MODULAR MULTIPLICATIVE INVERSES OF FIBONACCI NUMBERS

HYUN-JONG SONG

ABSTRACT. Let $F_n, n \in \mathbb{N}$ be the $n$-th Fibonacci number, and let $(p, q)$ be one of ordered pairs $(F_{n+2}, F_n)$ or $(F_{n+1}, F_n)$. Then we show that the multiplicative inverse of $q \mod p$ as well as that of $p \mod q$ are again Fibonacci numbers. For proof of our claim we make use of well-known Cassini, Catlan and dOcagne identities. As an application, we determine the number $N_{p,q}$ of nonzero term of a polynomial $\Delta_{p,q}(t) = \frac{(t^{pq} - 1)(t - 1)}{(t^p - 1)(t^q - 1)}$ through the Carlitz’s formula.

1. Preliminaries

Motivation of problems dealt in this paper arose from two intriguing observations made for a torus knots $t(p, q)$ in knot theory where $p, q$ are relative prime positive integers. One is that for each triple of consecutive Fibonacci numbers $F_n, F_{n+1}, F_{n+2}$, twisting $F_{n+1}$-parallel strands of a torus knot $t(F_{n+2}, F_n)$, we have a trivial knot. For more details see [2]. The other is that in 1966 an eminent number theorist Carlitz [1] provided a method of computing the number $N_{p,q}$ of non-zero terms of a polynomial $\Delta_{p,q}(t) = \frac{(t^{pq} - 1)(t - 1)}{(t^p - 1)(t^q - 1)}$, which turns out to be the Alexander polynomial of $t(p, q)$. Indeed the Alexander polynomial $\Delta_{p,q}$ is $\Phi_{pq}$, the $pq$-th cyclotomic polynomial if $p, q$ are distinct primes [3]. Explicit knowledge of $N_{p,q}$ is useful for a certain topological construction of $t(p, q)$ dealt in [5].

Definition 1. Let $(p, q)$ be an ordered pair of relative prime positive integers. Then an ordered pair of positive integers $(x, v)$ is said to be a pairwise modular multiplicative inverse of $(p, q)$ if and only if

1. $xq \equiv 1 \mod p$ ($1 \leq x \leq p - 1$) and
2. $vp \equiv 1 \mod q$ ($1 \leq x \leq -1$).

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Note that a pairwise modular multiplicative inverse of \((p, q)\) is uniquely determined.

From [4, Proposition 2.1] we have.

**Lemma 1.1.** Under the notations in Definition 1.1, the following statements are equivalent.

1. \((x, v)\) is the pairwise modular multiplicative inverse of \((p, q)\)
2. there exists a uniquely determined quadruple of positive integers \(u, v, x\) and \(y\) such that
   
   \[
   xv - yu = 1 \quad (1)
   \]
   \[
   p = x + y \quad (2)
   \]
   \[
   q = u + v \quad (3)
   \]

**Remark 1.** In Lemma we can replace equation (1.1) by one of following equations.

4. \(qx - pu = 1\)  \(\text{(4)}\)
5. \(pv - qy = 1\)  \(\text{(5)}\)

From [4, Corollary 2.6] we have.

**Lemma 1.2.** The number \(N_{p,q}\), denoted by \(N_{p,q}\), of all non-zero terms of \(\Delta_{p,q}(t)\) is equal to \(vx + uy = 2vx - 1\).

For simplicity we assume that \(p > q\).

We recall three well known identities which naturally reveals modular multiplicative inverse of a pair of Fibonacci numbers.

A: Cassinis identity

\[
F_{n-1}F_{n+1} - F_n^2 = (-1)^n \quad (6)
\]

is divided to two subcases: for each \(k \in \mathbb{N}\)

\[
F_{2k-1}F_{2k+1} - F_{2k}^2 = 1 \quad (7)
\]

B: Catalans identity

\[
F_n^2 - F_{n-2}F_{n+2} = (-1)^{n-2} \quad (8)
\]

is divided to two subcases: for each \(k \in \mathbb{N}\)

\[
F_{2k}^2 - F_{2k-2}F_{2k+2} = 1 \quad (9)
\]

\[
F_{2k-1}F_{2k+3} - F_{2k+1}^2 = 1 \quad (10)
\]

C: dOcagnes identity

\[
F_{n+2}F_{n+1} - F_nF_{n+3} = (-1)^n \quad (11)
\]

is divided to two subcases: for each \(k \in \mathbb{N}\)

\[
F_{2k+2}F_{2k+1} - F_{2k}F_{2k+3} = 1 \quad (12)
\]

\[
F_{2k-1}F_{2k+2} - F_{2k+1}F_{2k} = 1 \quad (13)
\]
2. The main results

The following theorem shows intriguing applications of the three well known identities among a sequence of Fibonacci numbers to detecting a pairwise modular multiplicative inverse for a suitable two Fibonacci numbers.

Theorem 2.1. For each \( k \in \mathbb{N} \) we have:

1. \((F_{2k-1}, F_{2k})\) is the pairwise multiplicative inverse of \((F_{2k+1}, F_{2k})\).
2. \((F_{2k+1}, F_{2k-1})\) is the pairwise multiplicative inverse of \((F_{2k+2}, F_{2k+1})\).
3. \((F_{2k}, F_{2k-1})\) is the pairwise multiplicative inverse of \((F_{2k+2}, F_{2k})\).
4. \((F_{2k+2}, F_{2k-1})\) is the pairwise multiplicative inverse of \((F_{2k+3}, F_{2k+1})\).

Proof. (1) For \((p, q) = (F_{2k+1}, F_{2k})\) and \((x, u) = (F_{2k-1}, F_{2k-2})\), applying the d’Ocagne’s identity (10) to equation (4) we have the pairwise multiplicative inverse \((F_{2k-1}, F_{2k-1})\) of \((F_{2k+1}, F_{2k})\). In this case equation (5) corresponds to the Cassini’s identity (6), since

\[
(F_{2k} - F_{2k-2})F_{2k+1} - (F_{2k+1} - F_{2k-1})F_{2k} = 1;
\]
\[
F_{2k-1}F_{2k+1} - F_{2k}^2 = 1
\]

(2) For \((p, q) = (F_{2k+2}, F_{2k})\) and \((x, u) = (F_{2k+1}, F_{2k})\), applying the Cassini identity (7) to equation (4) we have the pairwise multiplicative inverse \((F_{2k+1}, F_{2k-1})\) of \((F_{2k+2}, F_{2k})\). In this case equation (5) corresponds to d’Ocagne’s identity (11), since

\[
(F_{2k+1} - F_{2k})F_{2k+2} - (F_{2k+2} - F_{2k+1})F_{2k} = 1;
\]
\[
F_{2k-1}F_{2k+2} - F_{2k}F_{2k+1} = 1
\]

(3) For \((p, q) = (F_{2k+2}, F_{2k})\) and \((x, u) = (F_{2k}, F_{2k-2})\), applying the Catalan’s identity (8) to equation (4), we have the pairwise multiplicative inverse \((F_{2k}, F_{2k-1})\) of \((F_{2k+2}, F_{2k})\). In this case equation (1.5) corresponds to d’Ocagne’s identity (11.1), since

\[
(F_{2k} - F_{2k-2})F_{2k+2} - (F_{2k+2} - F_{2k})F_{2k} = 1;
\]
\[
F_{2k-1}F_{2k+2} - F_{2k}F_{2k+1} = 1
\]

(4) For \((p, q) = (F_{2k+3}, F_{2k+1})\) and \((x, u) = (F_{2k+2}, F_{2k})\), identifying the d’Ocagne’s identity (10) to equation (4) we have the pairwise multiplicative inverse \((F_{2k+2}, F_{2k-1})\) of \((F_{2k+3}, F_{2k+1})\). In this case equation (5) corresponds to the Catalan’s identity (9), since

\[
(F_{2k+1} - F_{2k})F_{2k+3} - (F_{2k+3} - F_{2k+2})F_{2k+1} = 1;
\]
\[
F_{2k-1}F_{2k+3} - F_{2k+1}^2 = 1
\]

□

As an application, we determine the number \( N_{p, q} \) of non-zero term of the Alexander polynomial \( \Delta_{p, q}(t) = \frac{(t^{n-1})(t^{-1})}{(t^r-1)(t^s-1)} \) of a torus knot \( t(p, q) \) as follows:
Corollary 2.2.

\begin{align*}
(1) \quad N_{F_{2k+1}, F_{2k}} &= 2F_{2k-1}^2 - 1 \\
(2) \quad N_{F_{2k+2}, F_{2k+1}} &= 2F_{2k+1}F_{2k-1} - 1 \\
(3) \quad N_{F_{2k+2}, F_{2k}} &= 2F_{2k}F_{2k-1} - 1 \\
(4) \quad N_{F_{2k+3}, F_{2k+1}} &= 2F_{2k+2}F_{2k-1} - 1
\end{align*}

Applying the method introduced in [5] to the Corollary 2.2, we shall determine \((1,1)\)-diagrams of torus knots \(t(F_n, F_{n+2})\) for each \(n \geq 3\).

References


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