/IM \). For converse, let \( p \) be a minimal prime ideal of Ass(\( M/IM \)) and set \( \text{grade}_M (IR_P) = k \). Then \( H_{n-k}(IR_P, M_P) \cong (yM : M I/yM)_P \) is the last non-vanishing homology module, where \( y = y_1, \ldots, y_k \) is an \( M_P \)-regular sequence in \( IR_P \). This module is a submodule of \( (M/yM)_P \), which is equidimensional and so is all of its submodules. Then by rigidity of the Koszul homology we cannot have any intermediate \( H_i(I, M)_P \) equal to 0 (see [15, Theorem 5.10]). Thus \( \text{Ann}(H_i(I, M)) \subseteq \sqrt{I + \text{Ann}(M)} \) and so \( \dim M/IM \leq \dim H_i(I, M) \). This completes the proof. □

By using the proof of Lemma 1.1, we conclude that \( \text{Ann}(H_i(I, M)) \subseteq \sqrt{I + \text{Ann}(M)} \). Vasconcelos, in [15, page 286], wrote that in general we have \( \text{Ann}(H_i(I, R)) \subseteq \sqrt{I} \). But the following example says that \( \text{Ann}(H_i(I, M)) \) is not contained in \( \sqrt{\text{Ann}(M/IM)} \) in general.

For the computation of all examples we use Macaulay 2 [3].

**Example 1.2.** Let \( R = k[x, y, z, u] \) be a polynomial ring with \( k \) be a field. Let \( I = (x, yz, yu) \) be an ideal of \( R \). Then \( \text{Ann}(H_1(I, M)) = (x, y) \), where \( M = R/I \).

The following example says that the assumption of Cohen-Maculayness of \( M \) in Lemma 1.1 is essential.

**Example 1.3.** Let \( R = k[x, y, z] \) be a polynomial ring with \( k \) is a field. Let \( I = (x, y) \) and \( M = R/(x) \oplus R/(x, y, z) \). Then \( H_2(I, M) \neq 0 \), \( \dim H_2(I, M) = 0 \) but we have \( \dim M/IM = 1 \).

**Corollary 1.4.** Let \( (R, m) \) be a local ring and let \( N \) be a CM \( R \)-modules with grade \( N \text{Ann}(M) = g \). Then \( \dim \text{Ext}^g(M, N) = \dim N/\text{Ann}(M) \). In particular, \( \dim \text{Ext}^g(M, N) = \dim N - g \).

**Proof.** Set \( \text{Ann}(M) = (x_1, \ldots, x_n) \). By [11, Corollary 2.2]

\[
\dim \text{Ext}^g(R/\text{Ann}(M), N) = \dim \text{Ext}^g(M, N)
\]
and also by [1, Theorem 1.6.16]

\[(1) \quad \dim \text{Ext}^g(R/\text{Ann}(M), N) = \dim H_{n-g}(\text{Ann}(M), N).\]

Thus by Lemma 1.1 and [1, Theorem 2.1.2] the result follows.

The following result easily follows by the proof of Lemma 1.1 and [13, Definition 2.1].

**Corollary 1.5.** Let \((R, \mathfrak{m})\) be a local ring and \(I\) be an ideal of \(R\). If \(\dim M \leq 1\), then \(M\) and \(H_i(I, M)\) are aCM for all \(i\).

**Proposition 1.6.** Let \(M\) be an aCM \(R\)-module and let \(I = (x_1, \ldots, x_n)\) be an ideal of \(R\) such that \(x = x_1\) is an \(M\)-regular element. Then \(H_i(I, M)\) is aCM for all \(i\) if and only if \(H_i(\overline{I}, \overline{M})\) is aCM for all \(i\), where \(\overline{I} = 1/(x)\) and \(\overline{M} = M/\mathfrak{m}M\).

**Proof.** From the exact sequence \(0 \to M \to M/\mathfrak{m}M \to 0\), we have a long exact sequence

\[
\cdots \to H_i(I, M) \to H_i(\overline{I}, \overline{M}) \to H_{i-1}(I, M) \to H_{i-1}(I, M) \cdots.
\]

Since \(xH_i(I, M) = 0 = xH_{i-1}(I, M)\), we have the exact sequence

\[
(\ast) \quad 0 \to H_i(I, M) \to H_i(\overline{I}, \overline{M}) \to H_{i-1}(I, M) \to 0
\]

for all \(i \geq 1\). Set \(\dim M/IM = d\). If \(H_i(I, M)\) are aCM, then \(\dim H_i(I, M) = 1 \leq d - 1 \leq \text{depth} H_i(I, M)\) for all \(i\) and so by the exact sequence (\(\ast\)), \(\dim H_i(\overline{I}, \overline{M}) = 1 \leq \text{depth} H_i(\overline{I}, \overline{M})\) for all \(i \geq 1\). For \(i = 0\), \(H_i(\overline{I}, \overline{M}) \cong \overline{M}/\overline{I}M = H_i(I, M)\) is aCM.

Conversely, suppose \(\dim H_i(\overline{I}, \overline{M}) \leq \dim \overline{M}/\overline{I}M = d\). Induct on \(i\) to show that \(\dim H_i(I, M) = 1 \leq \text{depth} H_i(I, M)\). For \(i = 0\), \(H_i(\overline{I}, \overline{M}) \cong M/IM = H_i(I, M)\), and by assumption this is aCM. Suppose we have shown \(H_{i-1}(I, M)\) is aCM. It follows from (\(\ast\)) that

\[
\text{depth} H_i(I, M) \geq \min\{\text{depth} H_i(\overline{I}, \overline{M}), \text{depth} H_{i-1}(I, M) + 1\}
\]

\[
\geq \min\{\text{depth} H_i(\overline{I}, \overline{M}), d\} = \text{depth} H_i(\overline{I}, \overline{M}).
\]

Therefore \(\text{depth} H_i(I, M) \geq \dim H_i(I, M) - 1\) and so \(H_i(I, M)\) is aCM, as required.

**Theorem 1.7.** Let \(M\) be an aCM \(R\)-module and \(I = (x_1, \ldots, x_n, x_{n+1})\) be an ideal of \(R\) such that \(x_1, \ldots, x_n\) is an \(M\)-regular sequence. If \(H_0(I, M)\) is an aCM \(R\)-module, then \(H_i(I, M)\) is an aCM \(R\)-module for all \(i \geq 0\).

**Proof.** Since \(\text{grade}(I, M) = n\), by [1, Theorem 1.6.17] we have \(H_0(I, M) = 0\) for all \(i \geq 2\). By assumption \(H_0(I, M) = M/IM\) is aCM. Thus it remains to show that \(H_1(I, M)\) is aCM. By [1, Theorem 1.6.16] we have \(H_1(I, M) \cong \text{Hom}_R(R/I, M/(x_1, \ldots, x_n)M)\). Consider the following exact sequences

\[(\dagger) \quad 0 \to M/(x_1, \ldots, x_n)M :M x_{n+1} \to M/(x_1, \ldots, x_n)M \to M/IM \to 0,
\]
and

\[ \begin{array}{c}
0 \to H_1(I, M) \to M/(x_1, \ldots, x_n)M \to M/(x_1, \ldots, x_n)M :x_{n+1} \to 0.
\end{array} \]

Since \( M/IM \) is aCM, we have \( \dim M/IM \geq \dim M - 1 \geq \dim M - \text{grade}_M I - 2 \), the second inequality follows by [12, Theorem 2.3]. Since \( M \) is aCM, we have

\[
\text{depth } M/(x_1, \ldots, x_n)M = \text{depth } M - n = \text{depth } M - \text{grade}_M I \\
\geq \dim M - \text{grade}_M I - 1.
\]

Hence the exact sequence (†) yields that

\[
\text{depth } M/(x_1, \ldots, x_n)M :x_{n+1} \geq \dim M - \text{grade}_M I - 1.
\]

Therefore the exact sequence (‡) yields that

\[
\text{depth } H_1(I, M) \geq \dim M - \text{grade}_M I - 1 \\
\geq \dim M/IM - 1 \geq \dim H_1(I, M) - 1.
\]

Hence \( H_1(I, M) \) is aCM.

\[ \square \]

**Theorem 1.8.** Let \((R, m)\) be an aCM local ring and \( I \) be an ideal of \( R \). If \( H_i(I, R) \) is aCM for all \( i \), then \( R/(0 : I) \) is aCM.

**Proof.** We can assume that \((0 : I) \neq 0\). If \( \dim R/I \leq 1 \), then by the proof of Lemma 1.1 \( \dim H_i(I, R) \leq 1 \) and so \( \dim R/(0 : I) \leq 1 \). Thus \( R/(0 : I) \) is aCM.

Now, we can assume that \( \dim R/I \geq 2 \) and so there exists a nonzero divisor \( z \) on \( H_i(I, R) \) and \( R \) for all \( i \). The exact sequence

\[ 0 \to R \xrightarrow{z} R \to R/zR \to 0 \]

gives a long exact sequence

\[ H_i(I, R) \xrightarrow{z} H_i(I, R) \to H_i(I, R/zR) \to H_{i-1}(I, R) \xrightarrow{z} H_{i-1}(I, R). \]

Since \( z \) is a nonzero divisor on \( H_{i-1}(I, R) \) and \( H_i(I, R) \), we obtain the exact sequence

\[ 0 \to H_i(I, R) \xrightarrow{z} H_i(I, R) \to H_i(I, R/zR) \to 0, \]

and so \( H_i(I, R)/zH_i(I, R) \cong H_i(I/zR, R/zR) \). Thus it follows that \( H_i(I/zR, R/zR) \) are aCM. Set \( \overline{I} = I/zI \) and \( \overline{R} = R/zR \). We induct on \( \dim R/I \) to prove \( R/(0 : I) \) is aCM. Since \( \dim R/I \geq 2 \) we choose \( z \) as above, \( \overline{R}/(0 : \overline{I}) \cong R/(z : I) \) is aCM. Let \( n \) be the number of generated of \( I \). Since \( z \) is not a zero divisor on \( H_n(I, R) \) we have \( (z : I)/z = H_n(T, \overline{R}) = H_n(I, R)/zH_n(I, R) = (0 : I)/z(0 : I) \). It follows that \( (z : I) = ((0 : I), z) \). Since \( z \) is not a zero divisor on \( R \), \( z \) is not a zero divisor on \( R/(0 : I) \). As \( R/(z : I) = R/((0 : I), z) \) is aCM, we conclude that \( R/(0 : I) \) is aCM, as required.

\[ \square \]

Huneke in [4] and [5] defined that a sequence \( x_1, \ldots, x_n \) in \( R \) is a \( d \)-sequence which satisfies in the following two conditions:

\[(i) \ x_i \notin (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \text{ for } 1 \leq i \leq n \text{ and} \]

and

\[(\dagger) \ 0 \to H_1(I, M) \to M/(x_1, \ldots, x_n)M \to M/(x_1, \ldots, x_n)M :x_{n+1} \to 0. \]
(ii) for all $k \geq i + 1$ and all $i \geq 0$, \((x_1, \ldots, x_i) : x_{i+1}x_k) = ((x_1, \ldots, x_i) : x_k).

In the following definition we generalize [7, Definition; page 297].

**Definition 1.9.** A $d$-sequence $x_1, \ldots, x_n$ is called aCM if the rings $R/(x_1, \ldots, x_i) : R I$ and $R/((x_1, \ldots, x_i) : R I) + I$ are aCM for all $0 \leq i \leq n - 1$, where $I = (x_1, \ldots, x_n)$.

In the sequel we recall the following example from [7, Example 2.1].

**Example 1.10.** Let $R = k[x_{ij}]$, where $R$ is a field and let $I$ be the ideal in $k[x_{ij}]$ generated by the $t$ minors of $X$. Set $R = k[x_{ij}]/I$. Then the images of any row or column of $X$ in $R$ form a CM $d$-sequence.

**Theorem 1.11.** Let $(R, m)$ be an $aCM$ local ring and $x_1, \ldots, x_n$ be an $aCM$ $d$-sequence. Then depth $H_i(x_1, \ldots, x_n; R) \geq i$ for all $i \geq 0$ whenever $H_i(x_1, \ldots, x_n; R) \neq 0$.

**Proof.** Let $I = (x_1, \ldots, x_n)$. We proceed by induction on $n$. Clearly, if $n = 1$, then by [1, Exercise 1.6.31], $H_i(I; R) = 0$ for all $i > 1$ and hence we have nothing to prove. Let $n > 1$ and the assertion holds for all $d$-sequence of length less than $n$. We consider two cases.

**Case 1:** Let $k := \text{grade } I > 0$. By [1, Exercise 1.6.31], $H_i(I; R) = 0$ for all $i > n - k$, and $H_i(I; R) \neq 0$ for all $0 \leq i \leq n - k$. Clearly, if $n = k$ we have nothing to prove. Let $n > k$. Since by [7, Remark 2.6], $x_1, \ldots, x_k$ is an $R$-regular sequence, then from [1, Theorem 1.6.16] it follows that $H_{n-k}(I; R) \cong ((x_1, \ldots, x_k) : R I)$. Hence, the exact sequence

$$0 \longrightarrow ((x_1, \ldots, x_k) : R I) \longrightarrow R/(x_1, \ldots, x_k) \longrightarrow R/((x_1, \ldots, x_k) : R I) \longrightarrow 0$$

yields that depth $H_{n-k}(I; R) \geq \dim R - k - 1$, because by this exact sequence, depth $R/(x_1, \ldots, x_k) \geq \dim R - k - 1$ and by Definition 1.9 we have depth $R/(x_1, \ldots, x_k) : R I) \geq \dim (x_1, \ldots, x_k) : R I) - 1$. From [12, Theorem 2.3] it follows that depth $R/(x_1, \ldots, x_k) : R I) \geq \dim R - \text{grade}((x_1, \ldots, x_k) : R I) - 2$. Note that grade $((x_1, \ldots, x_k) : R I) = k$. Indeed, since $(x_1, \ldots, x_k) \subseteq ((x_1, \ldots, x_k) : R I)$ then $\text{grade}((x_1, \ldots, x_k) : R I) \geq k$. Let $\text{grade}((x_1, \ldots, x_k) : R I) > k$. Thus, there exists $\alpha \in ((x_1, \ldots, x_k) : R I)$ such that $\alpha \notin Z_R(R/(x_1, \ldots, x_k))$. Now since $\alpha I \subseteq (x_1, \ldots, x_k)$ then there exists $x_{k+1} \in I \setminus (x_1, \ldots, x_k)$ such that $\alpha x_{k+1} \subseteq (x_1, \ldots, x_k)$. Since $\alpha \notin Z_R(R/(x_1, \ldots, x_k))$ then $x_{k+1} \notin (x_1, \ldots, x_k)$. But this is a contradiction with the definition of a d-sequence. Hence depth $H_{n-k}(I; R) \geq n - k - 1$.

Now, it remains to show that depth $H_i(I; R) \geq i - 1$ for all $0 \leq i < n - k$.

Consider the exact sequence

$$(2) \quad 0 \longrightarrow H_{n-k}(I; R) \longrightarrow H_{n-k}(I; R) / H_{n-k-1}(I; R) \longrightarrow 0.$$
where “"−" denotes the canonical homomorphism from $R$ to $R/(x_1)$ and $H_{n-k}(T, \overline{R})$ is the Koszul homology of the elements $0, x_2, \ldots, x_n$. Note that by induction hypothesis, for all $i$ we have depth $H_i(T, \overline{R}) \geq i - 1$ as $H_i(T, \overline{R}) \cong H_i(x_2, \ldots, x_n; \overline{R}) \oplus H_{i-1}(T_2, \ldots, T_n; \overline{R})$ (see [7, Remark 1.4]). So, the exact sequence (2) yields that depth $H_{n-k-1}(I; R) \geq n - k - 2$. Hence, the exact sequence

$$0 \rightarrow H_{n-k-1}(I; R) \rightarrow H_{n-k-1}(T, \overline{R}) \rightarrow H_{n-k-2}(I; R) \rightarrow 0$$

yields that depth $H_{n-k-2}(I; R) \geq n - k - 3$. Proceeding in this manner we get depth $H_i(I; R) \geq i - 1$ for all $0 \leq i < n - k$, as required.

Case 2: Let grade $I = 0$. By [7, Lemma 1.1], for all $i \geq 0$ we have the exact sequence

(2) $$0 \rightarrow \oplus(0 :_R I) \rightarrow H_i(I; R) \rightarrow H_i(T, \overline{R}) \rightarrow 0,$$

where “"−" denotes the homomorphism from $R$ to $R/(0 : I) = \overline{R}$. By this exact sequence, depth($0 :_R I$) $\geq i - 1$, because by Definition 1.9, $R/(0 :_R I)$ is aCM, and hence by [12, Theorem 2.3] we have depth $R/(0 :_R I) \geq \dim R/(0 :_R I) - 1 \geq \dim R - \text{grade}(0 :_R I) - 2$. Obversely grade($0 :_R I$) $= 0$ and so from the exact sequence 0 $\rightarrow (0 :_R I) \rightarrow R \rightarrow R/(0 :_R I) \rightarrow 0$ we have depth($0 :_R I$) $\geq \dim R - 1$. Consequently, from [7, Remark 2.4] it follows that depth($0 :_R I$) $\geq \dim R - 1 \geq n - 1 \geq i - 1$. Since grade$T \geq 1$, by using Case 1 and induction on $n$ we have depth $H_i(T, \overline{R}) \geq i - 1$. Now, the exact sequence (2) yields that depth $H_i(I; R) \geq i - 1$ for all $0 \leq i \leq n$. □

The following examples show that all almost Cohen-Macaulay $R$-modules are not necessarily Cohen-Macaulay $R$-module.

**Example 1.12.** (i) Let $k$ be a field. Set $R := k[[y]]$ and $M := k[x, y]/(x^2, xy)$. Then $M$ is a finitely generated $R$-module as the set $\{\overline{1}, \overline{x}\}$ generates $M$, where “"−" denotes the canonical homomorphism $R[[x]] \rightarrow M$. So, we have $\dim M = \dim R/\text{Ann}_R M = 1$ as $\text{Ann}_R M = 0$. Clearly, $(y) \subseteq Z_R(M)$, hence depth $M = 0$. Therefore, by [14, Lemma 1.2], it follows that $M$ is an almost Cohen-Macaulay $R$-module, however, it is not Cohen-Macaulay $R$-module.

(ii) Let $k$ be a field. Set $R := k[x, y, z]$, and $M := k[x, y, z]/(x, y) \cap (x, y, z)^2$. Clearly, $\dim R = 3$, $\dim M = 1$ and depth $M = 0$. Thus, by [14, Lemma 1.2], $M$ is an almost Cohen-Macaulay $R$-module but it is not Cohen-Macaulay.

(iii) All finitely generated $R$-modules with $\dim M \leq 1$ are almost Cohen-Macaulay.

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