CHARACTERIZATIONS FOR THE FOCK-TYPE SPACES

HONG RAE CHO, JEONG MIN HA, AND KYESOOK NAM

ABSTRACT. We obtain Lipschitz type characterization and double integral characterization for Fock-type spaces with the norm
\[ \|f\|_{F_{m,\alpha,t}^p} = \int_{\mathbb{C}^n} \left| f(z) e^{-\alpha|z|^m} \right|^p \frac{dV(z)}{(1+|z|)^t}, \]
where \( \alpha > 0 \), \( t \in \mathbb{R} \), and \( m \in \mathbb{N} \). The results of this paper are the extensions of the classical weighted Fock space \( F_{2,\alpha,t}^p \).

1. Introduction

For a fixed positive integer \( n \), let \( H(\mathbb{C}^n) \) be the space of all entire functions on the complex \( n \)-space \( \mathbb{C}^n \). For \( z = (z_1, \ldots, z_n) \) and \( w = (w_1, \ldots, w_n) \) in \( \mathbb{C}^n \), we write
\[ z \cdot w = z_1 w_1 + \cdots + z_n w_n, \quad |z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}. \]

For \( \alpha > 0 \), \( t \in \mathbb{R} \), and \( m \in \mathbb{N} \), we define \( dG_{m,\alpha,t} \) the \( t \)-weighted \((m,\alpha)\)-Gaussian measure on \( \mathbb{C}^n \) by
\[ dG_{m,\alpha,t}(z) = C_{m,\alpha,t} e^{-\alpha|z|^m} \frac{dV(z)}{(1+|z|)^t}, \]
where \( dV \) is the volume measure on \( \mathbb{C}^n \) and \( C_{m,\alpha,t} \) is the positive constant to be the normalized volume measure. For \( 0 < p < \infty \), we consider the Fock-type space \( F_{m,\alpha,t}^p(\mathbb{C}^n) \) consisting of all \( f \in H(\mathbb{C}^n) \), the class of all entire functions on \( \mathbb{C}^n \), where the norm is defined by
\[ \|f\|_{F_{m,\alpha,t}^p} = C_{m,\alpha,t} \int_{\mathbb{C}^n} \left| f(z) e^{-\alpha|z|^m} \right|^p \frac{dV(z)}{(1+|z|)^t}. \]
Then \( F_{m,\alpha,t}^p(\mathbb{C}^n) = L^p(dG_{m,\alpha,t}) \cap H(\mathbb{C}^n) \). When \( m = 2 \), \( F_{2,\alpha,t}^p(\mathbb{C}^n) \) is the classical weighted Fock space.

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In this paper, we characterize the Fock-type space $F_{p,m,\alpha,t}(\mathbb{C}^n)$. One of the main results is Lipschitz type characterization for $F_{p,m,\alpha,t}(\mathbb{C}^n)$ stated in Theorem 1.1 as follows.

**Theorem 1.1.** Let $\alpha > 0$, $0 < p < \infty$, $s \geq 0$, $m \in \mathbb{N}$ and $t \in \mathbb{R}$. Then the following statements are equivalent for entire functions $f$ on $\mathbb{C}^n$:

(a) $f \in F_{p,m,\alpha,t}(\mathbb{C}^n)$.

(b) There exists a nonnegative continuous function $g \in L^p(G_{m,\alpha p,t-sp(m-1)})$ such that

$$\frac{|f(z) - f(w)|}{|z - w|} \leq (1 + |z|^{m-1} + |w|^{m-1})^{1+s}(g(z) + g(w))$$

for each $z, w \in \mathbb{C}^n$ with $z \neq w$.

For $f \in H(\mathbb{C}^n)$, we define $L_f(z,w) = f(z) - f(w)$.

Let $m$ be an even positive integer and $s \in \mathbb{R}$. As a local type, we define

$$L_{s,r}f(z,w) = [f(z) - f(w)]e^{s(z \cdot w)^m} \chi_{E_r(z)}(w),$$

where $\chi_{E_r(z)}$ denotes the characteristic function in the Euclidean ball $E_r(z) = \{ w \in \mathbb{C}^n : |w - z| < \frac{r}{1+|z|^{m-1}} \}$ for $r > 0$. Hence, the other main result is the double integral characterization for $F_{p,m,\alpha,t}$ as follows.

**Theorem 1.2.** Let $\alpha > 0$, $0 < p < \infty$, and $t \in \mathbb{R}$. Let $m$ be an even positive integer. For $s \geq 0$, let $\beta = \frac{s + \alpha}{2}$. Then the following statements are equivalent for entire functions $f$ on $\mathbb{C}^n$:

(a) $f \in F_{p,m,\alpha,t}(\mathbb{C}^n)$.

(b) $Lf \in L^p(G_{m,\alpha p,t} \times G_{m,\alpha p,t})$.

(c) $L_{s,r}f \in L^p(G_{m,\beta p,\delta} \times G_{m,\beta p,\delta})$, where $\delta = \frac{t}{2} - n(m-1)$.

In Theorem 1.2, (a) and (b) are equivalent when $m \in \mathbb{N}$. On the other hand, (a) and (c) are equivalent when $m$ is an even positive integer. An even positive integer $m$ is necessary in order that the function $L_{s,r}f(z,w)$ is holomorphic with respect to $w$.

Lipschitz type characterization for weighted Bergman spaces with standard weights on the unit disc in the complex plane $\mathbb{C}$ in terms of the Euclidean, hyperbolic, and pseudo-hyperbolic metrics was introduced by Hasi Wulan and Kehe Zhu [5]. Moreover, they generalized these results to the unit ball in $\mathbb{C}^n$. As an application, they proved the boundedness of the symmetric lifting operator in [4]. Moreover, double integral characterizations for weighted Bergman spaces in the unit ball in $\mathbb{C}^n$ were proved in [3] and [4]. Our results were motivated by Lipschitz type characterization and double integral characterization for the weighted Fock spaces $F_{2,\alpha,t}(\mathbb{C}^n)$ in [1]. It’s well known that $F_{2,\alpha,t}(\mathbb{C}^n)$ is closed.
in $L^p(G_{2,\alpha,t})$. In particular, $F_{2,\alpha,t}^2(C^n)$ is a Hilbert space and the reproducing kernel $K_z$ at $z \in C^n$ for $F_{2,1/2,0}^2(C^n)$ is given by

$$K_z(w) = e^{z \cdot w}.$$ See [6]. In [1], the reproducing kernel $K_z$ was used to get a norm estimate for $F_{2,\alpha,t}^2(C^n)$. In this paper, we have the same norm estimate for the Fock-type space $F_{p,m,\alpha,t}^2(C^n)$ without using $K_z$.

For nonnegative quantities $X$ and $Y$, the notation $X \lesssim Y$ means that there exists a positive constant $C$ such that $X \leq CY$. The constant $C$ is independent of the relevant variables. Furthermore, the notation $X \approx Y$ means that $X \lesssim Y$ and $Y \lesssim X$. In this case, we say that $X$ and $Y$ are equivalent.

2. Preliminaries

In this section, we suppose $m \in \mathbb{N}$.

For $r > 0$, $E_r(z)$ is defined by

$$E_r(z) = \left\{ w \in C^n : |w - z| < \frac{r}{1 + |z|^{m-1}} \right\}.$$ \hfill (2.1)

Lemma 2.1. Let $m \in \mathbb{N}$ and $r > 0$. For any $w \in E_r(z)$, there exists a positive constant $C = C(m,r)$ such that

$$C^{-1} \leq \frac{e^{|w|^m}}{e^{|z|^m}} \leq C.$$ \hfill (2.2)

Moreover, there exists a positive constant $C = C(m,r)$ such that

$$C^{-1} \leq \frac{1 + |w|^{m-1}}{1 + |z|^{m-1}} \leq C.$$ \hfill (2.3)

Proof. Note that the asserted inequalities are clear when $m = 1$. So, let $m \geq 2$. For $w \in E_r(z)$, we have

$$|w| < \frac{r}{1 + |z|^{m-1}} + |z|$$

so that

$$|w|^m < |z|^m + \sum_{k=1}^{m} \binom{m}{k} r^k \frac{|z|^{m-k}}{(1 + |z|^{m-1})^k} < |z|^m + (1 + r)^m.$$ \hfill (2.4)

Similarly, we get

$$1 + |w|^{m-1} < (1 + |z|^{m-1})(1 + r)^{m-1}$$

which implies

$$\frac{r}{1 + |z|^{m-1}} < \frac{r(1 + r)^{m-1}}{1 + |w|^{m-1}}.$$
Thus, we have
\[ E_r(z) \subset E_{2r(1+r)^{m-1}}(w), \quad w \in E_r(z). \]  
Consequently, (2.4) and (2.6) give us (2.2). Moreover, (2.5) and (2.6) imply (2.3). The proof is complete. \[ \square \]

We denote a multi-index \( M = (m_1, \ldots, m_n) \) which is an \( n \)-tuple of non-negative integers and use the following notation
\[ |M| = m_1 + \cdots + m_n, \quad M! = m_1! \cdots m_n! \]
and \( \partial_j \) denotes partial differentiation with respect to the \( j \)-th component.

**Lemma 2.2.** Let \( r > 0, \ m \in \mathbb{N} \) and \( \alpha, t \in \mathbb{R} \). Given a multi-index \( M \), there is a positive constant \( C = C(\alpha, m, r, t, M) \) such that
\[
|\partial^M g(z)|^p e^{-\alpha|z|^m} \leq C \int_{E_r(z)} |g(w)|^p \frac{e^{-\alpha|w|^m}}{(1 + |w|)^t} dV(w), \quad z \in \mathbb{C}^n,
\]
for \( 0 < p < \infty \) and \( g \in H(\mathbb{C}^n) \).

**Proof.** Let \( 0 < p < \infty \) and \( g \in H(\mathbb{C}^n) \). Let \( z \in \mathbb{C}^n \). By subharmonicity we have
\[
|g(z)|^p \leq \frac{1}{V(E_r(z))} \int_{E_r(z)} |g(w)|^p dV(w) = \frac{(1 + |z|^{m-1})^{2n}}{\omega_n r^{2n}} \int_{E_r(z)} |g(w)|^p dV(w),
\]
where \( \omega_n \) is the volume of the unit ball of \( \mathbb{C}^n \). And hence the Cauchy Estimates over the ball \( E_r(z) \) implies that
\[
|\partial^M g(z)|^p \leq C(1 + |z|^{m-1})^{p|M| + 2n} \int_{E_r(z)} |g(w)|^p dV(w)
\]
for some \( C = C(m, r, M) > 0 \). Moreover, for \( z \in \mathbb{C}^n \), we know
\[
1 + |z|^{m-1} \approx (1 + |z|)^{m-1}.
\]
So, using (2.2) and (2.3), we complete the proof. \[ \square \]

For a function \( f \in H(\mathbb{C}^n) \), we define the radial derivative \( \mathcal{R} f \) of \( f \) at \( z \) by
\[
\mathcal{R} f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z).
\]
The complex gradient of \( f \) at \( z \) is defined by
\[
|\nabla f(z)| = \left[ \sum_{j=1}^n \left| \frac{\partial f}{\partial z_j}(z) \right|^2 \right]^{1/2}.
\]
For a radial function \( \phi(r) \), we deduce Proposition 2.5 from Definition 2.3.
Definition 2.3 ([2]). Assume \( \phi : [0, \infty) \to \mathbb{R}^+ \) is twice continuously differentiable and there exists \( \rho > 0 \) such that \( \phi'(r) \neq 0 \) for \( r \geq \rho \). We say that \( \phi \) is in the class \( \mathcal{W}_p \), if it satisfies the following conditions:

\[
\lim_{r \to \infty} \frac{r e^{-p \phi(r)}}{\phi'(r)} = 0,
\]
\[
\limsup_{r \to \infty} \frac{r}{\phi'(r)} < \frac{p}{n},
\]
\[
\liminf_{r \to \infty} \frac{r}{\phi'(r)} > -\infty.
\]

Lemma 2.4 ([2]). Let \( 0 < p < \infty \) and \( \phi \in \mathcal{W}_p \). Then for \( f \in H(\mathbb{C}^n) \),

\[
(2.7) \quad \int_{\mathbb{C}^n} |f(z)|^p e^{-p \phi(|z|)} dV(z) \approx |f(0)|^p + \int_{\mathbb{C}^n} \frac{|\mathcal{R} f(z)|^p}{(1 + |z| \phi'(|z|))^{p}} e^{-p \phi(|z|)} dV(z).
\]

Note that constants functions are contained in \( F^p_{m,\alpha,t}(\mathbb{C}^n) \) so we have \( f \in F^p_{m,\alpha,t}(\mathbb{C}^n) \) if and only if \( f - f(0) \in F^p_{m,\alpha,t}(\mathbb{C}^n) \). Furthermore Lemma 2.4 implies that \( F^p_{m,\alpha,t}(\mathbb{C}^n) \)-norm of \( f - f(0) \) is equivalent to the norm of the product of its radial derivative and \( \frac{1}{1 + |z|^{m}} \) in the \( n \)-dimensional complex space. We know that this factor \( \frac{1}{1 + |z|^{m}} \) is the supplementary amount to control the growth of the radial derivative. See the following.

Proposition 2.5. Let \( \alpha > 0 \), \( 0 < p < \infty \), \( m \in \mathbb{N} \), and \( t \in \mathbb{R} \). Then the following norms

\[
\|f - f(0)\|_{F_{m,\alpha,t}} \leq \left\| \frac{\mathcal{R} f(z)}{1 + |z|^m} \right\|_{L^p(G_{m,\alpha,p,t})} \leq \left\| \frac{\nabla f(z)}{1 + |z|^{m-1}} \right\|_{L^p(G_{m,\alpha,p,t})}
\]

are comparable to one another for \( f \in H(\mathbb{C}^n) \).

Proof. Let \( \phi(|z|) = \alpha |z|^m + \frac{1}{p} \ln(1 + |z|) \) for \( z \in \mathbb{C}^n \). Then \( \phi \) is in the class \( \mathcal{W}_p \) when \( m \in \mathbb{N} \) and \( t \in \mathbb{R} \). Also, by simple calculation, there exists a constant \( C = C(\alpha, m, r, t) > 0 \) such that

\[
C^{-1} \leq \frac{1 + |z|^m}{1 + |z| \phi'(|z|)} \leq C.
\]

Thus, applying Lemma 2.4 to \( f - f(0) \), we obtain

\[
\int_{\mathbb{C}^n} |f(z) - f(0)|^p e^{-p \phi(|z|)} dV(z) \approx \int_{\mathbb{C}^n} \frac{|\mathcal{R} f(z)|^p}{(1 + |z|^m)^t} e^{-p \phi(|z|)} dV(z).
\]

Also, due to \( |\mathcal{R} f(z)| \leq |z| |\nabla f(z)| \), this estimate implies

\[
\|f - f(0)\|_{F_{m,\alpha,t}} \lesssim \left\| \frac{\nabla f(z)}{1 + |z|^{m-1}} \right\|_{L^p(G_{m,\alpha,p,t})}.
\]
Now, it remains to show
\[ \left\| \frac{\nabla f(z)}{1 + |z|^{m-1}} \right\|_{L^p(G_{m,\alpha,p,t})} \lesssim \|f - f(0)\|_{F_{m,\alpha,t}}. \]

It follows from Lemma 2.2 that
\[ \frac{|\nabla f(z)|^p}{(1 + |z|^{m-1})^p} \lesssim (1 + |z|^{m-1})^{2n} \int_{E_r(z)} |f(w) - f(0)|^p \, dV(w). \]

Let \( r > 0 \) and \( r_1 := 2r(1 + r)^{m-1} \). Integrating both sides of the above against \( dG_{m,\alpha,p,t} \), we have from (2.6), (2.2) and (2.3)
\[ \|f - f(0)\|^p \]
\[ \lesssim \int_{\Omega_r} \int_{R_{r_1}(z)} |f(w) - f(0)|^p \, dV(w) \, dG_{m,\alpha,p,t-2n(m-1)}(z) \]
\[ \lesssim \int_{\Omega_r} \int_{R_{r_1}(w)} |f(w) - f(0)|^p \, dG_{m,\alpha,p,t-2n(m-1)}(z) \, dV(w) \]
\[ \approx \|f - f(0)\|_{F_{m,\alpha,t}}^p \]
as desired. \( \square \)

3. Lipschitz type characterization

In this section, we prove our first result Theorem 1.1. For \( r > 0 \), we set
\[ \Omega_r := \{(z, w) : |w - z|(1 + |z|^{m-1} + |w|^{m-1}) < r\}. \]

**Theorem 3.1.** Let \( \alpha > 0 \), \( 0 < p < \infty \), \( s \geq 0 \), \( m \in \mathbb{N} \), and \( t \in \mathbb{R} \). Then the following statements are equivalent for entire functions \( f \) on \( \mathbb{C}^n \):

(a) \( f \in F_{m,\alpha,t}(\mathbb{C}^n) \).

(b) There exists a nonnegative continuous function \( g \in L^p(G_{m,\alpha,p,t-\alpha p(m-1)}) \) such that
\[ \frac{|f(z) - f(w)|}{|z - w|} \leq (1 + |z|^{m-1} + |w|^{m-1})^{1+s}(g(z) + g(w)) \]
for each \( z, w \in \mathbb{C}^n \) with \( z \neq w \).

**Proof.** First, we assume that (b) holds. Fixing \( z \) and taking the limits \( w \to z \) along the directions parallel to the coordinate axes,
\[ |\partial_j f(z)| \lesssim (1 + |z|^{m-1})^{1+s} g(z) \]
for each \( j \). Thus, we have
\[ \frac{|\nabla f(z)|}{1 + |z|^{m-1}} \lesssim (1 + |z|^{m-1})^s g(z), \quad z \in \mathbb{C}^n \]
and thus
\[ \int_{\mathbb{C}^n} \frac{|\nabla f(z)|^p}{(1 + |z|^{m-1})^p} \, e^{-\alpha p |z|^m} \, dV(z) \lesssim \int_{\mathbb{C}^n} |g(z)|^p \frac{e^{-\alpha p |z|^m}}{(1 + |z|)^{t-s p(m-1)}} \, dV(z). \]
Since \( g(z) \in L^p(G_{m,\alpha,p,t-\phi(m-1)}) \), by Proposition 2.5, we conclude 
\[ f \in F^p_{m,\alpha,t}(\mathbb{C}^n). \]

Second, we assume that (a) holds. Fix any \( r > 0 \). We consider \( (z, w) \in \Omega_r \).
Then \( w \in E_r(z) \) and
\[ 1 + |z|^{m-1} + |w|^{m-1} \approx 1 + |z|^{m-1}. \]
By the fundamental theorem of calculus, we get
\[ |f(z) - f(w)| \leq |z - w| \int_0^1 |\nabla f(\rho z + (1 - \rho)w)| \, d\rho. \]
Since \( \rho z + (1 - \rho)w \) in \( E_r(z) \), it follows
\[ |f(z) - f(w)| \leq |z - w| \sup_{\zeta \in E_r(z)} |\nabla f(\zeta)|. \]

Furthermore, we note
\[
|\nabla f(\zeta)| \approx (1 + |z|^{m-1})^{1+s} \frac{|\nabla f(\zeta)|}{(1 + |\zeta|^{m-1})^{1+s}} \\
\approx (1 + |z|^{m-1} + |w|^{m-1})^{1+s} \frac{|\nabla f(\zeta)|}{(1 + |\zeta|^{m-1})^{1+s}}
\]
for \( \zeta \in E_r(z) \). Let 
\[ h_s(z) := \sup_{\zeta \in E_r(z)} \frac{|\nabla f(\zeta)|}{(1 + |\zeta|^{m-1})^{1+s}}. \]
Then we have by (3.1) and (3.2)
\[ |f(z) - f(w)| \lesssim |z - w|(1 + |z|^{m-1} + |w|^{m-1})^{1+s}(h_s(z) + h_s(w)) \]
for \( (z, w) \in \Omega_r \).
Next, we consider \( (z, w) \notin \Omega_r \). Then \( |w - z|(1 + |z|^{m-1} + |w|^{m-1}) \geq r \).
Therefore, for \( s \geq 0 \), we obtain
\[
|f(z) - f(w)| \\
\leq |z - w|((1 + |z|^{m-1} + |w|^{m-1})^{1+s})(|f(z)| + |f(w)|) \\
\leq |z - w|((1 + |z|^{m-1} + |w|^{m-1})^{1+s}) \left( \frac{|f(z)|}{(1 + |z|^{m-1})^s} + \frac{|f(w)|}{(1 + |w|^{m-1})^s} \right).
\]
Hence, by setting \( g(z) := h_s(z) + \frac{|f(z)|}{(1 + |z|^{m-1})^s} \) for \( z \in \mathbb{C}^n \), we have
\[
|f(z) - f(w)| \lesssim |z - w|(1 + |z|^{m-1} + |w|^{m-1})^{1+s}(g(z) + g(w))
\]
for each \( z, w \in \mathbb{C}^n \) with \( z \neq w \). Note that the constant suppressed above depends only on \( m, r \) and \( s \). Also, the function \( g(z) \) is continuous on \( \mathbb{C}^n \). It remains for us to show the function \( g(z) \) belongs to \( L^p(G_{m,\alpha,p,t-\phi(m-1)}) \). It is clear that \( \frac{|f(z)|}{(1 + |z|^{m-1})^s} \) is in \( L^p(G_{m,\alpha,p,t-\phi(m-1)}) \) for \( f \in F^p_{m,\alpha,t}(\mathbb{C}^n) \).
Now, we claim $h_s$ is in $L^p(G_{m,op,t-sp(m-1)})$. Let $\zeta \in E_r(z)$. Then $E_{r_0}(\zeta) \subset E_{r_1}(z)$ by (2.6) where $r_0 = 2r(1 + r)^{m-1}$ and $r_1 = 2r_0(1 + r_0)^{m-1}$. By Lemma 2.2 and (2.3), we get
\[
\frac{|\nabla f(\zeta)|}{(1 + |\zeta|^{m-1})^{1+sp}} \lesssim (1 + |\zeta|^{m-1})^{2n} \int_{E_{r_0}(\zeta)} |f(w)|^p dV(w) \lesssim (1 + |\zeta|^{m-1})^{2n} \int_{E_{r_1}(z)} |f(w)|^p dV(w).
\]
Taking the supremum over $\zeta \in E_r(z)$, we have
\[
|h_s(z)|^p \lesssim (1 + |z|^{m-1})^{2n} \int_{E_{r_1}(z)} |f(w)|^p dV(w)
\]
for all $z \in \mathbb{C}^n$. Let $r_2 = 2r_1(1 + r_1)^{m-1}$. By integrating both sides of the above against the measure $dG_{m,op,t-sp(m-1)}(z)$, it follows
\[
\|h_s\|_{L^p(G_{m,op,t-sp(m-1)})}^p \lesssim \int_{\mathbb{C}^n} (1 + |z|^{m-1})^{2n} \int_{E_{r_1}(z)} |f(w)|^p dV(w) dG_{m,op,t}(z)
\]
\[
= \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} (1 + |z|^{m-1})^{2n} |f(w)|^p \chi_{E_{r_1}(z)}(w) dG_{m,op,t}(z) \ dV(w)
\]
\[
< \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} (1 + |z|^{m-1})^{2n} |f(w)|^p \chi_{E_{r_2}(z)}(w) dG_{m,op,t}(z) \ dV(w)
\]
\[
= \int_{\mathbb{C}^n} |f(w)|^p \int_{E_{r_2}(w)} (1 + |z|^{m-1})^{2n} dG_{m,op,t}(z) \ dV(w),
\]
where $\chi$ denotes the characteristic function in its subscripted set. For $z \in E_{r_2}(w)$, we know (2.2) and (2.3). Hence, it follows that
\[
\|h_s\|_{L^p(G_{m,op,t-sp(m-1)})}^p \lesssim \int_{\mathbb{C}^n} |f(w)|^p (1 + |w|^{m-1})^{2n} \int_{E_{r_2}(w)} dV(z) dG_{m,op,t}(w)
\]
\[
< \int_{\mathbb{C}^n} |f(w)|^p (1 + |w|^{m-1})^{2n} \int_{E_{r_2}(w)} \frac{\omega_m \text{r}_2^n}{(1 + |w|^{m-1})^{2n}} dG_{m,op,t}(w)
\]
\[
\lesssim \|f\|_{F_{m,op,t-sp(m-1)}^p}^p.
\]
This finishes the proof. \hfill \Box

From the proof of Theorem 3.1, we have the following local version of Theorem 3.1 for arbitrary s real.

**Theorem 3.2.** Let $\alpha > 0$, $0 < p < \infty$, $r > 0$, $m \in \mathbb{N}$, and $s, t \in \mathbb{R}$. Then the following statements are equivalent for entire functions $f$ on $\mathbb{C}^n$:

(a) $f \in F_{m,\alpha,t}^p(\mathbb{C}^n)$. 

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(b) There exists a nonnegative continuous function $g \in L^p(G_{m,\alpha_p,t-sp(m-1)})$ such that
\[
\frac{|f(z) - f(w)|}{|z - w|} \leq (1 + |z|^{m-1} + |w|^{m-1})^{1+s}(g(z) + g(w))
\]
for $(z, w) \in \Omega_r$ with $z \neq w$.

4. Double integral characterization

In this section, we prove the main Theorem 1.2. By the same proof of Proposition 3.1 in [1], we have the following.

Theorem 4.1. Let $0 < p < \infty$ and $\phi \in W_p$. Then the estimate
\[
\int_{C^n} |f(z) - f(0)|^p e^{-p\phi(|z|)} \, dV(z)
\]
\[
\approx \int_{C^n} \int_{C^n} |Lf(z, w)|^p e^{-p\phi(|z|)} \, dV(z) \, e^{-p\phi(|w|)} \, dV(w)
\]
holds for $f \in H(C^n)$.

Letting $\phi(|z|) = \alpha |z|^m + \frac{1}{p} \ln(1 + |z|)$ for $z \in C^n$, Theorem 4.1 gives us the following characterization.

Theorem 4.2. Let $\alpha > 0$, $0 < p < \infty$, $m \in \mathbb{N}$, and $t \in \mathbb{R}$. Then the following statements are equivalent for entire functions $f$ on $C^n$:

(a) $f \in F^{p_m,\alpha,t}(C^n)$.

(b) $Lf \in L^p(G_{m,\alpha_p,t} \times G_{m,\alpha_p,t})$.

Moreover, the norms $\|f - f(0)\|_{F^{p_m,\alpha,t}}$ and $\|Lf\|_{L^p(G_{m,\alpha_p,t} \times G_{m,\alpha_p,t})}$ are comparable to each other.

Lemma 4.3. Let $0 < p < \infty$ and $s \geq 0$. Let $m$ be an even positive integer. For $w \in E_r(z)$, there exists a positive constant $C$ such that $|e^{s(z \cdot w)^{m/2}}| \leq e^{sp|z|^{m/2}}$.

Proof. We prove that
\[
|e^{s(z \cdot w)^{m/2}}| \leq e^{sp|z|^{m/2}|w|^{m/2}} \leq e^{sp|z|^{m/2}|z|^{m}} \leq e^{s|z|^{m}}
\]
by the Cauchy-Schwarz inequality, arithmetic-geometric mean inequality and (2.2) in Lemma 2.1 for $w \in E_r(z)$ in turn.

Theorem 4.4. Let $\alpha, r > 0$, $0 < p < \infty$, and $t \in \mathbb{R}$. Let $m$ be an even positive integer. For $s \geq 0$, let $\beta := \frac{s + \alpha}{2}$. Then the following statements are equivalent for entire functions $f$ on $C^n$:

(a) $f \in F^{p_m,\alpha,t}(C^n)$.
(b) \( L_r^s f \in L^p(G_m, \beta, \delta \times G_m, \beta, \delta) \), where \( \delta = \frac{r}{2} - n(m - 1) \).

Moreover, the norms

\[
\|f - f(0)\|_{L^p_{r, m, \alpha, t}} \quad \text{and} \quad \|L_r^s f\|_{L^p(G_m, \beta, \delta \times G_m, \beta, \delta)}
\]

are comparable to each other.

**Proof.** We assume that (b) holds. Fix \( f \in H(C^n) \) and let \( z \in C^n \). Define a function

\[
g_z(w) = |f(w) - f(z)| e^{s|w|^p}.
\]

Then \( g_z \in H(C^n) \) and \( \nabla g_z(z) = \nabla f(z) e^{s|z|^m} \). By applying Lemma 2.2, (2.2) and (2.3), we get

\[
\left( \frac{\nabla f(z) e^{s|z|^m}}{1 + |z|^{m-1}} \right)^p 
\]

\[
\lesssim (1 + |z|^{m-1})^{2n} \int_{E_v(z)} |f(w) - f(z)|^p |e^{s|w|^p}| dV(w) 
\]

\[
\approx \epsilon \frac{e^{s|z|^m}}{1 + |z|^{m-1}} n 
\]

\[
\times \int_{E_v(z)} |f(w) - f(z)|^p |e^{s|w|^p}| dV(w).
\]

By integrating both sides of the above against \( dG_m(s+n)p, t(z) \) and applying (2.3) for \( w \in E_v(z) \), we have

\[
\int_{C^n} \left( \frac{\nabla f(z)}{1 + |z|^{m-1}} \right)^p dG_m, c, p, t(z)
\]

\[
\lesssim \int_{C^n} \int_{E_v(z)} |f(w) - f(z)|^p |e^{s|w|^p}| dG_m, s, p, t - n|m - 1| (w) dG_m, s, p, t - n|m - 1| (z)
\]

\[
\approx \|L_r^s f\|_{L^p(G_m, \beta, \delta \times G_m, \beta, \delta)}.
\]

Thus, by Proposition 2.5, we obtain

\[
\|f - f(0)\|_{L^p_{r, m, \alpha, t}} \lesssim \|L_r^s f\|_{L^p(G_m, \beta, \delta \times G_m, \beta, \delta)}.
\]

The constant suppressed above is independent of \( f \). We complete that (b) implies (a).

Now, we assume that (a) holds. Let \( r_0 = 2r(1 + r)^{m-1} \).

\[
\|L_r^s f\|_{L^p(G_m, \beta, \delta \times G_m, \beta, \delta)}
\]

\[
= \int_{C^n} \int_{C^n} |f(w) - f(z)|^p |e^{s|w|^p}| \chi_{E_v(z)}(w) dG_m, s, p, \delta (w) dG_m, s, p, \delta (z)
\]

\[
\lesssim \int_{C^n} \int_{C^n} |f(z) - f(0)|^p |e^{s|w|^p}| \chi_{E_v(z)}(w) dG_m, s, p, \delta (w) dG_m, s, p, \delta (z)
\]

\[
+ \int_{C^n} \int_{C^n} |f(w) - f(0)|^p |e^{s|w|^p}| \chi_{E_v(w)}(z) dG_m, s, p, \delta (z) dG_m, s, p, \delta (w)
\]

...
\[ \lesssim \int_{\mathbb{C}^n} |f(z) - f(0)|^p \int_{E_{n_0}(z)} |e^{s_0(z_w)}|^\frac{m}{p} \, |dG_{m,\beta p,\delta}(w)| \, dG_{m,\beta p,\delta}(z). \]

For the first inequality, we used \( E_\tau(z) \subset E_{n_0}(z) \), \( E_\tau(z) \subset E_{n_0}(w) \) for \( w \in E_\tau(z) \) and Fubini’s theorem. And for \( w \in E_{n_0}(z) \), we have from (2.3)

\[
\|L^*_\tau f\|_{L^p(G_{m,\beta p,\delta} \times G_{m,\beta p,\delta})} \lesssim \int_{\mathbb{C}^n} |f(z) - f(0)|^p (1 + |z|^{m-1})^n \\
\times \int_{E_{n_0}(z)} |e^{s_0(z_w)}|^\frac{m}{p} \, |dG_{m,\beta p,-n(m-1)}(w)| \, dG_{m,\beta p,\tau}(z) \\
= \int_{\mathbb{C}^n} |f(z) - f(0)|^p I_\tau(z) \, dG_{m,\alpha p,\tau}(z),
\]

where \( I_\tau(z) := e^{(\tau s_0(z_w)) |z|^m} (1 + |z|^{m-1})^n \int_{E_{n_0}(z)} |e^{s_0(z_w)}|^\frac{m}{p} \, |dG_{m,\beta p,-n(m-1)}(w)|. \)

Now, we claim that \( I_\tau(z) \lesssim 1 \). For \( w \in E_{n_0}(z) \), we have Lemma 4.3, (2.2) and (2.3). It follows that

\[
\int_{E_{n_0}(z)} |e^{s_0(z_w)}|^\frac{m}{p} \, |dG_{m,\beta p,-n(m-1)}(w)| \lesssim \int_{E_{n_0}(z)} e^{sp|z|^m} \, dG_{m,\beta p,-n(m-1)}(w) \\
\lesssim e^{\frac{(\tau - \alpha) s_0(z_w)}{2} |z|^m} (1 + |z|^{m-1})^n V[E_{n_0}(z)].
\]

Since \( V[E_{n_0}(z)] \approx (1 + |z|^{m-1})^{-2n} \), we have \( I_\tau(z) \lesssim 1 \). Hence we complete that (a) implies (b).

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References

HONG RAE CHO  
DEPARTMENT OF MATHEMATICS  
PUSAN NATIONAL UNIVERSITY  
PUSAN 46241, KOREA  
Email address: chohr@pusan.ac.kr  

JEONG MIN HA  
DEPARTMENT OF MATHEMATICS  
PUSAN NATIONAL UNIVERSITY  
PUSAN 46241, KOREA  
Email address: jm.ha@pusan.ac.kr  

KYEonsook Nam  
FACULTY OF LIBERAL EDUCATION  
SEOUL NATIONAL UNIVERSITY  
SEOUL 08826, KOREA  
Email address: ksnam@snu.ac.kr