FINITE $p$-GROUPS ALL OF WHOSE SUBGROUPS OF CLASS 2 ARE GENERATED BY TWO ELEMENTS

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Abstract. We proved that finite $p$-groups in the title coincide with finite $p$-groups all of whose non-abelian subgroups are generated by two elements. Based on the result, finite $p$-groups all of whose subgroups of class 2 are minimal non-abelian (of the same order) are classified, respectively. Thus two questions posed by Berkovich are solved.

1. Introduction

In this note, the groups considered are finite $p$-groups (in brief, $p$-groups). $p$-groups is the groups of prime-power order. The subgroup of class 2 of a group means the subgroup of nilpotent class 2. Assume $G$ is a $p$-group. We use $c(G)$ and $d(G)$ to denote the nilpotent class and the minimal number of generators of $G$ respectively. Let $r(G) = \max\{d(H) \mid H \leq G\}$ and $r_i(G) = \max\{d(H) \mid H \leq G \text{ and } c(H) = i\}$.

Obviously, $r(G) = \max\{r_i(G) \mid 1 \leq i \leq c(G) = c\}$.

Moreover, if $p$ is an odd prime, then Laffey in [5] have proved that $r(G) = \max\{r_1(G), r_2(G)\}$.

Blackburn in [4, Theorem 4.1] classified $p$-groups $G$ with $r_1(G) = 2$ and $p > 2$. Obviously, $r_2(G) \geq 2$. A natural question is: what can be said about $p$-groups $G$ with $r_2(G) = 2$? The motivation of this note is to classify such $p$-groups. We prove that such $p$-groups coincide with the $p$-groups all of whose non-abelian subgroups are generated by two elements, which was classified by Xu et al. in [8]. The fact implies that $r_2(G) = 2 \iff r_i(G) = 2$ for all $i$ with $2 \leq i \leq c$.
If $r_2(G) \geq 3$, then is it true that $r_i(G) \leq r_2(G)$ for all $i$ with $3 \leq i \leq c$? We will give an example to show that there exists a group $G$ of order $2^8$ with $r_2(G) = 3$ and $r_3(G) = 4$. This above fact motivates us to consider such a question: how much difference are there between the $p$-groups determined by some property of their non-abelian subgroups and the $p$-groups determined by some property of their subgroups of class 2? Notice that if $G$ is a minimal non-abelian $p$-group, then $G$ is two-generator. Hence as a nontrivial application of the classification of the $p$-groups by Xu et al. in [8], $p$-groups all of whose subgroups of class 2 are minimal non-abelian (of the same order) are respectively classified in this note. Hence the following two questions posed by Berkovich are solved.

**Problem 6** ([3, p337]). Classify the $p$-groups all of whose subgroups of class 2 are two-generator.

**Problem 372** ([1]). Study the $p$-groups all of whose subgroups of class 2 are minimal non-abelian.

### 2. Preliminaries

Following Berkovich and Janko [2], for a positive integer $t$, a finite $p$-group $G$ is called an $A_t$-group if its every subgroup of index $p^t$ is abelian, but it has at least one non-abelian subgroup of index $p^{t-1}$. So $A_1$-groups are nothing but the minimal non-abelian $p$-groups. For $t \leq 3$, all $A_t$-groups are known (see [6,11,12]). We use $G \in A_t$ to denote $G$ is an $A_t$-group.

Following Xu et al. [8], $B_p$ denotes the class of $p$-groups whose non-abelian proper subgroups are two-generator, $B'_p$ denotes the class of groups consisting of groups in $B_p$ which are neither abelian nor minimal non-abelian, $D_p = \{ G \in B'_p \mid G \text{ has an abelian maximal subgroup} \}$ and $M_p = \{ G \in B'_p \mid G \text{ has no abelian maximal subgroup} \}$. $D_p(2) = \{ G \in D_p \mid d(G) = 2 \}$ and $D_p(3) = \{ G \in D_p \mid d(G) = 3 \}$, $D'_p(2) = \{ G \in D_p(2) \mid G \text{ is not of maximal class} \}$ and $M'_p = \{ G \in M_p \mid G \text{ is neither metacyclic nor 3-group of maximal class} \}$.

In terms of notation mentioned above, the [8, Main Theorem] can be restated as follows.

**Theorem 2.1.** Suppose that $G$ is a finite non-abelian $p$-group. If all non-abelian proper subgroups of $G$ are two-generator, then $G$ is one of the following groups:

1. $A_1$-groups;
2. $A_2$-groups;
3. $p$-groups of maximal class with an abelian maximal subgroup;
4. 3-groups of maximal class;
5. $D'_p(2)$-groups with $p \geq 3$;
6. $M'_2$-groups with a unique minimal non-abelian maximal subgroup;
7. $M'_p$-groups having no minimal non-abelian maximal subgroup, where $p \geq 3$;
(8) metacyclic groups.

Remark 2.2. From the argument in [8] or a simple check, it is not difficult to get the converse of Theorem 2.1 is also true.

Lemma 2.3 ([12, Lemma 2.6(1-2)]). Assume $G \in A_2$. Then $d(G) \leq 3$. If $d(G) = 3$, then $c(G) = 2$.

Lemma 2.4 ([8, Lemma 2.2]). Suppose that $G$ is a finite non-abelian $p$-group. Then the following conditions are equivalent.

1. $G$ is minimal non-abelian;
2. $d(G) = 2$ and $|G'| = p$;
3. $d(G) = 2$ and $\Phi(G) = Z(G)$.

Proposition 2.5 ([7]). Let $G$ be a metabelian group and $a, b \in G$. For any positive integers $i$ and $j$, let

$$[ia, jb] = [a, b, a, \ldots, a, b, \ldots, b]_{i-1, j-1}.$$

Then, for any positive integers $m$ and $n$,

1. $[a^m, b^n] = \prod_{i=1}^{m} \prod_{j=1}^{n} [ia, jb]_{i+j}^{(i+j)_i}$,
2. $(ab^{-1})^m = a^m \left( \prod_{i+j \leq m} [ia, jb]_{i+j} \right) b^{-m}$, $m \geq 2$.

Lemma 2.6 ([1, Theorem 9.6(e)]). Let $G$ be a group of maximal class and order $p^m$, $p > 2$, $m > p + 1$. Then one of maximal subgroups of $G$ is the fundamental subgroup and the others are the subgroups of maximal class.

Lemma 2.7 ([1, §9, Exercise 10]). Let $G$ be a 3-group of maximal class. Then the fundamental subgroup of $G$ is either abelian or minimal non-abelian.

Lemma 2.8 ([8, Theorem 5.4]). Let $G \in M'_p$, $|G| = p^n \geq p^6$, $p$ be an odd prime and $K$ be a maximal subgroup of $G$. Then

1. $K$ is not a group of maximal class;
2. $K \in A_1$ or $K \in D'_p(2)$;
3. $c(G) = n - 2$;
4. If every maximal subgroup of $G$ is not minimal non-abelian, then $|G| = p^6$.

Lemma 2.9 ([8, Theorem 3.2(1)]). Assume $G$ is a $D'_p(2)$-group and $c(G) = c$. If $M$ is a non-abelian subgroup of $G$ with $|G : M| = p^t$, then $c \geq 3$, $t \leq c - 2$, $c(M) = c - t$. 
3. The classification of finite $p$-groups $G$ with $r_2(G) = 2$ and its application

Assume $G$ is a finite non-abelian $p$-group. For convenience, we introduce the following notation.

$\mathcal{Q}_i = \{G \mid G$ is the $p$-group whose non-abelian subgroups have property $\mathcal{P}_i\}$;
$\mathcal{Q}^*_i = \{G \mid G$ is the $p$-group whose non-abelian proper subgroups have property $\mathcal{P}_i\}$;
$\mathcal{R}_i = \{G \mid G$ is the $p$-group whose subgroups of class 2 have property $\mathcal{P}_i\}$;
$\mathcal{R}^*_i = \{G \mid G$ is the $p$-group whose proper subgroups of class 2 have property $\mathcal{P}_i\}$.

In this note, $\mathcal{P}_1$ is “two-generator”, $\mathcal{P}_2$ is “minimal non-abelian” and $\mathcal{P}_3$ is “the same order”.

Obviously,

$\mathcal{Q}_i \subseteq \mathcal{Q}^*_i$, $\mathcal{R}_i \subseteq \mathcal{R}^*_i$, $\mathcal{Q}_i \subseteq \mathcal{R}_i$, $\mathcal{Q}^*_i \subseteq \mathcal{R}^*_i$ and $\mathcal{Q}^*_i \cup \mathcal{R}_i = \mathcal{R}^*_i$.

Moreover, in this note we will prove

$\mathcal{Q}_1 = \mathcal{R}_1$, $\mathcal{Q}^*_1 = \mathcal{R}^*_1$, $\mathcal{R}_3 \subseteq \mathcal{R}_2 \subseteq \mathcal{R}_1$ and $\mathcal{R}^*_3 \subseteq \mathcal{R}^*_2 \subseteq \mathcal{R}^*_1$.

A nature question is: is it true that $\mathcal{Q}_i = \mathcal{R}_i$ and $\mathcal{Q}^*_i = \mathcal{R}^*_i$ for $i = 2, 3$?

By determining the groups in $\mathcal{R}_2$ and $\mathcal{R}_3$, we can get the answer is false. That is,

$\mathcal{Q}_2 \not\subseteq \mathcal{R}_1$ and $\mathcal{Q}^*_2 \not\subseteq \mathcal{R}^*_1$ for $i = 2, 3$.

**Theorem 3.1.** (1) $\mathcal{Q}_1 = \mathcal{R}_1$; (2) $\mathcal{Q}^*_1 = \mathcal{R}^*_1$; (3) $\mathcal{R}_3 \subseteq \mathcal{R}_2 \subseteq \mathcal{R}_1$; (4) $\mathcal{R}^*_3 \subseteq \mathcal{R}^*_2 \subseteq \mathcal{R}^*_1$.

**Proof.** (1) Obviously, $\mathcal{Q}_1 \subseteq \mathcal{R}_1$. We prove $\mathcal{Q}_1 \supseteq \mathcal{R}_1$. If not, then there exists $G$ such that $G \in \mathcal{R}_1$ and $G \notin \mathcal{Q}_1$. Let $\mathcal{K} = \{K \leq G \mid d(K) \geq 3$ and $K' \neq 1\}$. Since $G \notin \mathcal{Q}_1$, $\mathcal{K} \neq \emptyset$. Hence there exists $K \in \mathcal{K}$ such that $|K|$ is of smallest order. It follows that $K \in \mathcal{Q}^*_1$. Thus $K$ is isomorphic to one of the groups in Theorem 2.1. By a simple check we get $d(K) = 2$ but $A_2$-groups. Hence $K$ is an $A_2$-group and $d(K) \geq 3$. It follows by Lemma 2.3 that $c(K) = 2$. Notice that if $G \in \mathcal{R}_1$, then $H \in \mathcal{R}_1$ for all $H \leq G$. Hence $K \in \mathcal{R}_1$. This contradicts $d(K) = 3$. Thus the conclusion follows.

(2) Obviously, $\mathcal{Q}^*_1 \subseteq \mathcal{R}^*_1$. We prove $\mathcal{Q}^*_1 \supseteq \mathcal{R}^*_1$. Let $G \in \mathcal{R}^*_1$ and $H$ is a non-abelian proper subgroup of $G$. Then $H \in \mathcal{R}_1$. It follows from (1) that $H \in \mathcal{Q}_1$. Hence $d(H) = 2$. Thus the conclusion follows.

(3) It follows from Lemma 2.4 that $\mathcal{R}_2 \subseteq \mathcal{R}_1$. We prove $\mathcal{R}_3 \subseteq \mathcal{R}_2$. Assume $G \in \mathcal{R}_3$, $H \leq G$ and $c(H) = 2$. Let $K < H$. Since $c(H) = 2$, $c(K) \leq 2$. Since $G \in \mathcal{R}_3$ and $c(H) = 2$, $c(K) \neq 2$. Hence $K$ is abelian. It follows that $H$ is minimal non-abelian. This means $G \in \mathcal{R}_2$. Thus the conclusion follows.

(4) It is a direct consequence of (3). □
Now the $p$-groups in $Q_1^*$ were classified by Xu et al. in [8]. Thus, by Theorem 3.1(1),(2), Lemma 2.3 and the argument of Theorem 3.1(1) we get:

**Theorem 3.2.** Suppose that $G$ is a finite non-abelian $p$-group. Then

1. $G \in \mathcal{R}_1$ if and only if $G$ is one of the groups in Theorem 2.1.

2. $G \in \mathcal{R}_1$ if and only if $G$ is one of the groups in Theorem 2.1 except for $A_2$-groups with three-generator.

In following we determine the groups in $\mathcal{R}_2$ and $\mathcal{R}_3$.

**Theorem 3.3.** $G \in \mathcal{R}_2$ if and only if all $A_2$-subgroups of $G$ are of class 3.

**Proof.** (⇒) Let $L \leq G$ and $L \in A_2$. Then $c(L) \leq 3$ by [12, Lemma 2.6(1)]. Since $G \in \mathcal{R}_2$, $c(L) = 3$.

(⇐) If not, then there exists $L$ such that $L \leq G$, $c(L) = 2$ and $L$ is not minimal non-abelian. Without loss of generality assume $L$ is an $A_1$-group with $t \geq 2$. Let $H$ be an non-abelian subgroup of smallest order of $L$. By the definition of $A_2$, we get $|L : H| = p^{t-1}$. Thus there exists $K$ satisfying $L \supseteq K \geq H$ and $|K : H| = p$. Thus $K$ is an $A_2$-group. Since $c(L) = 2$, $c(K) = 2$. This contradicts “all $A_2$-subgroups of $G$ are of class 3”.

**Lemma 3.4.** Assume $G \in \mathcal{R}_1$ and $|G'| \geq p^2$. Then $G \in \mathcal{R}_2$ if and only if all subgroups $H$ of $G$ with $|H'| = p^2$ are of class 3.

**Proof.** (⇒) By hypothesis we get $2 \leq c(H) \leq 3$. Since $|H'| = p^2$, $H$ is not minimal non-abelian by Lemma 2.4. It follow by $G \in \mathcal{R}_2$ that $c(H) = 3$.

(⇐) Let $L \leq G$ and $c(L) = 2$. We need to show $L \in A_1$. Since $G \in \mathcal{R}_1$, $d(L) = 2$. Assume $L = \langle a, b \rangle$ without loss of generality. Since $c(L) = 2$, $L' = \langle [a, b], g \in G \rangle = \langle [a, b] \rangle \leq Z(L)$. Let $|L'| = p^t$. If $t \geq 2$, then let $K = \langle a^{p^{t-2}}, b \rangle$. We get $K \leq L$ and $|K'| = p^2$. Hence $c(K) = 3$. This contradicts $c(L) = 2$. Hence $t = 1$. It follows by Lemma 2.4 that $L \in A_1$.

**Lemma 3.5.** Assume $G$ is a 3-group of maximal class which has no abelian subgroup of index 3. Then one of maximal subgroups of $G$ is minimal non-abelian and the others are of maximal class with an abelian maximal subgroup.

**Proof.** Notice that there exists an abelian maximal subgroup in a group of maximal class with order $3^4$. Hence $|G| \geq 3^5$. By Lemma 2.6, all maximal subgroups of $G$ are of maximal class except for the fundamental subgroup. The fundamental subgroup of $G$ is minimal non-abelian by Lemma 2.7. It follows that $\Phi(G)$ is abelian. Moreover, $\Phi(G)$ is maximal in all maximal subgroups of $G$.

**Lemma 3.6.** Suppose that $G$ is a finite non-abelian $p$-group. Then

1. if $G \in A_1$, then $G \in \mathcal{R}_3$;

2. if $G \in A_2$ and $c(G) = 3$, then $G \in \mathcal{R}_3$;

3. if $G \in A_2$ and $c(G) \neq 3$, then $G \not\in \mathcal{R}_2$;

4. if $G$ is a $p$-group of maximal class with an abelian maximal subgroup, then $G \in \mathcal{R}_3$;
(5) If $G$ is a 3-group of maximal class having no abelian maximal subgroup, then $G \in \mathcal{R}_2 \setminus \mathcal{R}_3$.

(6) If $G \in \mathcal{D}_p'(2)$, then $G \in \mathcal{R}_3$.

(7) If $G \in \mathcal{M}_p'$ and $G$ has no minimal non-abelian maximal subgroup, where $p \geq 3$, then $G \in \mathcal{R}_3$.

(8) If $G \in \mathcal{M}_3'$ and $G$ has a unique minimal non-abelian maximal subgroup, then $G \in \mathcal{R}_2 \setminus \mathcal{R}_3$.

Proof. (1) and (2) are trivial. It follows by the definition of $A_t$-groups.

(3) It follows by Theorem 3.3.

(4) By [9, Corollary 8.3.2] we know all non-abelian subgroups of $G$ are of maximal class. Hence all subgroups of class 2 are of order $p^3$. That is, $G \in \mathcal{R}_3$.

(5) Let $M$ be a subgroup of class 2 of $G$. Obviously, $c(G) > 2$. Hence $M$ is contained in a maximal subgroup of $G$. By Lemma 3.5, one of maximal subgroups of $G$ is minimal non-abelian and the others are of maximal class with an abelian maximal subgroup. If $M$ is contained in a minimal non-abelian subgroup, then $M$ is minimal non-abelian. If $M$ is contained in a subgroup of maximal class with an abelian maximal subgroup, then, by the argument of (4), $|M| = 3^3$. Hence $M$ is also minimal non-abelian. In either case, $G \in \mathcal{R}_2$.

Now $G$ has a subgroup of class 2 of order $3^3$ by the argument above paragraph. On the other hand, it follows by Lemma 3.5 that $G$ has a maximal subgroup which is minimal non-abelian. Moreover, $|G| \geq 3^5$ by the argument of Lemma 3.5. Hence $G$ has a subgroup of class 2 of order greater than $3^3$. So $G \not\in \mathcal{R}_3$.

(6) Let $M$ be a subgroup of class 2 of $G$. Then $|G : M| = p^{c(G) - 2}$ by Lemma 2.9. That is, all subgroups of class 2 of $G$ are of the same order. Thus $G \in \mathcal{R}_3$.

(7) Firstly, we claim that each maximal subgroup of $G$ is of class 3. In fact, let $K$ be a maximal subgroup of $G$. Since $G \in \mathcal{M}_p'$, we get $c(G) = 4$, $K \in \mathcal{D}_p'(2)$ and $c(K) \neq 4$ by Theorem 2.8. It follows by $c(G) = 4$ and $c(K) \neq 4$ that $c(K) \leq 3$. Since $K \in \mathcal{D}_p'(2)$, $c(K) = 3$ by Lemma 2.9.

Let $M$ be a subgroup of class 2 of $G$. Since $c(G) = 4$, $M$ is contained in a maximal subgroup $H$ of $G$. Thus $|H : M| = p^{c(H) - 2}$ by Lemma 2.9. Thus all subgroups of class 2 of $G$ are of the same order. So $G \in \mathcal{R}_3$.

(8) Let $M$ be a subgroup of class 2 of $G$. It follows by Lemma 2.8 that $c(G) > 2$, and one of maximal subgroups of $G$ is minimal non-abelian and the others are $\mathcal{D}_p'(2)$ groups. Hence $M$ is contained in a maximal subgroup of $G$. If $M$ is contained in a minimal non-abelian subgroup, then $M$ is minimal non-abelian. If $M$ is contained in $\mathcal{D}_p'(2)$ group, then, by (6) and Theorem 3.1(3), $M$ is also minimal non-abelian. In either case, $G \in \mathcal{R}_3$.

Since $G$ has a maximal subgroup which is minimal non-abelian, $G$ has a maximal subgroup $M_1$ of class 2. On the other hand, by the argument of above paragraph, we get that there exists $K \in \mathcal{D}_p'(2)$ and $K$ is maximal in $G$. Then $c(K) \geq 3$ by Theorem 2.9. Thus there exists a subgroup $M_2$ of class 2 which is a proper subgroup of $K$. Obviously, $|M_1| \neq |M_2|$. So $G \not\in \mathcal{R}_3$. □
Theorem 3.7. Suppose that $G$ is a finite nonabelian $p$-group. Then $G \in \mathcal{R}_2$ if and only if $G$ is one of the following groups:

(1) One of the groups (1) and (3)-(7) in Theorem 2.1;
(2) $A_2$-groups with class 3;
(3) metacyclic groups: $\langle a, b \mid a^{r+s+u+v+u} = 1, b^{r+s+t} = a^{r+s+u+v}, a^b = a^{-1+2^{r+s}}, \rangle$, where $r, s, v, t, t'$ and $u$ are non-negative integers satisfying $r \geq 2$, $t' \leq r$, $u \leq 1$, $tt' = sv = tv = 0$, $0 \leq s + t' + u \leq 2$, and $u = 0$ if $t' \geq r - 1$.

Proof. ($\Rightarrow$) By Theorem 3.1(3) we get $\mathcal{R}_2 \subseteq \mathcal{R}_1$. By Theorem 3.2(2), $G$ is one of the groups in Theorem 2.1 except for $A_2$-groups with three-generator. If $G$ is one of the groups (1)-(7) in Theorem 2.1, then, by Lemma 3.3, we get the groups (1)-(2) in the Theorem. The remains is the case of $G$ being metacyclic.

Assume $G$ is metacyclic. Then, by [10, Theorems 2.1, 2.2 and Remark 2.3], $G$ is one of the following groups:

(i) groups with a cyclic subgroup of index $p$;
(ii) $\langle a, b \mid a^{r+s+u+v} = 1, b^{r+s+t} = a^{r+s+u}, a^b = a^{1+p'}, \rangle$, where $r, s, t$ and $u$ are non-negative integers satisfying $u \leq r$, and $r \geq 2$ if $p = 2$; $r \geq 1$ if $p > 2$;
(iii) $\langle a, b \mid a^{r+s+u+v+u} = 1, b^{r+s+t} = a^{r+s+u+v}, a^{r+s+t} = a^{-1+2^{r+s}}, \rangle$, where $r, s, v, t, t'$ and $u$ are non-negative integers satisfying $r \geq 2$, $t' \leq r$, $u \leq 1$, $tt' = sv = tv = 0$, and $u = 0$ if $t' \geq r - 1$.

If $G$ is the group (i), then $G$ is minimal non-abelian or a group of maximal class with an abelian maximal subgroup by [1, Theorem 1.2]. They are one of the groups (1) in the Theorem.

If $G$ is the group (ii), then we will prove $r + s + u \leq 3$. If not, then let $K = \langle a, b^{r+s+u} \rangle$. By calculation, using Proposition 2.5(1), we get

$$[a, b^{r+s+u}] = [a, b]^{p^{r+s+u}} [a, b]^{(2^{r+s+u})}.$$

Since $r + s + u > 3$, $[a, b]^{(2^{r+s+u})} = 1$. Notice that $\langle [x, y] \rangle \leq G$ for any $x, y \in G$. Thus

$$K' = \langle [a, b^{r+s+u}] \rangle = \langle [a, b]^{p^{r+s+u}} \rangle = \langle a^{p^{r+s+u}} \rangle.$$

Then $|K'| = p^2$. It follows by Lemma 3.4 that $c(K) = 3$. Hence $K_3 \neq 1$, where $K_3$ is the third term of the lower center series of $G$. Notice that

$$K_3 = \langle [a, b^{p^{r+s+u}}, b^{p^{r+s+u}}] \rangle = \langle a^{2^{2r+2s+2u-4}} \rangle.$$

Hence $2r + 2s + 2u - 4 < r + s + u$. That is, $r + s + u \leq 3$. This is a contradiction.

Now it follows from $r + s + u \leq 3$ that $|G'| \leq p^2$. By Theorem 3.1(3), $G \in \mathcal{R}_2 \subseteq \mathcal{R}_1$. Hence non-abelian subgroups of $G$ are generated by two elements. If $|G'| = p$, then $G \in A_1$ by Lemma 2.4. Thus $G$ is one of the groups (1) in the Theorem. If $|G'| = p^2$, then it is easy to get $|M'| = p$ for each non-abelian maximal subgroup $M$ of $G$. It follows by Lemma 2.4 that $M \in A_1$. Hence $G \in A_2$. Since $G \in \mathcal{R}_2$, $G$ is the group (2) in the Theorem by Theorem 3.3.
If $G$ is the group (iii), then we will prove $s + t' + u \leq 2$. If not, then let $K = \langle a, b^{2s+t'+u-2} \rangle$. By calculation, using the formula in Proposition 2.5(1), we get
\[
[a, b^{2s+t'+u-2}] = a^{-1}a^{b^{2s+t'+u-2}}b^{a^{-1}a^{(1+2s+t')^2+t'+u-2}}.
\]
Since $s + t' + u > 2$,
\[
\langle a^{-1}a^{(1+2s+t')^2+t'+u-2} \rangle = \langle a^{p^{r+s+t'+u-2}} \rangle.
\]
Thus \((\langle a, b^{2s+t'+u-2} \rangle) = \langle a^{p^{r+s+t'+u-2}} \rangle\). Hence
\[
|K'| = |\langle a, b^{2s+t'+u-2} \rangle| = |\langle a^{p^{r+s+t'+u-2}} \rangle| = p^2.
\]
It follows by Lemma 3.4 that $c(K) = 3$. Hence $K_3 \neq 1$. Notice that
\[
K_3 = \langle a^{p^{r+s+t'+u-2}}, b^{p^{r+s+t'+u-2}} \rangle = \langle a^{p^{r+s+t'+u-2}} \rangle.
\]
Hence $2(r+s+v+t'+u-2) < r+s+v+t'+u$. That is, $r+s+v+t'+u \leq 3$. This is a contradiction. We get the groups (3) in the Theorem.

\((\Longleftarrow)\) If $G$ is one of the groups (1)-(2), then $G \in R_2$ by Theorem 3.6. We will prove all subgroups of class 2 in the groups (3) are minimal non-abelian. Assume $G$ is the group (3), $H \leq G$ and $|H'| = 4$. By Lemma 3.4 it is enough to show $c(H) = 3$.

It is easy to see that $H' = \langle a^{2^{r+s+t'+u-2}} \rangle$. Assume $H = \langle a^{i_1b^{j_1}}, a^{i_2b^{j_2}} \rangle$ without loss of generality, where $i_1, i_2, j_1, j_2$ are integer numbers. Let $M = \langle a, b^2 \rangle$. Then
\[
[a, b^2] = a^{-1}a^{b^2} = a^{(-1+2^{-s})^2-1}.
\]
Obviously, $2^{r+s+1} | (-1 + a^{-r})^2 - 1$. Since $s + t' + u \leq 2$, $|M'| \leq 2$. If $2 \mid j_1$ and $2 \mid j_2$, then $H \leq M$. This contradicts $|H'| = 4$. Hence $2 \nmid j_1$ or $2 \nmid j_2$. Assume $2 \nmid j_1$ without loss of generality. It easy to see that
\[
[a^{i_1}b^{j_1}, a^{2^{r+s+t'+u-2}}] = [b^{j_1}, a^{2^{r+s+t'+u-2}}].
\]
Since $a^{2^{r+s+t'+u-2}} \in Z(G)$, $[b^{j_1}, a^{2^{r+s+t'+u-2}}] \neq 1$. Hence $H_3 \neq 1$. So $c(H) = 3$. The proof is complete. \(\square\)

**Theorem 3.8.** Suppose that $G$ is a finite nonabelian $p$-group. Then $G \in R_3$ if and only if $G$ is one of the following groups:

1. One of the groups (1), (3), (5) and (7) in Theorem 2.1;
2. the groups (2) in Theorem 3.7;
3. the groups (3) in Theorem 3.7 with $s + t' + u \leq 1$.

**Proof.** \((\Longrightarrow)\) By Theorem 3.1(3) we get $R_3 \subseteq R_2$. Thus $G$ is one of the groups in Theorem 3.7. If $G$ is one of the groups (1)-(2) in Theorem 3.7, then, by Lemma 3.6, we get the groups (1)-(2) in the Theorem. If $G$ is the group (3) in Theorem 3.7, then we will prove $s + t' + u \leq 1$. If not, then let $H_1 = \langle a^{2^{r+s}}, b \rangle$ and $H_2 = \langle a, b^2 \rangle$. It is easy to get $|H_1'| = |H_2'| = 2$. Hence $H_1$ and $H_2$ are
of class 2. Since \( r \geq 2 \), \( H_1 \) is not maximal in \( G \). On the other hand, \( H_2 \) is maximal in \( G \). Hence \( |H_1| \neq |H_2| \). This contradicts \( G \in R_3 \). So \( s + t' + u \leq 1 \). We get the group (3) in the Theorem.

\( \langle \rangle \) If \( G \) is one of the groups (1)-(2), then \( G \in R_3 \) by Theorem 3.6. If \( G \) is the group (3), then each subgroup \( K \) of class 2 of \( G \) is minimal non-abelian. It follows by Lemma 2.4 that \( |K'| = 2 \). It is enough to show each subgroup \( H \) of \( G \) with \( |H'| = 2 \) is of the same order. Without loss of generality assume

\[
H = \langle b^i a^{i_1}, b^j a^{i_2} \rangle,
\]

where \( i_1, i_2, j_1, j_2 \) are integer numbers. Notice that

\[
[a, b^2] = a^{-1} b^2 a = a^{-1} a^{(r+v)2} = a^{r+v}.
\]

Obviously, \( 2^{r+v+1} | (-1 + a^{r+v})^2 - 1 \). Since \( s + t' + u \leq 1 \), \( b^2 \in Z(G) \). If \( 2 \not| j_1 \) and \( 2 \not| j_2 \), then \( H \) is abelian. This contradicts \( |H'| = 2 \). Hence \( 2 \not| j_1 \) or \( 2 \not| j_2 \). Assume \( 2 \not| j_1 \) without loss of generality. By calculation we have that there exists \( k_1 \) such that \( (b^i a^{i_1})^{i_1} = b^k_1 \). Then \( H = \langle b^{k_1}, b^j a^{i_2} \rangle \). Moreover, there exists \( k_2 \) such that \( (b^{k_1})^{i_2} b^j a^{i_2} = a^{k_2} \). Thus \( H = \langle b^{k_1}, a^{k_2} \rangle \). Now

\[
H' = \langle [b^{k_1}, a^{k_2}] \rangle = \langle [b, a^{k_2}] \rangle = \langle a^{k_2} \rangle.
\]

On the other hand, since \( |H'| = 2 \), \( H' = \langle a^{2^{r+v+1} + u - 1} \rangle \).

Let \( n = r + s + v + t' + u \). Then \( 2k_2 \equiv 2^{n-1}(mod\ 2^n) \). That is, \( k_2 \equiv 2^{n-2}(mod\ 2^n - 1) \). Hence

\[
H = \langle b^{k_1}, a^{2^{n-2}} \rangle.
\]

By calculation we get

\[
(b^{k_1})^2 = b^2 a^{k_1} 2^{r+v} \neq 1, (b^{k_1})^4 = (b^2 a^{k_1} 2^{r+v})^2 = b^4 a^{k_1} 2^{r+v+1} = b^4.
\]

Hence

\[
|H| = |\langle b^{k_1}, a^{2^{n-2}} \rangle| = \frac{|\langle a^{2^{n-2}} \rangle| |\langle b^{k_1} \rangle|}{|\langle a^{2^{n-2}} \rangle \cap \langle b^{k_1} \rangle|} = \frac{|\langle a^{2^{n-2}} \rangle| |\langle b \rangle|}{|\langle a^{2^{n-2}} \rangle \cap \langle b \rangle|}.
\]

By the arbitrary of \( H \), the conclusion follows. \( \square \)

**Corollary 3.9.** Suppose that \( G \) is a finite non-abelian \( p \)-group. Then

1. If \( G \) is non-metacyclic, then \( G \in R_2 \) if and only if \( G \in R_1 \);
2. If \( G \) has no minimal non-abelian maximal subgroup, then \( G \in R_3 \) if and only if \( G \in R_2 \).

**Proof.** (1) By Theorem 3.2 and Theorem 3.7, it is enough to check non-metacyclic \( A_2 \)-groups \( G \) with \( d(G) \neq 3 \) are of class 3. \( A_2 \)-groups are listed in [11] or [12, Lemma 2.5]. This is a routine work.

(2) It follows by Theorem 3.7, Theorem 3.8 and Lemma 3.5. \( \square \)

**Corollary 3.10.** \( Q_i \subseteq R_i \) and \( Q_i^* \subseteq R_i^* \) for \( i = 2, 3 \).
Proof. Let $G$ be a maximal class group of order $3^5$ and $G$ have an abelian maximal subgroup. Then $G \in \mathcal{R}_i$ for $i = 2, 3$ by Theorem 3.7 and Theorem 3.8. Thus $G \in \mathcal{R}_i^*$ for $i = 2, 3$. It is obvious that $|Z(G)| = p$. Thus there is a non-abelian subgroup $H$ of order $3^4$ of $G$. By [9, Corollary 8.3.2] we know all non-abelian subgroups of $G$ are of maximal class. Hence $c(H) = 3$. So $H$ is not a minimal non-abelian group by Lemma 2.4. Then $G \notin \mathcal{Q}_2^*$. It follows by $\mathcal{Q}_3^* \subseteq \mathcal{Q}_2^*$ that $G \notin \mathcal{Q}_3^*$. Obviously, $G \notin \mathcal{Q}_i$ for $i = 2, 3$. □

4. An example of a $p$-group $G$ with $r_2(G) = 3$ and $r_3(G) = 4$

Theorem 3.1(1) means such a fact that $r_2(G) = 2 \iff r_i(G) = 2$ for all $i$ with $2 \leq i \leq c(G)$. In other words, if $r_2(G) = 2$, then $r_i(G) \leq r_2(G)$ for all $i$ with $3 \leq i \leq c(G)$. However, if $r_2(G) \geq 3$, then the fact is not true. Here we give an example to show that there exists a group $G$ of order $2^8$ with $r_2(G) = 3$ and $r_3(G) = 4$. First we give a lemma as follows.

Lemma 4.1. Let $G = \langle a, b, c, d \rangle | a^4 = b^4 = c^4 = 1, d^2 = b^2c^2, [a, b] = [a, c] = 1, [a, d] = [b, d] = a^2, [b, c] = a^2b^2, [c, d] = a^2c^2 \rangle$. Then $d(H) \leq 3$ for $H < G$.

Proof. By a simple checking we know that $G \in \mathcal{A}_4$ and $|G| = 2^7$, and $\Omega_1(G) = \Omega_i(G) = Z(G) = G' \cong C_2^3$. It follows that $d(H) \leq 3$ if $H$ is abelian. By Lemma 2.4 we get $d(H) = 2$ if $H \in \mathcal{A}_1$. It follows that $d(H) \leq 3$ if $H \in \mathcal{A}_2$. So it needs only to show $d(H) \leq 3$ for any $\mathcal{A}_2$-subgroup $H$ of $G$. If not, then there exists $M \in \mathcal{A}_3$ and $d(M) \geq 4$. Let $\overline{G} = G/(a^2)$. Then $\overline{G} = \langle \bar{a} \rangle \times \langle \bar{b}, \bar{c}, \bar{d} \rangle$, where $\langle \bar{b}, \bar{c}, \bar{d} \rangle$ is a minimal non-metacyclic group of order $2^8$. Obviously, all maximal subgroups of $\overline{G}$ are three-generator. It follows that $d(M) = 3$. It follows from $d(M) > d(M)$ that $a^2 \notin \Phi(M)$. Hence $a \notin M$. Thus $M = \langle ba^i, ca^j, da^k \rangle$, where $i, j, k \in \{0, 1\}$. Let $K = \langle ba^i, ca^j, da^k \rangle$. Since $d(M) \geq 4$, $a^2 \notin K$. On the other hand, $[ca^j, da^k, (ca)^2] = (c^2a^2a^2)(c^2a^2) = a^2 \in K$. This is a contradiction. □

Example 4.2. Let $G = \langle a, b, c, d \rangle | a^8 = b^4 = c^4 = 1, d^2 = a^4b^2c^2, [a, b] = [a, c] = [b, c^2] = 1, [a, d] = [b, d] = a^2, [b, c] = a^2b^2, [c, d] = a^{-2}c^2 \rangle$ and $H$ be a non-abelian proper subgroup of $G$. Then $|G| = 2^8, e(G) = 3, d(G) = 4$ and $d(H) \leq 3$.

Proof. Let $K = \langle a, b, c^2 \rangle | a^8 = b^4 = c^4 = 1, [a, b] = [a, c^2] = [b, c^2] = 1$. Then $K \cong C_8 \times C_4 \times C_2$. Let $M = \langle K, e \rangle = \langle a, b, c \rangle | a^8 = b^4 = c^4 = 1, [a, b] = [a, c] = [b, c^2] = 1, [b, c] = a^2b^2 \rangle$.

Then $M$ is an extension of $K$ by $C_2$. It is easy to verify that $G$ is an extension of $M$ by $C_2$. Thus $|G| = 2^8$.

By calculation we get $G' = \Omega_1(G) = \Phi(G) = \langle a^2 \rangle \times \langle b^2 \rangle \times \langle c^2 \rangle \cong C_4 \times C_2 \times C_2$ and $G_3 = \langle a^4 \rangle \cong C_2$. 


where $G_3$ is the third term of the lower central series of $G$. Thus $d(G) = 4$ and $c(G) = 3$.

In following we prove $d(H) \leq 3$. First we have the following facts:

(1) $\Omega_1(G) \cong C_3^2$;

(2) $\Omega_2(C_G(\Omega_1(G))) \cong C_4 \times C_2$;

(3) $\Omega_2(G) = G_1 = \langle a^3 \rangle \cong C_2$;

(4) $\overline{G} = \overline{G}/\Omega_2(G) \cong L$, where $L$ is the group described in Lemma 4.1.

Assume the conclusion is false. Then there exists $H < G$ such that $d(H) \geq 4$. If $\overline{G}(2) \not\subset H$ or $\overline{G}(2) \not\subset \Phi(H)$, then it follows by Lemma 4.1 that $d(H) \leq 3$. This contradicts $d(H) \geq 4$. If $\overline{G}(2) \subset H \setminus \Phi(H)$, then we may assume $H = K \times \overline{G}(2)$. Since $d(H) \geq 4$, $d(K) \geq 3$. Then $K$ has a normal subgroup $N$ of type $(2, 2)$. It follows from $N/C$-theorem that $|K : C_K(N)| \leq 2$. Notice that $\Omega_1(G) = N \times \overline{G}(2)$. Then $\overline{G}(2) \not\subset C_K(N) \subset C_K(\Omega_1(G))$. In particular, $C_K(N) \subset \overline{G}(2) \subset C_K(\Omega_1(G))$. From (2) we get $\overline{G}(2) \not\subset \overline{G}(2)(\Omega_1(G))$. Hence $\overline{G}(2)(\Omega_1(G)) \not\subset \overline{G}(2)$. This means $C_K(N) \not\subset \overline{G}(2)$. It follows that $|K| \leq 2^4$. From (1) we know $H$ is non-abelian. Hence $K$ is non-abelian. Since $d(K) \geq 3$, $K$ has an $A_4$-subgroup of order 8. Moreover, $K \cong K\overline{G}(2)/\overline{G}(2) \subset \overline{G} \cong L$. This contradicts $L \in A_4$.

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