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A Ratio Variation of Nicomachus's Theorem

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Abstract

In this paper, we study a ratio variation of Nicomachus' theorem by computing $\sum_{k=1}^{n}[rk]$ and $\sum_{k=1}^{n}[rk]^3$, where r is a rational number.

Keywords : Nicomachus' Theorem, Triangular Number, Sum of Integer Parts

1. Introduction

A well-known Nicomachus' theorem states that the sum of the first n cubes is the square of the nth triangular number, that is

$$1^3+2^3+\cdots+n^3=(1+2+\cdots+n)^2. \tag{1}$$

Expanding this, Warnaar [2] observed and proved the q-analog of (1). In this note, we expand (1) in another way by considering the ratio

$$F_n(\alpha):=\frac{\sum_{k=1}^{n}[\alpha k]^3}{\left(\sum_{k=1}^{n}[\alpha k]\right)^2}$$

where α is a nonzero real number and $[x]$ denotes the integer part of a real number x. We observe that $F_n(1)=1$ and

$$\lim_{n\to\infty}F_n(\alpha)=\alpha, \tag{2}$$

which follows from $[\alpha n]=\alpha n+O(1)$ and Nicomachus' theorem. From a personal communication to the second author of [1], Stolarsky observed that if $\alpha=\phi$ or ϕ^2, where ϕ is a golden ratio, then with a choice of n along the Fibonacci sequence, the limit relation (2) can be 'quantified'. Related results and more about this were recently published, and for details, see Theorems 2.1 and 2.5 in [1].

In this study, we consider $F_n(r)$ when r is a nonzero rational number while [1] dealt with r as the golden ratio that is an irrational number. For this, we study

$$\sum_{k=1}^{n}[rk] \text{ and } \sum_{k=1}^{n}[rk]^3 \tag{3}$$

where $r=p/q$ for some $p,q\in\mathbb{Z}$ with $\gcd(p,q)=1$ and $q\neq 0$ so that the limit in (2) follows. It seems to be very hard to obtain the closed forms of the both in (3) where r is a rational number. So in this notes, we study some special cases so that we come to close to obtain 'nice forms' of the both in (3).

2. Results and questions

Let $r=p/q$ be the lowest form of a rational number r, that is, $p,q\in\mathbb{Z}$ with $\gcd(p,q)=1$ and $q\neq 0$. Then the following gives a closed form of $\sum_{k=1}^{n}[rk]$ when n is a multiple of q.

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Proposition 1 Let $p, q \in \mathbb{Z}$ with $\gcd(p,q)=1$ and $q \neq 0$. If n is a multiple of q, then

$$\sum_{k=1}^{n}\left[\frac{p}{q}k\right]=\binom{n}{2}\frac{p}{q}-\frac{n+1}{2q}(q-1).$$

Proof Suppose that $p, q \in \mathbb{Z}$ with $\gcd(p,q)=1$, $q \neq 0$ and n is a multiple of q. Then

$$\sum_{k=1}^{n-1}\left[\frac{p}{q}k\right]=\sum_{k=1}^{n-1}\left(\frac{p}{q}k\right)-\sum_{k=1}^{n-1}\left\{\frac{p}{q}k\right\}=$$

$$\frac{p}{q}\frac{n(n-1)}{2}-\sum_{k=1}^{n-1}\left\{\frac{p}{q}k\right\}, \tag{4}$$

where $\{x\}$ denotes the fractional part of a real number x. But since q divides n,

$$\sum_{k=1}^{n-1}\left\{\frac{p}{q}k\right\}=\frac{n}{q}\sum_{k=1}^{q-1}\left\{\frac{p}{q}k\right\}.$$

The congruence $pk \equiv a \pmod{q}$ has a unique solution for every a with $0 \leq a \leq q-1$. Reordering the solutions gives

$$\frac{n}{q}\sum_{k=1}^{q-1}\left\{\frac{p}{q}k\right\}=\frac{n}{q}\sum_{k=1}^{q-1}\left\{\frac{1}{q}k\right\}=\frac{n}{q}\sum_{k=1}^{q-1}\frac{k}{q}=\frac{n(q-1)}{2q}.$$

Hence back to (4), we have

$$\sum_{k=1}^{n-1}\left[\frac{p}{q}k\right]=\frac{p}{q}\frac{n(n-1)}{2}-\frac{n(q-1)}{2q}=\frac{n}{2q}(pn-p-q+1)$$

and so

$$\sum_{k=1}^{n}\left[\frac{p}{q}k\right]=\frac{n+1}{2q}(pn-q+1)=\binom{n}{2}\frac{p}{q}-\frac{n+1}{2q}(q-1).$$

Remark 2 In general, for $n \in \mathbb{Z}$ with $n \geq 2$, and t a positive real number,

$$\sum_{k=1}^{n}a_k^t \asymp_{n,t}\left(\sum_{k=1}^{n}a_k\right)^t \quad (a_1, a_2, \cdots, a_n > 0),$$

where $\asymp_{n,t}$ denotes the order of magnitude estimate whose estimation depends on n and t. So for $p, q \in \mathbb{Z}^+$ with $\gcd(p,q)=1$. Then

$$\sum_{k=1}^{n}\left[\frac{p}{q}k\right]^3 \asymp_n \left(\sum_{k=1}^{n}\left[\frac{p}{q}k\right]\right)^3,$$

that is, $\sum_{k=1}^{n}\left[\frac{p}{q}k\right]^3$ and $\left(\sum_{k=1}^{n}\left[\frac{p}{q}k\right]\right)^3$ have the same order of magnitude. So for n being a multiple of q, we have

$$\frac{\sum_{k=1}^{n}\left[\frac{p}{q}k\right]^3}{\left(\sum_{k=1}^{n}\left[\frac{p}{q}k\right]\right)^2} \asymp_n \sum_{k=1}^{n}\left[\frac{p}{q}k\right]=\binom{n}{2}\frac{p}{q}-\frac{n+1}{2q}(q-1).$$

In the next proposition, we provides recurrence relations of both $\sum_{k=1}^{n}[rk]$ and $\sum_{k=1}^{n}[rk]^3$.

Proposition 3 Let $p, q \in \mathbb{Z}$ with $\gcd(p,q)=1$ and $q \neq 0$, and

$$U_n=\sum_{k=1}^{n}\left[\frac{p}{q}k\right],\ V_n=\sum_{k=1}^{n}\left[\frac{p}{q}k\right]^2,\ W_n=\sum_{k=1}^{n}\left[\frac{p}{q}k\right]^3.$$

Then

(a) $U_n = U_{n-1}+U_{n-q}-U_{n-q-1}+p,$
(b) $V_n = V_{n-1}+2V_{n-q}-2V_{n-q-1}- V_{n-2q}+V_{n-2q-1}+2p^2$
(c) $W_n = W_{n-1}+3W_{n-q}-3W_{n-q-1}- 3W_{n-2q}+3W_{n-2q-1}+ W_{n-3q}-W_{n-3q-1}+6p^2.$

Proof We only prove (b). The (a) and (c) can be proved in the same way. The proof follows from the identities below:

$$\begin{aligned}
&V_{n-1}+2V_{n-q}-2V_{n-q-1}-V_{n-2q}+V_{n-2q-1}+2p^2\\
&=\sum_{k=1}^{n-1}\left[\frac{p}{q}k\right]^2+2\left(\sum_{k=1}^{n-q}\left[\frac{p}{q}k\right]^2-\sum_{k=1}^{n-q-1}\left[\frac{p}{q}k\right]^2\right)-\\
&\quad\left(\sum_{k=1}^{n-2q}\left[\frac{p}{q}k\right]^2-\sum_{k=1}^{n-2q-1}\left[\frac{p}{q}k\right]^2\right)+2p^2\\
&=\sum_{k=1}^{n-1}\left[\frac{p}{q}k\right]^2+2\left[\frac{p}{q}(n-q)\right]^2-\left[\frac{p}{q}(n-2q)\right]^2+2p^2\\
&=\sum_{k=1}^{n-1}\left[\frac{p}{q}k\right]^2+2\left[\frac{np}{q}-p\right]^2-\left[\frac{np}{q}-2p\right]^2+2p^2\\
&=\sum_{k=1}^{n-1}\left[\frac{p}{q}k\right]^2+2\left(\left[\frac{np}{q}\right]-p\right)^2-\left(\left[\frac{np}{q}\right]-2p\right)^2+2p^2\\
&=\sum_{k=1}
\end{aligned}$$

Remark 4 We may compute that

$$U_n - U_{n-1} = \begin{cases} U_r - U_{r-1} + sp & \text{if } n = qs + r,\ 2 \le r \le q-1, \\ U_q - U_{q-1} + (s-1)p & \text{if } n = qs + r,\ 0 \le r \le 1, \end{cases}$$
$$= \begin{cases} \left[\dfrac{p}{q} r\right] + sp & \text{if } n = qs + r,\ 2 \le r \le q-1, \\ (s+1)p & \text{if } n = qs + r,\ 0 \le r \le 1, \end{cases}$$

from (a) of Proposition 3.

The closed forms $\sum_{k=1}^{n} [rk]$ and $\sum_{k=1}^{n} [rk]^3$ might be very hard to obtain. But some specific cases can be computed.

Example 5 We can prove that

$$\sum_{k=1}^{n} \left[\frac{5}{3}k\right] = \frac{1}{18}\left(15n^2 + 9n - 4 + 4\cos\frac{2n\pi}{3}\right)$$
$$= \begin{cases} \dfrac{3}{2} m(5m-1), & n = 3m, \\ \dfrac{1}{2}(3m+2)(5m+1), & n = 3m+1, \\ \dfrac{1}{2}(m+1)(15m+8), & n = 3m+2, \end{cases}$$

and

$$\sum_{k=1}^{n} \left[\frac{5}{3}k\right]^3 = \begin{cases} \dfrac{3}{4} m(5m+1)(25m^2+5m+4), & n = 3m, \\ \dfrac{1}{4}(5m+1)(75m^3+115m^2+52m+4), & n = 3m+1, \\ \dfrac{1}{4}(m+1)(375m^3+775m^2+500m+112), & n = 3m+2. \end{cases}$$

Mathematical induction can be used to prove the above equalities. Such proofs are quite straightforward, so we skip them. From the above,

$$\frac{\sum_{k=1}^{36n} \left[\frac{5}{3}k\right]^3}{\left(\sum_{k=1}^{36n} \left[\frac{5}{3}k\right]\right)^2} = \frac{36n(60n+1)(900n^2+15n+1)}{[18n(60n+1)]^2}$$
$$= \frac{900n^2+15n+1}{9n(60n+1)} = \frac{5}{3} + \frac{1}{9n(60n+1)}$$

and

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{36n} \left[\frac{5}{3}k\right]^3}{\left(\sum_{k=1}^{36n} \left[\frac{5}{3}k\right]\right)^2} = \frac{5}{3}$$

which explains (2).

We have studied certain $\sum_{k=1}^{n} [rk]$ and $\sum_{k=1}^{n} [rk]^3$. Finally, we leave a general question.

Questions 6 Using mathematical induction, we can prove that

$$\sum_{k=1}^{n} \left[\frac{1}{\alpha}k\right] = \frac{1}{2}\left[\frac{n}{\alpha}\right]\left(2n - \alpha + 2 - \alpha\left[\frac{n}{\alpha}\right]\right) \quad (\alpha \in \mathbb{Z}^+)$$

If $2|\alpha$, then substitution $\alpha/2$ for α so that

$$\sum_{k=1}^{n} \left[\frac{2}{\alpha}k\right] = \frac{1}{2}\left[\frac{2n}{\alpha}\right]\left(2n - \frac{\alpha}{2} + 2 - \frac{\alpha}{2}\left[\frac{2n}{\alpha}\right]\right).$$

A next natural question would be whether the closed forms of

both $\sum_{k=1}^{n} \left[\frac{t}{\alpha}k\right]$ and $\sum_{k=1}^{n} \left[\frac{t}{\alpha}k\right]^3$, where $2 \le t \le \alpha - 1$,

exist or not.

References

[1] F. Luca, G. Polanco, and W. Zudilin, "A variation on the theme of Nicomachus", Bull. Aust. Math. Soc., Vol. 97, pp. 367-373, 2018.

[2] S. O. Warnaar, "On the q -analogue of the sum of cubes", Electron. J. Combin. 11(1), Note 13, 2 pages, 2004.