Abstract. In this paper, we introduce quasiuniform convergence structure induced by operators on ecl-premonoid \((L, *, \odot)\). Moreover, we obtain \((L, *, \odot)\)-quasiuniform convergence structure induced by two \((L, *, \odot)\)-quasiuniform convergence structures and gives their examples.

1. Introduction

Gähler [2,3] introduced the notions of fuzzy filters in a frame \(L\). Höhle and Sostak [4] introduced the concept of \(L\)-filters for a complete quasimonoidal lattice \(L\). For the case that the lattice is a stsc quantale, \(L\)-filters were introduced in [12]. Jäger [5-6] developed stratified \(L\)-convergence structures based on the concepts of \(L\)-filters where \(L\) is a complete Heyting algebra. Yao [15] extended stratified \(L\)-convergence structures to complete residuated lattices and investigated between stratified \(L\)-convergence structures and \(L\)-fuzzy topological spaces. As an extension of Yao [15], Fang [7-11] introduced \(L\)-ordered convergence structures and (pre, quasi,semi) uniform convergence spaces on \(L\)-filters and investigated their relations. Ko and Kim [13] introduced the \((L, *, \odot)\)-quasiuniform convergence spaces as an extension of Fang’s uniform convergence spaces on ecl-premonoid in Orpen’s sense [14].

In this paper, we introduce quasiuniform convergence structure induced by operators on ecl-premonoid \((L, *, \odot)\) and gives their examples. Moreover, we obtain \((L, *, \odot)\)-quasiuniform convergence structure induced by two \((L, *, \odot)\)-quasiuniform convergence structures.
2. Preliminaries

Definition 2.1 ([14]). A complete lattice \((L, \leq, \bot, \top)\) is called a GL-monoid \((L, \leq, *, \bot, \top)\) with a binary operation \(* : L \times L \to L\) satisfying the following conditions:

(G1) \(a * \top = a\), for all \(a \in L\),

(G2) \(a * b = b * a\), for all \(a, b \in L\),

(G3) \(a * (b * c) = (a * b) * c\), for all \(a, b \in L\),

(G4) if \(a \leq b\), there exists \(c \in L\) such that \(b * c = a\),

(G5) \(a * \bigvee_{i \in \Gamma} b_i = \bigvee_{i \in \Gamma} (a * b_i)\).

We can define an implication operator:

\[ a \Rightarrow b = \bigvee \{c \mid a * c \leq b\}. \]

Remark 2.2 ([1, 4, 14]). (1) A continuous t-norm \([0, 1], \leq, *\) is a GL-monoid.

(2) A frame \((L, \leq, \wedge)\) is a GL-monoid.

Definition 2.3 ([1, 4, 14]). A complete lattice \((L, \leq, \bot, \top)\) is called a cl-premonoid \((L, \leq, \odot)\) with a binary operation \(\odot : L \times L \to L\) satisfying the following conditions:

(CL1) \(a \leq a \odot \top\) and \(a \leq \top \odot a\), for all \(a \in L\),

(CL2) if \(a \leq b\) and \(b \leq d\), then \(c \leq b \odot d\),

(CL3) \(a \odot \bigvee_{i \in \Gamma} b_i = \bigvee_{i \in \Gamma} (a \odot b_i)\) and \(\bigvee_{j \in \Gamma} a_j \odot b = \bigvee_{j \in \Gamma} (a_j \odot b)\).

We can define an implication operator:

\[ a \rightarrow b = \bigvee \{c \mid a \odot c \leq b\}. \]

Definition 2.4 ([1, 4, 14]). A complete lattice \((L, \leq, \bot, \top)\) is called an ecl-premonoid \((L, \leq, \odot, *)\) with a GL-monoid \((L, \leq, *, \bot, \top)\) and a cl-premonoid \((L, \leq, \odot)\) which satisfy the following condition:

(D) \((a \odot b) * (c \odot d) \leq (a * c) \odot (b * d)\), for all \(a, b, c, d \in L\).

An ecl-premonoid \((L, \leq, \odot, \ast)\) is called an M-ecl-premonoid if it satisfies the following condition:

(M) \(a \leq a \odot a\) for all \(a \in L\).

In this paper, we always assume that \((L, \leq, \odot, \ast)\) is an ecl-premonoid unless otherwise specified.

Lemma 2.5 ([1, 4, 13]). Let \((L, \leq, \odot, \ast)\) be an ecl-premonoid. For each \(a, b, c, d, a_i, b_i \in L\) and for \(\uparrow \in \{\to, \Rightarrow\}\), we have the following properties.

(1) If \(b \leq c\), then \(a \odot b \leq a \odot c\) and \(a * b \leq a * c\).
(2) $a \odot b \leq c$ iff $a \leq b \rightarrow c$. Moreover, $a * b \leq c$ iff $a \leq b \Rightarrow c$.

(3) If $b \leq c$, then $a \uparrow b \leq a \uparrow c$ and $c \uparrow a \leq b \uparrow a$.

(4) $a \leq b$ iff $a \Rightarrow b = \top$.

(5) $a * b \leq a \odot b$, $a \rightarrow b \leq a \Rightarrow b$ and $a * (b \odot c) \leq (a * b) \odot c$.

(6) $(a \uparrow b) \odot (c \uparrow d) \leq (a \odot c) \uparrow (b \odot d)$.

(7) $(b \uparrow c) \leq (a \odot b) \uparrow (a \odot c)$.

(8) $(b \uparrow c) \leq (a \uparrow b) \uparrow (a \uparrow c)$ and $(b \uparrow a) \leq (a \uparrow c) \uparrow (b \uparrow c)$.

(9) $(b \rightarrow c) \leq (a \uparrow b) \rightarrow (a \uparrow c)$ and $(b \uparrow a) \leq (a \rightarrow c) \rightarrow (b \uparrow c)$.

(10) $a_i \uparrow b_i \leq (\bigwedge_{i \in I} a_i) \uparrow (\bigwedge_{i \in I} b_i)$.

(11) $a_i \uparrow b_i \leq (\bigvee_{i \in I} a_i) \uparrow (\bigvee_{i \in I} b_i)$.

(12) $(c \uparrow a) * (b \rightarrow d) \leq (a \rightarrow b) \rightarrow (c \uparrow d)$.

**Definition 2.6 ([4, 13]).** For $L^X = \{f \mid f : X \rightarrow L \text{ is a function}\}$, a mapping $\mathcal{F} : L^X \rightarrow L$ is called an $(L, *)$-filter on $X$ if it satisfies the following conditions:

(F1) $\mathcal{F}(\bot_X) = \bot$ and $\mathcal{F}(\top_X) = \top$, where $\bot_X(x) = \bot, \top_X(x) = \top$ for $x \in X$.

(F2) $\mathcal{F}(f * g) \geq \mathcal{F}(f) \ast \mathcal{F}(g)$, for each $f, g \in L^X$.

(F3) if $f \leq g$, $\mathcal{F}(f) \leq \mathcal{F}(g)$.

The pair $(X, \mathcal{F})$ is called an $(L, *)$-filter space. We denote by $F_*(X)$ the set of all $(L, *)$-filters on $X$.

**Theorem 2.7 ([13]).** Let $\mathcal{U}, \mathcal{V} \in F_*(X \times X)$. We define $\mathcal{U} \circ \mathcal{V} : L^{X \times X} \rightarrow L$ as follows:

$$(\mathcal{U} \circ \mathcal{V})(w) = \bigvee \{\mathcal{U}(u) \circ \mathcal{V}(v) \mid u \circ v \leq w\}$$

where $u \circ v(x, z) = \bigvee_{y \in X} (u(x, y) \ast v(y, z))$.

(1) $u \circ v = \bot_{X \times X}$ implies $\mathcal{U}(u) \circ \mathcal{V}(v) = \bot$ iff $(\mathcal{U} \circ \mathcal{V}) \in F_*(X \times X)$.

(2) If $\mathcal{U}(1_\Delta) = \top$ where $1_\Delta(x, x) = \top$ and $1_\Delta(x, y) = \bot$ for $x \neq y \in X$, $\mathcal{U} \circ \mathcal{U} \geq \mathcal{U}$.

(3) $[(x, x)] \circ_* [(x, y)] = [(x, x)]$.

(4) $\bigwedge_{x \in X} [(x, x)] \circ_* \bigwedge_{x \in X} [(x, y)] = \bigwedge_{x \in X} [(x, y)]$.

**Definition 2.8 ([13]).** A map $\Lambda : F_*(X \times X) \rightarrow L$ is called an $(L, *, \circ)$-quasiuniform convergence structure on $X$ if it satisfies the following conditions:

(QC1) $\Lambda([(x, x)]) = \top$, for each $x \in X$.

(QC2) If $\mathcal{U} \leq \mathcal{V}$, then $\Lambda(\mathcal{U}) \leq \Lambda(\mathcal{V})$.

(QC3) $\Lambda(\mathcal{U}) \circ \Lambda(\mathcal{V}) \leq \Lambda(\mathcal{U} \circ \mathcal{V})$.

(QC4) $\Lambda(\mathcal{U}) \circ \Lambda(\mathcal{V}) \leq \Lambda(\mathcal{U} \circ \mathcal{V})$ where $\mathcal{U} \circ \mathcal{V} \in F_*(X \times X)$.

The pair $(X, \Lambda)$ is called an $(L, *, \circ)$-quasiuniform convergence space.
An $(L, *, \odot)$-quasiuniform convergence space is called an $(L, *, \odot)$-uniform convergence space if it satisfies the following condition:

$$(U) \quad \Lambda(U) \leq \Lambda(U^{-1}) \text{ where } U^{-1}(u) = U(u^{-1}) \text{ and } u^{-1}(x, y) = u(y, x) \text{ for } x, y \in X.$$ 

We say $\Lambda_1$ is finer than $\Lambda_2$ (or $\Lambda_2$ is coarser than $\Lambda_1$) iff $\Lambda_1 \leq \Lambda_2$.

We define $\top, \bot : F_\ast(X \times X) \to [0, 1]$ as follows:

$$\top(W) = \begin{cases} \top, & \text{if } W \subseteq [(x, x)], \forall x \in X \\ \bot, & \text{otherwise.} \end{cases} \quad \bot(W) = \top, \forall W \in F_\ast(X \times X)$$

Then $\top$ (resp. $\bot$) is the finest (resp. coarsest) $(L, *, \odot)$-quasiuniform convergence structure.

Let $(X, \Lambda_X)$ and $(Y, \Lambda_Y)$ be $(L, *, \odot)$-quasiuniform convergence spaces. A map $\psi : (X, \Lambda_X) \to (Y, \Lambda_Y)$ is called quasiuniformly continuous if for all $U \in F_\ast(X \times X)$, $\Lambda_X(U) \leq \Lambda_Y((\psi \times \psi)^R(U))$.

### 3. $(L, *, \odot)$-Quasiuniform Convergence Spaces Induced by Operators

**Theorem 3.1.** Let $M : F_\ast(X \times X) \to L^{X \times X}$ be maps satisfying the following conditions:

(M1) $M([(x, x)]) \uparrow [(x, x)] = \top$, for each $\uparrow \in \{\to, \Rightarrow\}$ and $x \in X$.

(M2) If $U \leq V$, then $M(U) \geq M(V)$.

(M3) $M(U \odot V) \leq M(U) \odot M(V)$.

(M4) $M(U \circ \odot V) \leq M(U) \circ M(V)$.

For each $\uparrow \in \{\to, \Rightarrow\}$, we define a map $\Lambda^M : F_\ast(X \times X) \to L$ as follows:

$$\Lambda^M(U) = \bigwedge_{u \in L^{X \times X}} (M(U)(u) \uparrow U(u)).$$

Then the following properties hold.

(1) $\Lambda^M$ is an $(L, *, \odot)$ quasiuniform convergence structure.

(2) If $\psi : (X, M_X) \to (Y, M_Y)$ is a map such that $M_Y((\psi \times \psi)^R(U))(v) \leq M_X(U)((\psi \times \psi)^R(v))$ for each $U \in F_\ast(X \times X)$, then $\psi : (X, \Lambda^M_X) \to (Y, \Lambda^M_Y)$ is quasiuniformly continuous.

**Proof.** (1) (QC1) Since $M([(x, x)]) \uparrow [(x, x)] = \top$, 

$$\Lambda^M([(x, x)]) = \bigwedge_{u \in L^{X \times X}} (M([(x, x)])(u) \uparrow [(x, x)](u)) = \top.$$
(QC3) For each $U, V \in F_*(X \times X)$, by Lemma 2.5(6),
\[
\Lambda^{M}(U) \cap \Lambda^M(V) \\
\leq \Lambda_{u \in L} (M(U)(u) \cap M(V)(u)) \\
\leq \Lambda_{u \in L} (M(U)(u) \circ M(V)(u)) \\
= \Lambda^M(U) \circ \Lambda^M(V).
\]

(2) For each $U \in F_*(X \times X)$, by Lemma 2.5(8),
\[
\Lambda_X(U) \cup \Lambda_Y(U) \\
\leq \Lambda_{v \in L} (M_X(U)(v) \circ M_Y(U)(v)) \\
\leq \Lambda_{v \in L} (M_X(U)(v) \circ M_Y(U)(v)) \\
\leq \Lambda_{v \in L} (M_Y(U)(v) \circ M_X(U)(v)) \\
= \Lambda^Y(U) \circ \Lambda^X(U).
\]

Since $M_Y((\psi \times \psi)^{\circ}(U)) \leq M_X(U)((\psi \times \psi)^{\circ}(U))$ for each $v \in L^Y, U \in F_*(X \times X)$, by Lemma 2.5(4),
\[
\bigwedge_{v \in L^Y \times Y} \left( M_Y((\psi \times \psi)^\vartriangleleft(U))(v) \Rightarrow M_X(U)((\psi \times \psi)^\triangledown(v)) \right) = \top.
\]

Hence \( \Lambda_X^{M^\uparrow}(U) \Rightarrow \Lambda_X^{M^\uparrow}((\psi \times \psi)^\vartriangleleft(U)) = \top \). Thus \( \psi : (X, \Lambda_X^{M^\uparrow}) \rightarrow (Y, \Lambda_Y^{M^\uparrow}) \) is quasi-uniformly continuous. \( \Box \)

**Example 3.2.** Let \( (L = [0, 1], \leq, \odot, *, 0, 1) \) be an M-ecl-premonoid. Let a map \( M_X : F_*(X \times X) \rightarrow [0, 1]^{[0, 1]X} \) defined as \( M_X(U) = \bigwedge_{x \in X} [(x, x)] \),

(1) Let \( (L = [0, 1], \leq, \land, *, 0, 1) \) be an M-ecl-premonoid. Since

\[
M_X(U) = \bigwedge_{x \in X} [(x, x)],
\]

\( M_X(U \odot V) = \bigwedge_{x \in X} [(x, x)] \leq \bigwedge_{x \in X} [(x, x)] \odot \bigwedge_{x \in X} [(x, x)] = M_X(U) \odot M_X(V) \) and

\[
(M_X(U) \odot \land M_X(V))(u) \geq M_X(U)(u) \odot M_X(U)(1_\Delta)
\]

\[
= \bigwedge_{x \in X} [(x, x)](u) \odot \bigwedge_{x \in X} [(x, x)](1_\Delta) \geq \bigwedge_{x \in X} [(x, x)](u),
\]

it satisfies the following conditions (M1), (M2) and (M3). For each \( \uparrow \in \{-, \Rightarrow\} \),

\[
\Lambda_X^{M^\uparrow}(U) = \bigwedge_{u \in L^X \times X} \bigwedge_{x \in X} [(x, x)](u) \uparrow U(u) = \bigwedge_{u \in L^X \times X} \bigwedge_{x \in X} u(x, x) \uparrow U(u).
\]

Then \( \Lambda_X^{M^\uparrow} \) is an \((L, *, \odot)\)-quasi-uniform convergence structure.

Let \( \psi : (X, M_X) \rightarrow (Y, M_Y) \) be a map with \( M_Y(V) = \bigwedge_{y \in Y} [(y, y)] \) for all \( V \in F(Y \times Y) \). Since \( M_Y((\psi \times \psi)^\vartriangleleft(U))(v) = \bigwedge_{y \in Y} v(y, y) \leq \bigwedge_{x \in X} v(\psi(x), \psi(x)) = M_X(U)((\psi \times \psi)^\triangledown(v)) \) for each \( v \in L^Y \times Y \), then \( \psi : (X, \Lambda_X^{M^\uparrow}) \rightarrow (Y, \Lambda_Y^{M^\uparrow}) \) is uniformly continuous.

**Example 3.3.** Let \( X = \{a, b, c\} \) be a set and \( (L = [0, 1], \leq, \land, *, 0, 1) \) an M-ecl-premonoid with \( a \ast b = (a + b - 1) \lor 0 \). Put \( u \in [0, 1]^{X \times X} \) as follows:

\[
u(a, a) = u(b, b) = 1, u(c, c) = 0.4, \ u(a, b) = u(b, a) = 0.6,
\]

\[
u(a, c) = u(c, a) = 0.5, u(b, c) = u(c, b) = 0.4.
\]

Define \([0, 1]-\)filter as \( \mathcal{U} : [0, 1]^{X \times X} \rightarrow [0, 1] \) as follows:

\[
\mathcal{U}(w) = \begin{cases} 1, & \text{if } w \geq 1_\Delta, \\ 0.2, & \text{if } u \leq w \geq 1_\Delta, \\ 0, & \text{otherwise.} \end{cases}
\]

Since \( v \circ 1_\Delta = v \), we obtain \( \mathcal{U} \odot \land \mathcal{U} = \mathcal{U} = \mathcal{U}^{-1} \) and \( 0.2 = \mathcal{U}(u) \leq [(c, c)](u) = 0.4 \). Put \( M_X(\mathcal{U}) = \mathcal{U} \) for all \( \mathcal{U} \in F_*(X \times X) \). Then \( M_X \) satisfies the conditions (M1)-(M4). For each \( \uparrow \in \{-, \Rightarrow\} \), we obtain an \((L, *, \land)\) uniform convergence structure.
\(\Lambda^{M^\uparrow}: F_*(X \times X) \rightarrow [0, 1]\) as follows:

\[
\Lambda^{M^\uparrow}(\mathcal{W}) = \bigwedge_{v \in L^X \times X} (M_X(\mathcal{W})(v) \uparrow \mathcal{W}(v)) = \bigwedge_{v \in L^X \times X} (\mathcal{V}(v) \uparrow \mathcal{W}(v))
\]

\[
\Lambda^{M^\uparrow}(\mathcal{W}^{-1}) = \bigwedge_{v \in L^X \times X} (M_X(\mathcal{W}^{-1})(v) \uparrow \mathcal{W}^{-1}(v)) = \bigwedge_{v \in L^X \times X} (\mathcal{V}^{-1}(v) \uparrow \mathcal{W}(v^{-1}))
\]

where \(a \Rightarrow b = (1 - a + b) \wedge 1\) and

\[
a \rightarrow b = \begin{cases} 
1, & \text{if } a \leq b, \\
b, & \text{if } a \nleq b.
\end{cases}
\]

**Example 3.4.** Let \(X = \{a, b, c\}\) be a set, \((L = [0, 1], \leq, \odot, *, 0, 1)\) an \(M\)-ecl-premonoid with \(a \ast b = a \cdot b, a \odot b = a^\frac{1}{2} \cdot b^\frac{1}{2}\) and \(u \in [0, 1]^X \times X\) defined as follows:

\[
u(a, a) = u(b, b) = u(c, c) = 1, \quad u(a, b) = 0.5, u(b, a) = 0.6, \\
u(a, c) = u(c, a) = 0.5, u(b, c) = 0.6, u(c, b) = 0.4.
\]

Define \([0, 1]\)-filter as \(\mathcal{U} : [0, 1]^{X \times X} \rightarrow [0, 1]\) as follows:

\[
\mathcal{U}(w) = \begin{cases} 
1, & \text{if } w = 1_{X \times X}, \\
0.6^n, & \text{if } u^n \leq w \ngeq u^{n-1}, n \in N, \\
0, & \text{otherwise}.
\end{cases}
\]

where \(u^{n+1} = u^n \ast u\) and \(u^0 = 1_{X \times X}\).

Since \(u^n \circ u^n = u^n\), we obtain

\[
(\mathcal{U} \odot \mathcal{U})(w) = \begin{cases} 
1, & \text{if } w = 1_{X \times X}, \\
0.6^n \odot 0.6^n, & \text{if } u^n \leq w \ngeq u^{n-1}, n \in N, \\
0, & \text{otherwise}.
\end{cases}
\]

\[
(\mathcal{U} \circ \mathcal{U})(w) = \begin{cases} 
1, & \text{if } w = 1_{X \times X}, \\
0.6^n \odot 0.6^n, & \text{if } u^n \leq w \ngeq u^{n-1}, n \in N, \\
0, & \text{otherwise}.
\end{cases}
\]

Put \(M_X(\mathcal{W}) = \mathcal{U}\) for all \(\mathcal{W} \in F_*(X \times X)\).

(1) Let \((L = [0, 1], \leq, \wedge, *, 0, 1)\) be an \(M\)-ecl-premonoid with \(a \ast b = a \cdot b\) with

\[
a \Rightarrow b = \begin{cases} 
1, & \text{if } a \leq b, \\
b, & \text{if } a \nleq b, \\
\end{cases} a \rightarrow b = \begin{cases} 
1, & \text{if } a \leq b, \\
b, & \text{if } a \nleq b.
\end{cases}
\]

Since \(\mathcal{U} \circ \mathcal{U} = \mathcal{U} \wedge \mathcal{U} = \mathcal{U}, M\) satisfies the conditions \((M1)-(M4)\). For each \(\uparrow \in \{\rightarrow, \Rightarrow\}\), we obtain an \((L, *, \wedge)\) quasi-uniform convergence structures \(\Lambda^{M^\uparrow}: F_*(X \times X) \rightarrow [0, 1]\) as follows:

\[
\Lambda^{M^\uparrow}(\mathcal{W}) = \bigwedge_{v \in L^X \times X} (M_X(\mathcal{W})(v) \uparrow \mathcal{W}(v))
\]

\[
= \bigwedge_{v \in L^X \times X} (\mathcal{U}(v) \uparrow \mathcal{W}(v))
\]

\[
= \bigwedge_{n \in N} (0.6^n \uparrow \mathcal{W}(u^n)).
\]
So, we have

\[
\Lambda^{M_X \Rightarrow} (\mathcal{W}) = \begin{cases} 
1, & \text{if } 0.6^n \leq \mathcal{W}(u^n), \forall n \in N \\
\frac{\mathcal{W}(u^n)}{0.6^n}, & \text{if } 0.6^n \not\in \mathcal{W}(u^n),
\end{cases}
\]

\[
\Lambda^{M_X \rightarrow} (\mathcal{W}) = \begin{cases} 
1, & \text{if } 0.6^n \leq \mathcal{W}(u^n), \forall n \in N \\
\mathcal{W}(u^n), & \text{if } 0.6^n \not\in \mathcal{W}(u^n).
\end{cases}
\]

Since \(1 = \Lambda^{M_X \Rightarrow} (\mathcal{U}) = 0.6 \rightarrow \mathcal{U}(u) \not\leq \Lambda^{M_X \Rightarrow} (\mathcal{U}^{-1}) = 0.6 \rightarrow \mathcal{U}(u^{-1}) = 0.6 \rightarrow 0.36 = 0.36, \Lambda^{M_X \Rightarrow} \) is not an \((L, *, \wedge)\) uniform convergence structure on \(X\). Since \(1 = \Lambda^{M_X \Rightarrow} (\mathcal{U}) = 0.6 \Rightarrow \mathcal{U}(u) \not\leq \Lambda^{M_X \Rightarrow} (\mathcal{U}^{-1}) = (0.6 \Rightarrow \mathcal{U}(u^{-1})) = \frac{1}{b}, \Lambda^{M_X \Rightarrow} \) is not an \((L, *, \wedge)\) uniform convergence structure on \(X\).

Let \(\psi : (X, M^{\omega(x)}) \rightarrow (Y, M^\Psi(x))\) be a map with \(M_Y(\mathcal{V}) \equiv (\psi \times \psi)(\mathcal{U})\) for all \(\mathcal{V} \in F_{\psi}(Y \times Y)\). Then \(M_Y((\psi \times \psi)(\mathcal{U}))(v) = (\psi \times \psi)(\mathcal{U})(v) = \mathcal{U}((\psi \times \psi)^{-1}(v)) = M_X(\mathcal{U})((\psi \times \psi)^{-1}(v))\) for each \(\mathcal{U} \in F_{\psi}(X \times X)\). Thus \(\psi : (X, \Lambda^{M_X \Rightarrow}) \rightarrow (Y, \Lambda^{M_X \Rightarrow})\) is uniformly continuous.

(2) Let \((L = [0, 1], \leq, \circ, *, \circ)\) be an \(M\)-ecl-premonoid with \(a * b = a \cdot b, a \circ b = a^\frac{1}{3} \cdot b^\frac{4}{3}\) with

\[
a \Rightarrow b = \begin{cases} 
1, & \text{if } a \leq b, \\
\frac{b}{a}, & \text{if } a \not\leq b,
\end{cases}
\]

We obtain an \((L, *, \circ)\) quasi-uniform convergence structures \(\Lambda^{M_X \Rightarrow}, \Lambda^{M_X \rightarrow} : F_{\ast}(X \times X) \rightarrow [0, 1]\) as follows:

\[
\Lambda^{M_X \Rightarrow} (\mathcal{W}) = \begin{cases} 
1, & \text{if } 0.6^n \leq \mathcal{W}(u^n), \forall n \in N \\
\frac{\mathcal{W}(u^n)}{0.6^n}, & \text{if } 0.6^n \not\in \mathcal{W}(u^n),
\end{cases}
\]

\[
\Lambda^{M_X \rightarrow} (\mathcal{W}) = \begin{cases} 
1, & \text{if } 0.6^n \leq \mathcal{W}(u^n), \forall n \in N \\
\left(\frac{\mathcal{W}(u^n)}{0.6^n}\right)^{\frac{1}{2}}, & \text{if } 0.6^n \not\in \mathcal{W}(u^n).
\end{cases}
\]

**Theorem 3.5.** Let \(\Lambda_1\) and \(\Lambda_2\) be \((L, *, \circ)\)-quasi-uniform convergence spaces on \(X\). We define a map \(\Lambda_1 \circ_\ast \Lambda_2 : F_{\ast}(X \times X) \rightarrow L\) as follows:

\[
(\Lambda_1 \circ_\ast \Lambda_2)(\mathcal{U}) = \sqrt{\{\Lambda_1(\mathcal{U}_1) \circ \Lambda_2(\mathcal{U}_2) \mid \mathcal{U}_1 \ast \mathcal{U}_2 \leq \mathcal{U}\}}.
\]

Then \(\Lambda_1 \circ_\ast \Lambda_2\) is an \((L, *, \circ)\)-quasi-uniform convergence space on \(X\) which is coarser than \(\Lambda_1\) and \(\Lambda_2\). Moreover, \(\Lambda_1 \ast \ast \Lambda_2\) is the finest \((L, *, \ast)\)-quasi-uniform convergence spaces on \(X\) which is coarser than \(\Lambda_1\) and \(\Lambda_2\).

**Proof.** (QUC1) Since \([(x, x)] \ast [(x, x)] \leq [(x, x)],

\[
(\Lambda_1 \circ_\ast \Lambda_2)([(x, x)]) \geq \Lambda_1([(x, x)]) \circ \Lambda_1([(x, x)]) = T
\]
Similarly, \( (\mathcal{U}_1 \circ \mathcal{V}_1) \ast (\mathcal{U}_2 \circ \mathcal{V}_2) \leq (\mathcal{U}_1 \ast \mathcal{U}_2) \circ (\mathcal{V}_1 \ast \mathcal{V}_2) \),

\[
(\Lambda_1 \ast_\circ \Lambda_2)(\mathcal{U}) \circ (\Lambda_1 \ast_\circ \Lambda_2)(\mathcal{V}) = \bigvee \{ \Lambda_1(\mathcal{U}_1) \circ \Lambda_2(\mathcal{U}_2) \mid \mathcal{U}_1 \ast \mathcal{U}_2 \leq \mathcal{U} \} \circ \bigvee \{ \Lambda_1(\mathcal{V}_1) \circ \Lambda_2(\mathcal{V}_2) \mid \mathcal{V}_1 \ast \mathcal{V}_2 \leq \mathcal{V} \}
\]

Since \( \mathcal{U} \circ ([x, x]) \leq \mathcal{U} \), then \( (\Lambda_1 \circ_\circ \Lambda_2)(\mathcal{U}) \geq (\Lambda_1(\mathcal{U}) \circ \Lambda_2([x, x])) = \Lambda_1(\mathcal{U}) \).

Similarly, \( \Lambda_1 \circ_\circ \Lambda_2 \geq \Lambda_2 \).

If \( \circ = * \) and \( \Lambda_i \leq \Lambda \) for \( i \in \{1, 2\} \), we have \( \Lambda_1 \ast_\circ \Lambda_2 \leq \Lambda \) from:

\[
(\Lambda_1 \ast_\circ \Lambda_2)(\mathcal{U}) = \bigvee \{ \Lambda_1(\mathcal{U}_1) \ast \Lambda_2(\mathcal{U}_2) \mid \mathcal{U}_1 \ast \mathcal{U}_2 \leq \mathcal{U} \}
\]

\[
\leq \bigvee \{ \Lambda(\mathcal{U}_1) \ast \Lambda(\mathcal{U}_2) \mid \mathcal{U}_1 \ast \mathcal{U}_2 \leq \mathcal{U} \} \leq \Lambda(\mathcal{U}).
\]

\Box

REFERENCES


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