SOME REMARKS ON SUMSETS AND RESTRICTED SUMSETS

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Abstract. Let $A$ be a finite set of integers. For any integer $h \geq 1$, let $h$-fold sumset $hA$ be the set of all sums of $h$ elements of $A$ and let $h$-fold restricted sumset $h^\wedge A$ be the set of all sums of $h$ distinct elements of $A$. In this paper, we give a survey of problems and results on sumsets and restricted sumsets of a finite integer set. In details, we give the best lower bound for the cardinality of restricted sumsets $2^\wedge A$ and $3^\wedge A$ and also discuss the cardinality of restricted sumset $h^\wedge A$.

1. Introduction

Let $\mathbb{N}$ denote the set of all nonnegative integers. Let $A$ be a finite nonempty integer set and let $l(A)$ denote the difference of the largest and the smallest elements of $A$. For any finite set of integers $A$ and any positive integer $h \geq 1$, define

$$hA = \{a_1 + \cdots + a_h : a_i \in A(1 \leq i \leq h)\},$$

$$h^\wedge A = \{a_1 + \cdots + a_h : a_i \in A(1 \leq i \leq h), a_i \neq a_j \text{ for all } i \neq j\}.$$ 

Here, $h^\wedge A = \emptyset$ if $|A| < h$. Let $A, B$ be sets of integers, define

$$A + B = \{a + b : a \in A, b \in B\}.$$ 

Sumsets are one of the central objects of study in additive number theory. Nathanson [12] proved the following fundamental and important results:

Theorem A ([12], Theorem 1.3). Let $h \geq 2$ be an integer and $A$ a finite set of integers with $|A| = k$. Then

$$|hA| \geq hk - h + 1.$$ 

Theorem B ([12], Theorem 1.6). Let $h \geq 2$ be an integer and $A$ a finite set of integers with $|A| = k$. Then

$$|hA| = hk - h + 1.$$ 

Received May 17, 2018; Revised October 18, 2018; Accepted January 8, 2019.
2010 Mathematics Subject Classification. 11B13.
Key words and phrases. restricted sumset, pigeonhole principle.
This work was supported by National Natural Science Foundation of China, Grant No. 11471017.
if and only if $A$ is a $k$-term arithmetic progression.

In 1995, Nathanson [11] considered the set of all sums of distinct elements of $A$. He obtained a lower bound for $|h^\wedge A|$ and determined the structure of the finite sets $A$ of integers for which $|h^\wedge A|$ is minimal.

**Theorem C ([11], Theorem 1).** Let $A$ be a set of $k$ integers and let $1 \leq h \leq k$. Then

$$|h^\wedge A| \geq hk - h^2 + 1.$$  

**Theorem D ([11], Theorem 2).** Let $k \geq 5$ and let $2 \leq h \leq k - 2$. If $A$ is a set of $k$ integers such that

$$|h^\wedge A| = hk - h^2 + 1,$$

then $A$ is an arithmetic progression.


In 1959, Freiman [2] proved the following result:

**Theorem E.** Let $k \geq 3$. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a set of integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$. We have

(i) If $a_{k-1} \geq 2k - 3$ and $\gcd(a_1, \ldots, a_{k-1}) = 1$, then $|2A| \geq 3k - 3$.

(ii) If $a_{k-1} = k - 1 + r \leq 2k - 3$ with $r \in [0, k - 2]$, then $|2A| \geq 2k - 1 + r = k + a_{k-1}$.

Theorem E shows that if $|A| = k$ and $|2A| \leq 3k - 4$, then $A$ is a subset of a short arithmetic progression. Moreover, Theorem E(ii) can be extend to $h \geq 2$ under the condition $a_{k-1} \leq 2k - 3$ (see [12], Exercise 1.9.17).

**Theorem F.** Let $h \geq 2$ and $k \geq 3$. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a set of integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$. If $a_{k-1} = k - 1 + r \leq 2k - 3$ with $r \in [0, k - 2]$, then $|hA| \geq k + (h - 1)a_{k-1}$.


**Theorem G.** Let $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_l\}$ be two sets of integers. If $a_k \leq k + l - 3$, then $|A + B| \geq a_k + l$. If $a_k \geq k + l - 2$ and $(a_1, \ldots, a_k, b_1, \ldots, b_l) = 1$, then $|A + B| \geq k + l + \min(k, l) - 3$.

There is a certain number of beautiful articles on this topic, see ([1], [4,5,7,8], [10], [13,14]).

In this paper, we give the best lower bound for the cardinality of restricted sumsets $2^\wedge A$ and $3^\wedge A$ under the condition $l(A) \leq 2|A| - 5$. The paper is organized as follows. In Section 2, we focus on the cardinality of restricted sumset $2^\wedge A$. In Section 3, we focus on the cardinality of restricted sumset $3^\wedge A$. In Section 4, we give a remark on the cardinality of restricted sumset $h^\wedge A$.
2. The cardinality of restricted sumset \(2^A\)

The proof of Theorem 2.1 (which is actually an exercise in Nathanson’s book [12]) have been already appeared in the article [6]. Here we give a simple combinatorial proof which was given by the anonymous referee in commenting on the first version of our manuscript.

**Theorem 2.1.** Let \(A\) be a finite nonempty integer set with \(|A| \geq 4\). If \(l(A) \leq 2|A| - 5\), then \(|2^A| \geq |A| + l(A) - 2\).

**Proof.** Without loss of generality, we may assume that \(\{0, l(A)\} \subseteq A \subseteq [0, l(A)]\) and \(l(A) \leq 2|A| - 5\). Put \(B := [0, l(A)] \setminus A\). We shall show that for each \(b \in B\) one has \(\{b, b + l(A)\} \cap 2^A \neq \emptyset\). Suppose that there exists an integer \(b \in B\) such that neither \(b\) nor \(b + l(A)\) lie in \(2^A\), then by the pigeonhole principle, we have

\[
|\{0, b\} \cap A| \leq \frac{b}{2} + 1 \quad \text{and} \quad |\{b, l(A)\} \cap A| \leq \frac{l(A) - b}{2} + 1.
\]

Thus

\[
|A| \leq \frac{l(A)}{2} + 2,
\]

which contradicts with the assumption \(l(A) \leq 2|A| - 5\).

Since \(\{a, a + l(A)\} \cup \{l(A)\} \subseteq 2^A\) whenever \(a \in A \setminus \{0, l(A)\}\), this gives \(|2^A| \geq 2|A| - 3\). And when \(b \in B\), we have \(\{b, b + l(A)\} \cap 2^A \neq \emptyset\), hence

\[
|2^A| \geq 2|A| - 3 + |B| = 2|A| - 3 + l(A) + 1 - |A| = l(A) + |A| - 2.
\]

**Remark 2.1.** The lower bound in Theorem 2.1 is best possible. For example, let \(A = \{0, 2, 3, 4, 5\}\), we have \(2^A = \{2, 3, 4, 5, 6, 7, 8, 9\}\) and \(|2^A| = 8 = |A| + l(A) - 2\).

**Remark 2.2.** The assumption \(l(A) \leq 2|A| - 5\) can not be relaxed in Theorem 2.1. For example, let \(A = \{0, 1, l(A) - 2, l(A) - 1, l(A)\}\) with \(l(A) \geq 2|A| - 4 = 6\). Then

\[
2^A = \{1, l(A) - 2, l(A) - 1, l(A), l(A) + 1, 2l(A) - 3, 2l(A) - 2, 2l(A) - 1\}
\]

and \(|2^A| = 8 < |A| + l(A) - 2\).

3. The cardinality of restricted sumset \(3^A\)

**Theorem 3.1.** Let \(A\) be a finite nonempty integer set with \(|A| \geq 5\). If \(l(A) \leq 2|A| - 5\), then \(|3^A| \geq 2|A| + l(A) - 7\).

**Proof.** Let \(|A| = k\), we may assume that \(A = \{a_0, a_1, \ldots, a_{k-1}\}\) with \(0 = a_0 < a_1 < \cdots < a_{k-1} \leq 2|A| - 5\). Then \(l(A) = a_{k-1}\). Define \(r\) by \(a_{k-1} = k - 1 + r\), and let \(B = [0, a_{k-1}] \setminus A\).
Consider the set
\[ T = \{ a_i + a_i : i = 2, \ldots, k - 2 \} \cup \{ a_i + a_{k-1} : i = 1, \ldots, k - 2 \} \cup \{ a_i + a_{k-2} + a_{k-1} : i = 1, \ldots, k - 3 \}. \]

Then \(|T| = 3|A| - 8\). Since \(B = [0, a_{k-1}] \setminus A\), we have \(|B| = a_{k-1} + 1 - k = r\).

By the proof of Theorem 2.1, we have \(2^\Delta A \cap \{ b, b + a_{k-1} \} \neq \emptyset\) for each \(b \in B\).

If \(b \in 2^\Delta A\), then \(b = a_i + a_j\), where \(a_i, a_j < a_{k-1}\) and \(a_i \neq a_j\). Thus \(b + a_{k-1} = a_i + a_j + a_{k-1} \in 3^\Delta A\).

If \(b + a_{k-1} \in 2^\Delta A\), then \(b + a_{k-1} = a_i + a_j\), where \(a_i, a_j < a_{k-1}\). Thus \(b + 2a_{k-1} = a_i + a_j + a_{k-1} \in 3^\Delta A\). Hence \(3^\Delta A \cap \{ b + a_{k-1}, b + 2a_{k-1} \} \neq \emptyset\) for each \(b \in B\).

Next, we shall prove \(|3^\Delta A \setminus T| \geq r\).

**Case 1.** \(b + a_{k-1} \in 3^\Delta A\). If \(b + a_{k-1} \notin T\), then \(b + a_{k-1} \in 3^\Delta A \setminus T\). Noting that \(b + a_{k-1} 
eq a_i + a_{k-1} (i = 1, \ldots, k - 2)\), we consider the following four cases.

**Case 1.1.** \(b + a_{k-1} = a_i + a_j (i = 2, \ldots, k - 3)\). Then \(b + 2a_{k-1} = a_1 + a_i + a_{k-1} (i = 2, \ldots, k - 3)\). Since \(a_{k-2} + a_{k-1} < b + 2a_{k-1} < a_1 + a_{k-2} + a_{k-1}\), we have \(b + 2a_{k-1} \in 3^\Delta A \setminus T\).

**Case 1.2.** \(b + a_{k-1} = a_1 + a_{k-2} + a_{k-1}\). Then \(b + 2a_{k-1} = a_1 + a_{k-2} + a_{k-1}\). We show that \(2a_{k-1} \in 3^\Delta A\). Suppose that \(2a_{k-1} \notin 3^\Delta A\), then except for \(a_{k-1} = 2a_i\) for some \(1 \leq i \leq k - 2\), we have \(\{ 2a_{k-1} - (a_j + a_{k-1}) : j = 1, \ldots, k - 2 \} \cap \{ a_1, \ldots, a_{k-2} \} = \emptyset\).

Write \(A_1 = \{ 2a_{k-1} - (a_j + a_{k-1}) : j = 1, \ldots, k - 2 \}, A_2 = \{ a_1, \ldots, a_{k-2} \}\).

Then sets \(A_1, A_2\) are pairwise disjoint except for at most one exception. Thus \(|A_1 \cup A_2| \geq 2k - 5\), which contradicts with the fact that \(A_1, A_2 \subseteq \{ 1, \ldots, a_{k-1} - 1 \} \subseteq \{ 1, \ldots, 2k - 6 \}\). Noting that \(a_{k-2} + a_{k-1} < 2a_{k-1} < a_1 + a_{k-2} + a_{k-1}\), we have \(2a_{k-1} \in 3^\Delta A \setminus T\).

**Case 1.3.** \(b + a_{k-1} = a_1 + a_{k-2} + a_{k-1}\). Then \(b = a_1 + a_{k-2}\). We show that \(a_{k-1} \in 3^\Delta A\). Suppose that \(a_{k-1} \notin 3^\Delta A\), then except for \(a_{k-1} = 2a_i\) for some \(1 \leq i \leq k - 2\), we have \(\{ a_{k-1} - a_j : j = 1, \ldots, k - 2 \} \cap \{ a_1, \ldots, a_{k-2} \} = \emptyset\).

Write \(B_1 = \{ a_{k-1} - a_j : j = 1, \ldots, k - 2 \}, B_2 = \{ a_1, \ldots, a_{k-2} \}\).

Then sets \(B_1, B_2\) are pairwise disjoint except for at most one exception. Thus \(|B_1 \cup B_2| \geq 2k - 5\), which contradicts with the fact that \(B_1, B_2 \subseteq \{ 1, \ldots, a_{k-1} - 1 \} \subseteq \{ 1, \ldots, 2k - 6 \}\). Noting that \(a_1 + a_{k-2} = b < a_{k-1} < a_1 + a_{k-1}\), then \(a_{k-1} \in 3^\Delta A \setminus T\).
Case 1.4. \( b + a_{k-1} = a_i + a_{k-2} + a_{k-1} (i = 2, \ldots, k-3) \). Then \( b = a_i + a_{k-2} (i = 2, \ldots, k-3) \). Thus \( b \in 3^k A \). Moreover,
\[
a_1 + a_{k-2} < b < a_1 + a_{k-1},
\]
we have \( b \in 3^k A \setminus T \).

Case 2. \( b + 2a_{k-1} \in 3^k A \). If \( b + 2a_{k-1} \notin T \), then \( b + 2a_{k-1} \in 3^k A \setminus T \).

Noting that \( b + 2a_{k-1} > a_{k-2} + a_{k-1} \), we consider the following two cases.

Case 2.1. \( b + 2a_{k-1} = a_1 + a_{k-2} + a_{k-1} \). This is same as Case 1.2. We have \( 2a_{k-1} \in 3^k A \). Moreover,
\[
a_{k-2} + a_{k-1} < 2a_{k-1} < a_1 + a_{k-2} + a_{k-1},
\]
we have \( 2a_{k-1} \in 3^k A \setminus T \).

Case 2.2. \( b + 2a_{k-1} = a_i + a_{k-2} + a_{k-1} (i = 2, \ldots, k-3) \). Then \( b + a_{k-1} = a_i + a_{k-2} (i = 2, \ldots, k-3) \). Thus \( b + a_{k-1} \in 3^k A \). Moreover,
\[
a_1 + a_{k-2} < b + a_{k-1} < a_{k-3} + a_{k-1}
\]
and \( b + a_{k-1} \neq a_j + a_{k-1} \) for each \( a_j \in A \). We have \( b + a_{k-1} \in 3^k A \setminus T \).

By Case 1, Case 2 and the fact that
\[
\bigcup_{b \in B} \{ a_{k-1} \}, \{ 2a_{k-1} \}, \bigcup_{b \in B} \{ b + a_{k-1}, b + 2a_{k-1} \}
\]
are pairwise disjoint, we have \( |3^k A \setminus T| \geq r \). Hence, \( |3^k A| \geq 3k - 8 + r = 2|A| + l(A) - 7 \).

This completes the proof of Theorem 3.1. \( \square \)

Remark 3.1. The assumption \( l(A) \leq 2|A| - 5 \) can not be relaxed in Theorem 3.1. For example, let \( A = \{ 0, 1, l(A) - 2, l(A) - 1, l(A) \} \) with \( l(A) \geq 2|A| - 4 = 6 \). Then
\[
3^k A = \{ l(A) - 1, l(A), l(A) + 1, 2l(A) - 3, 2l(A) - 2, 2l(A) - 1, 2l(A), 3l(A) - 3 \}
\]
and \( |3^k A| = 8 < 2|A| + l(A) - 7 \).

Remark 3.2. The estimate for \( |3^k A| \) is sharp. For example, let \( A = \{ 0, 2, 3, 4, 5, 6, 7 \} \). We have \( 3^k A = \{ 5, 6, \ldots, 18 \} \), and hence \( |3^k A| = 14 = 2|A| + l(A) - 7 \).

4. Concluding remark

Remark 4.1. Let \( h \geq 3 \) and \( A \) be a finite nonempty integer set with \( |A| \geq 5 \. If \( l(A) \leq 2|A| - 2h + 1 \), then \( |h^k A| \geq (h - 1)|A| + l(A) - h^2 + 2 \).

Theorem 3.1 implies the result holds for \( h = 3 \). Now, let \( h \geq 4 \). Write \( A = \{ a_0, a_1, \ldots, a_{k-1} \} \) with \( 0 = a_0 < a_1 < \cdots < a_{k-1} = l(A) \) and \( k = |A| \).

Assume that the result holds for \( h - 1 \), that is if \( l(A) \leq 2|A| - 2h + 3 \), then \( |(h - 1)^k A| \geq (h - 2)|A| + l(A) - (h - 1)^2 + 2 \). Now we shall prove the result holds for \( h \). Write \( B = A \setminus \{ a_1 \} \). Since \( l(A) \leq 2|A| - 2h + 1 \), we have
\[
l(B) = l(A) \leq 2|A| - 2h + 1 = 2|B| - 2h + 3.
\]
It follows from the induction hypothesis that

\[(h - 1)^{\nu} B + a_1 \geq (h - 2)|B| + l(B) - (h - 1)^2 + 2 = (h - 2)|A| + l(A) - (h - 1)^2 - h + 4.\]

Notice also that \((h - 1)^{\nu} B + a_1 \subset h^\nu A\) and \(\max((h - 1)^{\nu} B + a_1) = a_1 + a_{k-h+1} + \cdots + a_{k-1}\). Consequently, the set \((h - 1)^{\nu} B + a_1\) is disjoint from the set

\[C = \{a_i + a_{k-h+1} + \cdots + a_{k-1} : 2 \leq i \leq k - h\} \subset h^\nu A.\]

Therefore

\[
|h^\nu A| \geq |(h - 1)^{\nu} B + a_1| + |C|
\geq (h - 2)|A| + l(A) - (h - 1)^2 - h + 4 + (|A| - h - 1)
= (h - 1)|A| + l(A) - h^2 + 2.
\]

Hence, by induction the result holds for all \(h \geq 3\).

**Acknowledgment.** We are grateful to the anonymous referees for their valuable comments on this work.

**References**


