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ON NEW INEQUALITIES OF SIMPSON'S TYPE FOR GENERALIZED CONVEX FUNCTIONS

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Abstract. In this paper, using local fractional integrals on fractal sets R^{α} $(0 < \alpha \leq 1)$ of real line numbers, we establish new some inequalities of Simpson's type based on generalized convexity.

1. Introduction

The following inequality is one of the best-known results in the literature as Simpson's inequality.

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a, b) and $\left\| f^{(4)} \right\|_{\infty} = \sup\limits_{x\in(a,b)} \left| f^{(4)}(x) \right| < \infty$. Then, the following inequality holds:*

$$\left| \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{1}{2880}\left\| f^{(4)} \right\|_{\infty}(b-a)^4 .$$

For recent refinements, counterparts, generalizations and new Simpson's type inequalities, see ([1], [3], [4], [6], [7], [8]).

In [3], Dragomir et. al. proved the following some recent developments on Simpson's inequality for which the remainder is expressed in terms of lower derivatives than the fourth.

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Theorem 2. *Suppose $f:[a,b]\to\mathbb{R}$ is a differentiable mapping whose derivative is continuous on (a,b) and $f'\in L[a,b]$. Then the following inequality*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{3}\left\|f'\right\|_1 \tag{1.1}$$

holds, where $\left\|f'\right\|_1=\int_a^b\left|f'(x)\right|dx$.

The bound of (1.1) for L-Lipschitzian mapping was given in [3] by $\frac{5}{36}L(b-a)$.

Also, the following inequality was obtained in [3].

Theorem 3. *Suppose $f:[a,b]\to\mathbb{R}$ is an absolutely continuous mapping on $[a,b]$ whose derivative belongs to $L_p[a,b]$. Then the following inequality holds,*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{1}{6}\left[\frac{2^{q+1}+1}{3(q+1)}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\left\|f'\right\|_p \tag{1.2}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Alomari et. al proved some inequalities of Simpson type for s-convex functions by the following Lemma:

Lemma 1. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be an absolutely continuous mapping on interior I° of an interval I and $a,b\in I$ with $a<b$, then the following equality holds:*

$$\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx=(b-a)\int_0^1 m(t)f'(bt+(1-t)a)dt$$

where

$$m(t)=\begin{cases} t-\frac{1}{6} & t\in[0,\frac{1}{2}) \\ t-\frac{5}{6} & t\in[\frac{1}{2},1]. \end{cases}$$

2. Preliminaries

Recall the set R^α of real line numbers and use the Gao-Yang-Kang's idea to describe the definition of the local fractional derivative and local fractional integral, see [9,10] and so on.

Recently, the theory of Yang's fractional sets [10] was introduced as follows.

For $0 < \alpha \leq 1$, we have the following α-type set of element sets:

Z^α : The α-type set of integer is defined as the set $\{0^\alpha, \pm 1^\alpha, \pm 2^\alpha, ..., \pm n^\alpha, ...\}$.

Q^α : The α-type set of the rational numbers is defined as the set $\{m^\alpha = \left(\frac{p}{q}\right)^\alpha : p, q \in Z, \, q \neq 0\}$.

J^α : The α-type set of the irrational numbers is defined as the set $\{m^\alpha \neq \left(\frac{p}{q}\right)^\alpha : p, q \in Z, \, q \neq 0\}$.

R^α : The α-type set of the real line numbers is defined as the set $R^\alpha = Q^\alpha \cup J^\alpha$.

If a^α, b^α and c^α belongs the set R^α of real line numbers, then

(1) $a^\alpha + b^\alpha$ and $a^\alpha b^\alpha$ belongs the set R^α;
(2) $a^\alpha + b^\alpha = b^\alpha + a^\alpha = (a+b)^\alpha = (b+a)^\alpha$;
(3) $a^\alpha + (b^\alpha + c^\alpha) = (a+b)^\alpha + c^\alpha$;
(4) $a^\alpha b^\alpha = b^\alpha a^\alpha = (ab)^\alpha = (ba)^\alpha$;
(5) $a^\alpha (b^\alpha c^\alpha) = (a^\alpha b^\alpha) c^\alpha$;
(6) $a^\alpha (b^\alpha + c^\alpha) = a^\alpha b^\alpha + a^\alpha c^\alpha$;
(7) $a^\alpha + 0^\alpha = 0^\alpha + a^\alpha = a^\alpha$ and $a^\alpha 1^\alpha = 1^\alpha a^\alpha = a^\alpha$.

The definition of the local fractional derivative and local fractional integral can be given as follows.

Definition 1. [10] A non-differentiable function $f : R \to R^\alpha$, $x \to f(x)$ is called to be local fractional continuous at x_0, if for any $\varepsilon > 0$, there exists $\delta > 0$, such that

$$|f(x) - f(x_0)| < \varepsilon^\alpha$$

holds for $|x - x_0| < \delta$, where $\varepsilon, \delta \in R$. If $f(x)$ is local continuous on the interval (a, b), we denote $f(x) \in C_\alpha(a, b)$.

Definition 2. [10] The local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined by

$$f^{(\alpha)}(x_0) = \left. \frac{d^\alpha f(x)}{dx^\alpha} \right|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha \left(f(x) - f(x_0)\right)}{(x - x_0)^\alpha},$$

where $\Delta^\alpha \left(f(x) - f(x_0)\right) \cong \Gamma(\alpha+1) \left(f(x) - f(x_0)\right)$.

If there exists $f^{(k+1)\alpha}(x) = \overbrace{D_x^\alpha ... D_x^\alpha}^{k+1 \text{ times}} f(x)$ for any $x \in I \subseteq R$, then we denoted $f \in D_{(k+1)\alpha}(I)$, where $k = 0, 1, 2, ...$

Definition 3. [10] Let $f(x)\in C_\alpha[a,b]$. Then the local fractional integral is defined by,

$$ {}_aI_b^\alpha f(x)=\frac{1}{\Gamma(\alpha+1)}\int_a^b f(t)(dt)^\alpha=\frac{1}{\Gamma(\alpha+1)}\lim_{\Delta t\to 0}\sum_{j=0}^{N-1}f(t_j)(\Delta t_j)^\alpha, $$

with $\Delta t_j=t_{j+1}-t_j$ and $\Delta t=\max\{\Delta t_1,\Delta t_2,...,\Delta t_{N-1}\}$, where $[t_j,t_{j+1}]$, $j=0,...,N-1$ and $a=t_0<t_1<...<t_{N-1}<t_N=b$ is partition of interval $[a,b]$.

Here, it follows that ${}_aI_b^\alpha f(x)=0$ if $a=b$ and ${}_aI_b^\alpha f(x)=-{}_bI_a^\alpha f(x)$ if $a<b$. If for any $x\in[a,b]$, there exists ${}_aI_x^\alpha f(x)$, then we denoted by $f(x)\in I_x^\alpha[a,b]$.

Definition 4 (Generalized convex function). [10] Let $f:I\subseteq R\to R^\alpha$. For any $x_1,x_2\in I$ and $\lambda\in[0,1]$, if the following inequality

$$ f(\lambda x_1+(1-\lambda)x_2)\leq\lambda^\alpha f(x_1)+(1-\lambda)^\alpha f(x_2) $$

holds, then f is called a generalized convex function on I.

Here are two basic examples of generalized convex functions:

(1) $f(x)=x^{\alpha p}$, $x\geq 0$, $p>1$;

(2) $f(x)=E_\alpha(x^\alpha)$, $x\in R$ where $E_\alpha(x^\alpha)=\sum\limits_{k=0}^{\infty}\frac{x^{\alpha k}}{\Gamma(1+k\alpha)}$ is the Mittag-Lrffer function.

Theorem 4. [5] *Let $f\in D_\alpha(I)$, then the following conditions are equivalent*

a) f is a generalized convex function on I

b) $f^{(\alpha)}$ is an increasing function on I

c) for any $x_1,x_2\in I$,

$$ f(x_2)-f(x_1)\geq\frac{f^{(\alpha)}(x_1)}{\Gamma(1+\alpha)}(x_2-x_1)^\alpha. $$

Corollary 1. [5] *Let $f\in D_{2\alpha}(a,b)$. Then f is a generalized convex function (or a generalized concave function) if and only if*

$$ f^{(2\alpha)}(x)\geq 0\ \left(or\ f^{(2\alpha)}(x)\leq 0\right) $$

for all $x\in(a,b)$.

Lemma 2. [10]

(1) (Local fractional integration is anti-differentiation) Suppose that $f(x) = g^{(\alpha)}(x) \in C_\alpha[a,b]$, then we have

$$ {}_aI_b^\alpha f(x) = g(b) - g(a). $$

(2) (Local fractional integration by parts) Suppose that $f(x), g(x) \in D_\alpha[a,b]$ and $f^{(\alpha)}(x)$, $g^{(\alpha)}(x) \in C_\alpha[a,b]$, then we have

$$ {}_aI_b^\alpha f(x)g^{(\alpha)}(x) = f(x)g(x)|_a^b - {}_aI_b^\alpha f^{(\alpha)}(x)g(x). $$

Lemma 3. [10] *We have*

i) $\dfrac{d^\alpha x^{k\alpha}}{dx^\alpha} = \dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)} x^{(k-1)\alpha}$;

ii) $\dfrac{1}{\Gamma(\alpha+1)} \int\limits_a^b x^{k\alpha}(dx)^\alpha = \dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)}\left(b^{(k+1)\alpha} - a^{(k+1)\alpha}\right)$, $k \in R$.

Lemma 4 (Generalized Hölder's inequality). [10] *Let $f, g \in C_\alpha[a,b]$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$ \frac{1}{\Gamma(\alpha+1)} \int\limits_a^b |f(x)g(x)|(dx)^\alpha $$

$$ \leq \left(\frac{1}{\Gamma(\alpha+1)} \int\limits_a^b |f(x)|^p (dx)^\alpha \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\alpha+1)} \int\limits_a^b |g(x)|^q (dx)^\alpha \right)^{\frac{1}{q}}. $$

In [5], Mo et al. proved the following generalized Hermite-Hadamard inequality for generalized convex function:

Theorem 5. *Let $f(x) \in I_x^{(\alpha)}[a,b]$ be a generalized convex function on $[a,b]$ with $a < b$. Then*

$$ f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} \, {}_aI_b^\alpha f(t) \leq \frac{f(a)+f(b)}{2^\alpha}. \tag{2.1} $$

The interested reader is refer to [2], [5], [9]- [13] for local freactional theory.

The main aim of this paper is to establish new Simpson's type inequalities for the class of functions whose derivatives in absolute value at certain powers are generalized convex functions.

3. Main Results

The next theorems gives a new result of the Simpson inequality for generelized convex functions:

THEOREM 6. *Let $I \subseteq \mathbb{R}$ be an interval, $f : I^0 \subseteq \mathbb{R} \to \mathbb{R}^\alpha$ (I^0 is the interior of I) such that $f \in D_\alpha(I^0)$ and $f^{(\alpha)} \in C_\alpha[a,b]$ for $a, b \in I^0$ with $a < b$. Then, for all $x \in [a,b]$, we have the identity*

$$\frac{1}{6^\alpha}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\,{}_aI_b^\alpha f(t) \tag{3.1}$$

$$= \frac{(b-a)^\alpha}{\Gamma(1+\alpha)}\int_0^1 p(t)f^{(\alpha)}(bt+(1-t)a)\,(dt)^\alpha$$

where

$$p(t)=\begin{cases}\left(t-\frac{1}{6}\right)^\alpha, & t\in\left[0,\frac{1}{2}\right]\\[2mm] \left(t-\frac{5}{6}\right)^\alpha, & t\in\left(\frac{1}{2},1\right].\end{cases}$$

Proof. Using the local fractional integration by parts, we have

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1 p(t)f^{(\alpha)}(bt+(1-t)a)\,(dt)^\alpha$$

$$= \frac{1}{\Gamma(1+\alpha)}\int_0^{\frac{1}{2}}\left(t-\frac{1}{6}\right)^\alpha f^{(\alpha)}(bt+(1-t)a)\,(dt)^\alpha$$

$$+\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^1\left(t-\frac{5}{6}\right)^\alpha f^{(\alpha)}(bt+(1-t)a)\,(dt)^\alpha$$

$$= \left(t-\frac{1}{6}\right)^{\alpha}\frac{f(bt+(1-t)a)}{(b-a)^{\alpha}}\bigg|_{0}^{\frac{1}{2}} -\frac{1}{(b-a)^{\alpha}\Gamma(1+\alpha)}\int_{0}^{\frac{1}{2}}\Gamma(1+\alpha)f(bt+(1-t)a)(dt)^{\alpha}$$

$$+\left(t-\frac{5}{6}\right)^{\alpha}\frac{f(bt+(1-t)a)}{(b-a)^{\alpha}}\bigg|_{\frac{1}{2}}^{1} -\frac{1}{(b-a)^{\alpha}\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{1}\Gamma(1+\alpha)f(bt+(1-t)a)(dt)^{\alpha}$$

$$= \left(\frac{2}{6}\right)^{\alpha}\frac{f\left(\frac{a+b}{2}\right)}{(b-a)^{\alpha}}-\left(-\frac{1}{6}\right)^{\alpha}\frac{f(a)}{(b-a)^{\alpha}}+\left(\frac{1}{6}\right)^{\alpha}\frac{f(b)}{(b-a)^{\alpha}}$$

$$-\left(-\frac{2}{6}\right)^{\alpha}\frac{f\left(\frac{a+b}{2}\right)}{(b-a)^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{2\alpha}\Gamma(1+\alpha)}\int_{a}^{b}f(t)(dt)^{\alpha}.$$

If we devide the resulting equality with $(b-a)^{\alpha}$, then we complete the proof. □

THEOREM 7. *The assumptions of Theorem 6 are satisfied. If* $\left|f^{(\alpha)}\right|$ *is a generalized convex on* $[a,b]$, *then we have the inequality*

$$\left|\frac{1}{6^{\alpha}}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_{a}I_{b}^{\alpha}f(t)\right|$$

$$\leq \frac{(b-a)^{\alpha}}{12^{\alpha}}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]\left[\left|f^{(\alpha)}(a)\right|+\left|f^{(\alpha)}(b)\right|\right].$$

Proof. Taking madulus in Theorem 6, we find that

$$\left|\frac{1}{6^{\alpha}}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_{a}I_{b}^{\alpha}f(t)\right|$$

$$\leq \frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)}\int_{0}^{1}|p(t)|\left|f^{(\alpha)}(bt+(1-t)a)\right|(dt)^{\alpha}$$

$$= (b-a)^{\alpha}\left[\frac{1}{\Gamma(1+\alpha)}\int_{0}^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^{\alpha}\left|f^{(\alpha)}(bt+(1-t)a)\right|(dt)^{\alpha}\right.$$

$$\left.+\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{1}\left|t-\frac{5}{6}\right|^{\alpha}\left|f^{(\alpha)}(bt+(1-t)a)\right|(dt)^{\alpha}\right].$$

Since $\left|f^{(\alpha)}\right|$ is generalized convexity on $[a,b]$, we have

$$\left|f^{(\alpha)}(bt+(1-t)a)\right| \le t^{\alpha}\left|f^{(\alpha)}(b)\right| + (1-t)^{\alpha}\left|f^{(\alpha)}(a)\right| \tag{3.2}$$

From (3.2), it follows that

$$\left|\frac{1}{6^{\alpha}}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_{a}I_{b}^{\alpha}f(t)\right| \tag{3.3}$$

$$\le (b-a)^{\alpha}\left\{\frac{\left|f^{(\alpha)}(b)\right|}{\Gamma(1+\alpha)}\left[\int_{0}^{\frac{1}{6}}\left(\frac{1}{6}-t\right)^{\alpha}t^{\alpha}(dt)^{\alpha}+\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t-\frac{1}{6}\right)^{\alpha}t^{\alpha}(dt)^{\alpha}\right.\right.$$

$$\left.+\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-t\right)^{\alpha}t^{\alpha}(dt)^{\alpha}+\int_{\frac{5}{6}}^{1}\left(t-\frac{5}{6}\right)^{\alpha}t^{\alpha}(dt)^{\alpha}\right]$$

$$+\frac{\left|f^{(\alpha)}(a)\right|}{\Gamma(1+\alpha)}\left[\int_{0}^{\frac{1}{6}}\left(\frac{1}{6}-t\right)^{\alpha}(1-t)^{\alpha}(dt)^{\alpha}+\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t-\frac{1}{6}\right)^{\alpha}(1-t)^{\alpha}(dt)^{\alpha}\right.$$

$$\left.\left.+\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-t\right)^{\alpha}(1-t)^{\alpha}(dt)^{\alpha}+\int_{\frac{5}{6}}^{1}\left(t-\frac{5}{6}\right)^{\alpha}(1-t)^{\alpha}(dt)^{\alpha}\right]\right\}.$$

Using Lemma 3, we obtain

$$\frac{1}{\Gamma(1+\alpha)}\int_{0}^{\frac{1}{6}}\left(\frac{1}{6}-t\right)^{\alpha}t^{\alpha}(dt)^{\alpha}=\left(\frac{-1}{216}\right)^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}+\left(\frac{1}{216}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}, \tag{3.4}$$

(3.5)

$$\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t-\frac{1}{6}\right)^{\alpha}t^{\alpha}(dt)^{\alpha}=\left(\frac{26}{216}\right)^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}-\left(\frac{8}{216}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)},$$

(3.6)

$$\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-t\right)^{\alpha}t^{\alpha}(dt)^{\alpha}=\left(\frac{-98}{216}\right)^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}+\left(\frac{80}{216}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)},$$

(3.7)

$$\frac{1}{\Gamma(1+\alpha)}\int_{\frac{5}{6}}^{1}\left(t-\frac{5}{6}\right)^{\alpha}t^{\alpha}(dt)^{\alpha}=\left(\frac{91}{216}\right)^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}-\left(\frac{55}{216}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}.$$

Now, using he change of the variable $u=1-t$, we write

$$\begin{aligned}(3.8)\qquad &\frac{1}{\Gamma(1+\alpha)}\int_{0}^{\frac{1}{6}}\left(\frac{1}{6}-t\right)^{\alpha}(1-t)^{\alpha}(dt)^{\alpha}\\ &=\frac{1}{\Gamma(1+\alpha)}\int_{\frac{5}{6}}^{1}\left(u-\frac{5}{6}\right)^{\alpha}u^{\alpha}(du)^{\alpha}\\ &=\left(\frac{91}{216}\right)^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}-\left(\frac{55}{216}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\end{aligned}$$

and similarly,

(3.9)

$$\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t-\frac{1}{6}\right)^{\alpha}(1-t)^{\alpha}(dt)^{\alpha}=\left(\frac{-98}{216}\right)^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}+\left(\frac{80}{216}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)},$$

(3.10)

$$\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-t\right)^{\alpha}(1-t)^{\alpha}(dt)^{\alpha}=\left(\frac{26}{216}\right)^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}-\left(\frac{8}{216}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)},$$

(3.11)
$$\frac{1}{\Gamma(1+\alpha)}\int_{\frac{5}{6}}^{1}\left(t-\frac{5}{6}\right)^{\alpha}(1-t)^{\alpha}(dt)^{\alpha}=\left(\frac{-1}{216}\right)^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}+\left(\frac{1}{216}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}.$$

Substituting the equalities (3.4)-(3.11) in (3.3), we obtain required inequality. □

THEOREM 8. *The assumptions of Theorem 6 are satisfied. If* $\left|f^{(\alpha)}\right|^{q}$, $q>1$ *is a generalized convex on* $[a,b]$, *then we have the inequality*

$$\left|\frac{1}{6^{\alpha}}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_{a}I_{b}^{\alpha}f(t)\right| \tag{3.12}$$
$$\leq \frac{(b-a)^{\alpha}}{(\Gamma(1+\alpha))^{\frac{1}{q}}}\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}}\left[\frac{1+2^{p+1}}{6^{p+1}}\right]^{\frac{\alpha}{p}}$$
$$\times\left(\left[\frac{\left|f^{(\alpha)}(a)\right|^{q}+\left|f^{(\alpha)}\left(\frac{a+b}{2}\right)\right|^{q}}{2^{2\alpha}}\right]^{\frac{1}{q}}+\left[\frac{\left|f^{(\alpha)}\left(\frac{a+b}{2}\right)\right|^{q}+\left|f^{(\alpha)}(b)\right|^{q}}{2^{2\alpha}}\right]^{\frac{1}{q}}\right)$$

where $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Taking madulus in Theorem 6 and using generalized Hölder's inequality, we have

$$\left|\frac{1}{6^{\alpha}}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_{a}I_{b}^{\alpha}f(t)\right| \tag{3.13}$$
$$\leq \frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)}\int_{0}^{1}|p(t)|\left|f^{(\alpha)}(tb+(1-t)a)\right|(dt)^{\alpha}$$

(3.14)

$$
\begin{aligned}
&= (b-a)^{\alpha}\left[\frac{1}{\Gamma(1+\alpha)}\int_{0}^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^{\alpha}\left|f^{(\alpha)}(tb+(1-t)a)\right|(dt)^{\alpha}\right.\\
&\left.+\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{1}\left|t-\frac{5}{6}\right|^{\alpha}\left|f^{(\alpha)}(tb+(1-t)a)\right|(dt)^{\alpha}\right]\\
&\leq (b-a)^{\alpha}\left[\left(\frac{1}{\Gamma(1+\alpha)}\int_{0}^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^{\alpha p}(dt)^{\alpha}\right)^{\frac{1}{p}}\right.\\
&\times\left(\frac{1}{\Gamma(1+\alpha)}\int_{0}^{\frac{1}{2}}\left|f^{(\alpha)}(tb+(1-t)a)\right|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}}\\
&+\left(\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{1}\left|t-\frac{5}{6}\right|^{\alpha p}(dt)^{\alpha}\right)^{\frac{1}{p}}\\
&\left.\times\left(\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{1}\left|f^{(\alpha)}(tb+(1-t)a)\right|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Using Lemma 3, we have

$$
\begin{aligned}
(3.15)\quad &\frac{1}{\Gamma(1+\alpha)}\int_{0}^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^{\alpha p}(dt)^{\alpha}\\
&= \frac{1}{\Gamma(1+\alpha)}\int_{0}^{\frac{1}{6}}\left(\frac{1}{6}-t\right)^{\alpha p}(dt)^{\alpha}+\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t-\frac{1}{6}\right)^{\alpha p}(dt)^{\alpha}\\
&= \frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\left[\frac{1+2^{p+1}}{6^{p+1}}\right]^{\alpha}
\end{aligned}
$$

and

$$(3.16)\quad \frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{1}\left|t-\frac{5}{6}\right|^{\alpha p}(dt)^{\alpha}$$

$$= \frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-t\right)^{\alpha p}(dt)^{\alpha}+\frac{1}{\Gamma(1+\alpha)}\int_{\frac{5}{6}}^{1}\left(t-\frac{5}{6}\right)^{\alpha p}(dt)^{\alpha}$$

$$= \frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\left[\frac{1+2^{p+1}}{6^{p+1}}\right]^{\alpha}.$$

Since $\left|f^{(\alpha)}\right|^{q}$ is a generalized convex on $[a,b]$, by generalized Hermite-Hadamard inequality (Theorem 5), we have

$$(3.17)\quad \int_{0}^{\frac{1}{2}}\left|f^{(\alpha)}(tb+(1-t)a)\right|^{q}(dt)^{\alpha} = \frac{1}{(b-a)^{\alpha}}\int_{a}^{\frac{a+b}{2}}\left|f^{(\alpha)}(u)\right|^{q}(du)^{\alpha} \leq \frac{\left|f^{(\alpha)}(a)\right|^{q}+\left|f^{(\alpha)}\left(\frac{a+b}{2}\right)\right|^{q}}{2^{2\alpha}},$$

and

$$(3.18)\quad \int_{\frac{1}{2}}^{1}\left|f^{(\alpha)}(tb+(1-t)a)\right|^{q}(dt)^{\alpha} = \frac{1}{(b-a)^{\alpha}}\int_{\frac{a+b}{2}}^{b}\left|f^{(\alpha)}(u)\right|^{q}(du)^{\alpha} \leq \frac{\left|f^{(\alpha)}\left(\frac{a+b}{2}\right)\right|^{q}+\left|f^{(\alpha)}(b)\right|^{q}}{2^{2\alpha}}.$$

Putting the equalities (3.15)-(3.16) and inequalities (3.17)-(3.18) in (3.13), we obtain

$$\left|\frac{1}{6^{\alpha}}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_{a}I_{b}^{\alpha}f(t)\right|$$

$$
\begin{aligned}
&\leq (b-a)^{\alpha}\left[\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}}\left[\left(\frac{1}{6}\right)^{(p+1)\alpha}+\left(\frac{1}{3}\right)^{(p+1)\alpha}\right]^{\frac{1}{p}}\right. \\
&\times\frac{1}{(\Gamma(1+\alpha))^{\frac{1}{q}}}\left[\frac{\left|f^{(\alpha)}(a)\right|^{q}+\left|f^{(\alpha)}\left(\frac{a+b}{2}\right)\right|^{q}}{2^{2\alpha}}\right]^{\frac{1}{q}} \\
&+\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}}\left[\left(\frac{1}{6}\right)^{(p+1)\alpha}+\left(\frac{1}{3}\right)^{(p+1)\alpha}\right]^{\frac{1}{p}} \\
&\left.\times\frac{1}{(\Gamma(1+\alpha))^{\frac{1}{q}}}\left[\frac{\left|f^{(\alpha)}\left(\frac{a+b}{2}\right)\right|^{q}+\left|f^{(\alpha)}(b)\right|^{q}}{2^{2\alpha}}\right]^{\frac{1}{q}}\right]
\end{aligned}
$$

which completes the proof. □

THEOREM 9. *The assumptions of Theorem 6 are satisfied. If* $\left|f^{(\alpha)}\right|^{q}$, $q>1$ *is a generalized convex on* $[a,b]$, *then we have the inequality*

$$
\begin{aligned}
&\left|\frac{1}{6^{\alpha}}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_{a}I_{b}^{\alpha}f(t)\right| \\
&\leq (b-a)^{\alpha}\left(\frac{5}{36}\right)^{\frac{\alpha}{p}}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{\frac{1}{p}} \\
&\times\left\{\left(\left[\left(\frac{25}{216}\right)^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}-\left(\frac{7}{216}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right]\left|f^{(\alpha)}(b)\right|^{q}\right.\right. \\
&\left.+\left[\left(\frac{-7}{216}\right)^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}+\left(\frac{25}{216}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right]\left|f^{(\alpha)}(a)\right|^{q}\right)^{\frac{1}{q}} \\
&+\left(\left[\left(\frac{-7}{216}\right)^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}+\left(\frac{25}{216}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right]\left|f^{(\alpha)}(b)\right|^{q}\right. \\
&\left.\left.+\left[\left(\frac{25}{216}\right)^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}-\left(\frac{7}{216}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right]\left|f^{(\alpha)}(a)\right|^{q}\right)^{\frac{1}{q}}\right\}
\end{aligned}
$$

where $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Taking madulus in Theorem 6, we have

$$\left|\frac{1}{6^{\alpha}}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(t)\right|$$

$$\leq \frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)}\int_0^1 |p(t)|\left|f^{(\alpha)}(tb+(1-t)a)\right|(dt)^{\alpha}$$

$$= (b-a)^{\alpha}\left[\frac{1}{\Gamma(1+\alpha)}\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^{\alpha}\left|f^{(\alpha)}(tb+(1-t)a)\right|(dt)^{\alpha}\right.$$

$$\left.+\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|^{\alpha}\left|f^{(\alpha)}(tb+(1-t)a)\right|(dt)^{\alpha}\right].$$

Because of $\frac{1}{p}+\frac{1}{q}=1$, $\alpha\left(\frac{1}{p}+\frac{1}{q}\right)$ can be written instead of α. Using the generalized Holder's inequality, we find that

$$\left|\frac{1}{6^{\alpha}}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(t)\right| \tag{3.19}$$

$$\leq (b-a)^{\alpha}\left[\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^{\alpha}(dt)^{\alpha}\right)^{\frac{1}{p}}\right.$$

$$\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^{\alpha}\left|f^{(\alpha)}(tb+(1-t)a)\right|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|^{\alpha}(dt)^{\alpha}\right)^{\frac{1}{p}}$$

$$\left.\times\left(\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|^{\alpha}\left|f^{(\alpha)}(tb+(1-t)a)\right|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}}\right].$$

Since $\left|f^{(\alpha)}\right|^q$ is generalized convexity on $[a,b]$, we have

$$\frac{1}{\Gamma(1+\alpha)}\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^{\alpha}\left|f^{(\alpha)}(tb+(1-t)a)\right|^q(dt)^{\alpha} \tag{3.20}$$

$$\leq \frac{1}{\Gamma(1+\alpha)}\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^{\alpha}\left[t^{\alpha}\left|f^{(\alpha)}(b)\right|^q(1-t)^{\alpha}\left|f^{(\alpha)}(a)\right|^q\right](dt)^{\alpha}$$

$$= \left|f^{(\alpha)}(b)\right|^q\left[\left(\frac{25}{216}\right)^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}-\left(\frac{7}{216}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right]$$

$$+\left|f^{(\alpha)}(a)\right|^q\left[\left(\frac{-7}{216}\right)^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}+\left(\frac{25}{216}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right]$$

and

$$\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{1}\left|t-\frac{5}{6}\right|^{\alpha}\left|f^{(\alpha)}(tb+(1-t)a)\right|^q(dt)^{\alpha} \tag{3.21}$$

$$\leq \frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{1}\left|t-\frac{5}{6}\right|^{\alpha}\left[t^{\alpha}\left|f^{(\alpha)}(b)\right|^q(1-t)^{\alpha}\left|f^{(\alpha)}(a)\right|^q\right](dt)^{\alpha}$$

$$= \left|f^{(\alpha)}(b)\right|^q\left[\left(\frac{-7}{216}\right)^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}+\left(\frac{25}{216}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right]$$

$$+\left|f^{(\alpha)}(a)\right|^q\left[\left(\frac{25}{216}\right)^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}-\left(\frac{7}{216}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right].$$

Also, we note that

$$\frac{1}{\Gamma(1+\alpha)}\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^{\alpha}(dt)^{\alpha}=\left(\frac{5}{36}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \tag{3.22}$$

and

$$\frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{1}\left|t-\frac{5}{6}\right|^{\alpha}(dt)^{\alpha}=\left(\frac{5}{36}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}. \tag{3.23}$$

If we substitute (3.20)-(3.23) in (3.19), we obtain required result, which completes the proof. □

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