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$L-$FUZZY SEMI-PRIME IDEALS IN UNIVERSAL ALGEBRAS

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Abstract. In this paper, we introduce the notion of $L-$fuzzy semi-prime ideals and the radical of $L-$fuzzy ideals in universal algebras and make a theoretical study on their basic properties.

1. Introduction

The theory of fuzzy sets introduced by Zadeh [23] has evoked tremendous interest among researchers working in different branches of mathematics. Rosenfield in his pioneering paper [16] introduced the notions of fuzzy subgroups of a group. Since then, many researches have been studying fuzzy subalgebras of several algebraic structures (see [7, 13–15, 20]). As suggested by Gougen [9], the unit interval $[0, 1]$ is not sufficient to take the truth values of general fuzzy statements. U. M. Swamy and D. V. Raju [18, 19] studied the general theory of algebraic fuzzy systems by introducing the notion of a fuzzy $\mathfrak{L}-$ subset of a set X corresponding to a given class $\mathfrak{L}$ of subsets of X having truth values in a complete lattice satisfying the infinite meet distributive law.

In recent times, the theory of ideals has been taking place in a more general context. Gumm and Ursini [11] introduced the concept of ideals

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in universal algebras having a constant 0 as a generalization of those familiar structures: normal subgroups (in groups), ideals (in rings), submodules (in modules), subspaces (in vector spaces) and filters (in implication algebras or Heyting algebras). It has been further studied in a series of papers [3–6, 21].

In [1], we have introduced the concept of $L-$fuzzy ideals in universal algebras and we gave a necessary and sufficient condition for a variety of algebras to be an ideal determined. In [2], we continued our study and gave an internal characterization for $L-$fuzzy prime idelas and maximal $L-$fuzzy ideals of universal algebras analogous to the characterization of Swamy and Swamy [17] in the case of rings. In the present paper, we define $L-$fuzzy semi-prime ideals and the radical of $L-$fuzzy ideals in universal algebras in the frame work of $L-$fuzzy ideals given in [1]. We make a theoretical study on their properties and give several characterizing theorems.

2. Preliminaries

This section contains some definitions and results which will be used in the paper. We refer to the readers [8, 10], for the standard concepts in universal algebras. Throughout this paper $A \in \mathcal{K}$, where $\mathcal{K}$ is a class of algebras of a fixed type Ω and assume that there is an equationaly definable constant in all algebras of $\mathcal{K}$ denoted by 0. For a positive integer n, we write $\overrightarrow{a}$ to denote the $n-$tuple $\langle a_1, a_2, ..., a_n\rangle \in A^n$.

Definition 2.1. [11] A term $P(\overrightarrow{x}, \overrightarrow{y})$ is said to be an ideal term in $\overrightarrow{y}$ if and only if $P(\overrightarrow{x}, \overrightarrow{0}) = 0$.

Definition 2.2. [11] A nonempty subset I of A is called an ideal of A if and only if $P(\overrightarrow{a}, \overrightarrow{b}) \in I$ for all $\overrightarrow{a} \in A^n$, $\overrightarrow{b} \in I^m$ and any ideal term $P(\overrightarrow{x}, \overrightarrow{y})$ in $\overrightarrow{y}$.

We denote the class of all ideals of A, by $\mathcal{I}(A)$.

Definition 2.3. [11, 21] A term $t(\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z})$ is said to be a commutator term in $\overrightarrow{y}, \overrightarrow{z}$ if and only if it is an ideal term in $\overrightarrow{y}$ and an ideal term in $\overrightarrow{z}$.

Definition 2.4. [11] In an ideal determined variety, the commutator $[I, J]$ of ideals I and J is the zero congruence class of the commutator congruence $[I^\delta, J^\delta]$.

It is characterized in [11] as follows:

Theorem 2.5. [11, 21] *In an ideal determined variety,*

$$\begin{aligned}[I,J] \;=\; \{t(\overrightarrow{a},\overrightarrow{i},\overrightarrow{j}) : \overrightarrow{a}\in A^n, \overrightarrow{i}\in I^m \text{ and } \overrightarrow{j}\in J^k \\ \text{where } t(\overrightarrow{x},\overrightarrow{y},\overrightarrow{z}) \text{ is a commutator term in } \overrightarrow{y},\overrightarrow{z}\}\end{aligned}$$

For subsets H, G of A, $[H, G]$ denotes the product $[\langle H\rangle, \langle G\rangle]$. In particular, for $a, b \in A$, $[\langle a\rangle, \langle b\rangle]$ is denoted by $[a, b]$.

Definition 2.6. For each $I \in \mathcal{I}(A)$, we define by induction:

$$I^{(1)} = I = I^1;$$

$$I^{(n+1)} = [I^{(n)}, I^{(n)}] \text{ and } I^{n+1} = [I^{(n)}, I]$$

I will be called nilpotent (resp. solvable) if $I^n = (0)$ (resp. $I^{(n)} = (0)$) for some $n \in Z_+$.

Definition 2.7. [21] A proper ideal P of A is called prime if and only if for all $I, J \in \mathcal{I}(A)$:

$$[I, J] \subseteq P \Rightarrow I \subseteq P \text{ or } J \subseteq P$$

Theorem 2.8. [21] *A proper ideal P of A is prime if and only if:*

$$[a, b] \subseteq P \Rightarrow a \in P \text{ or } b \in P$$

for all $a, b \in A$.

Definition 2.9. [21] An ideal Q of A is called semi-prime if and only if for all $I \in \mathcal{I}(A)$:

$$[I, I] \subseteq Q \Rightarrow I \subseteq Q$$

Definition 2.10. [21] A subset M of A is called an $m-$system (resp. $n-$system) if: for all $a, b \in M$, $[a, b] \cap M \neq \emptyset$, (resp. for all $a \in M$, $[a, a] \cap M \neq \emptyset$).

Definition 2.11. [21] The prime radical of an ideal I of A, denoted by $\sqrt{I}$ is the intersection of all prime ideals of A containing I.

Throughout this paper $L = (L, \wedge, \vee, 0, 1)$ is a complete Brouwerian lattice; i.e., L is a complete lattice satisfying the infinite meet distributive law. By an $L-$fuzzy subset of A, we mean a mapping $\mu : A \to L$. For each $\alpha \in L$, the $\alpha-$level set of μ denoted by μ_α is a subset of A given by:

$$\mu_\alpha = \{x \in A : \alpha \leq \mu(x)\}$$

For fuzzy subsets μ and ν of A, we write $\mu \leq \nu$ to mean $\mu(x) \leq \nu(x)$ for all $x \in A$in the ordering of L.

Definition 2.12. [22] For each $x \in A$ and $0 \neq \alpha$ in L, the fuzzy subset x_α of A given by:

$$x_\alpha(z) = \begin{cases} \alpha & \textit{if } z = x \\ 0 & \textit{otherwise} \end{cases}$$

is called the fuzzy point of A. In this case x is called the support of x_α and α its value.

For a fuzzy subset μ of A and a fuzzy point x_α of A, we write $x_\alpha \in \mu$ whenever $\mu(x) \geq \alpha$.

Definition 2.13. [1] An $L-$fuzzy subset μ of A is said to be an $L-$fuzzy ideal of A (or shortly a fuzzy ideal of A) if and only if the following conditions are satisfied:

1. $\mu(0) = 1$, and
2. If $P(\overrightarrow{x}, \overrightarrow{y})$ is an ideal term in $\overrightarrow{y}$ and $\overrightarrow{a} \in A^n$, $\overrightarrow{b} \in A^m$, then

$$\mu(P(\overrightarrow{a}, \overrightarrow{b})) \geq \mu^m(\overrightarrow{b})$$

We denote by $\mathcal{FI}(A)$, the class of all fuzzy ideals of A.

Definition 2.14. [2] The commutator of fuzzy ideals μ and σ of A denoted by $[\mu, \sigma]$ is a fuzzy subset of A defined by:

$$\begin{aligned} [\mu, \sigma](x) &= \bigvee \{\alpha \wedge \beta : \alpha, \beta \in L, x \in [\mu_\alpha, \sigma_\beta]\} \\ &= \bigvee \{\lambda \in L : x \in [\mu_\lambda, \sigma_\lambda]\} \end{aligned}$$

for all $x \in A$.

Theorem 2.15. [2] *For each $x \in A$, and fuzzy ideals μ and σ of A:*

$$\begin{aligned}[\mu,\sigma](x) &= \bigvee\{\mu^m(\overrightarrow{b}) \wedge \sigma^k(\overrightarrow{c}) : x = t(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}),\\ &\quad \text{where } \overrightarrow{a} \in A^n, \overrightarrow{b} \in A^m, \overrightarrow{c} \in A^k, \text{ and}\\ &\quad t(\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}) \text{ is a commutator term in } \overrightarrow{y}, \overrightarrow{z}\}\end{aligned}$$

Definition 2.16. [2] For each $\mu \in \mathcal{FI}(A)$, we define by induction:

$$\mu^{(1)} = \mu = \mu^1;$$

$$\mu^{(n+1)} = [\mu^{(n)}, \mu^{(n)}] \text{ and } \mu^{n+1} = [\mu^{(n)}, \mu]$$

μ will be called fuzzy nilpotent (resp. fuzzy solvable) if $\mu^n = \chi_{(0)}$ (resp. $\mu^{(n)} = \chi_{(0)}$) for some $n \in Z_+$.

Definition 2.17. [2] A non-constant fuzzy ideal μ of A is called a fuzzy prime ideal if and only if:

$$[\nu, \sigma] \leq \mu \Rightarrow \nu \leq \mu \text{ or } \sigma \leq \mu$$

for all $\nu, \sigma \in \mathcal{FI}(A)$.

Theorem 2.18. [2] *A non-constant fuzzy ideal μ is a fuzzy prime ideal if and only if $Img(\mu) = \{1, \alpha\}$, where α is a prime element in L and the set $\mu_* = \{x \in A : \mu(x) = 1\}$ is a prime ideal of A.*

3. Fuzzy semi-prime ideals

Definition 3.1. A fuzzy ideal μ of A is called fuzzy semi-prime if:

$$[\theta, \theta] \leq \mu \Rightarrow \theta \leq \mu$$

for all $\theta \in \mathcal{FI}(A)$.

It is clear that every fuzzy prime ideal is fuzzy semi-prime.

Theorem 3.2. *A fuzzy ideal μ of A is fuzzy semi-prime if and only if μ_α is semi-prime for all $\alpha \in L$.*

Proof. Suppose that μ is fuzzy semi-prime and let $\alpha \in L$. Let I be an ideal of A such that $[I, I] \subseteq \mu_\alpha$. We show that $I \subseteq \mu_\alpha$. Define a fuzzy subset σ of A as follows:

$$\sigma(x) = \begin{cases} 1 & \text{if } x = 0 \\ \alpha & \text{if } x \in I - \{0\} \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in A$. Then it is easy to check that σ is a fuzzy ideal of A. Moreover, for each $x \in A$ we have:

$$[\sigma,\sigma](x) = \begin{cases} 1 & \textit{if } x = 0 \\ \alpha & \textit{if } x \in [I,I] - \{0\} \\ 0 & \textit{otherwise} \end{cases}$$

It follows from our hypothesis; $[I,I] \subseteq \mu_\alpha$ that $[\sigma,\sigma] \leq \mu$. Since μ is fuzzy semi-prime, $\sigma \leq \mu$. Thus the level ideal σ_α which is precisely I will be included in μ_α and hence μ_α is semi-prime. The converse part is clear. □

THEOREM 3.3. *An ideal I of A is semi-prime if and only if its characteristic function χ_I is fuzzy semi-prime.*

Proof. Suppose that I is semi-prime. Let μ be a fuzzy ideal of A such that $[\mu,\mu] \leq \chi_I$. We show that $\mu \leq \chi_I$. Suppose not. There exists $x \in A - I$ such that $\mu(x) > 0$. Since I is semi-prime, $[x,x] \nsubseteq I$. Choose an element a in $[x,x]$ and $a \notin I$. We can verify that $[\mu,\mu](a) \geq \mu(x) > 0$, which is a contradiction. Therefore χ_I is fuzzy semi-prime. The converse part is straight forward. □

THEOREM 3.4. *A non-constant fuzzy ideal μ of A is fuzzy semi-prime if and only if for any fuzzy point x_α of A:*

$$[x_\alpha, x_\alpha] \leq \mu \Rightarrow x_\alpha \in \mu$$

Proof. Suppose that μ satisfies the condition:

$$[x_\alpha, x_\alpha] \leq \mu \Rightarrow x_\alpha \in \mu$$

for each fuzzy point x_α of A. We show that μ is fuzzy semi-prime. Let σ be a fuzzy ideal of A such that $[\sigma,\sigma] \leq \mu$. Suppose on contrary that $\sigma \nleq \mu$. Then there exists $x \in A$ such that $\sigma(x) \nleq \mu(x)$. If we put $\alpha = \sigma(x)$, then x_α is a fuzzy point of A such that $x_\alpha \in \sigma$ but $x_\alpha \notin \mu$. So $[x_\alpha, x_\alpha] \leq [\sigma,\sigma] \leq \mu$, but $x_\alpha \notin \mu$. This contradicts to our hypothesis. Thus $\sigma \leq \mu$ and therefore μ is fuzzy semi-prime. The converse part is clear. □

THEOREM 3.5. *A fuzzy ideal μ of A is fuzzy semi-prime if and only if:*

(1) $$\mu(a) \geq \bigwedge\{\mu(x) : x \in [a,a]\}$$

for all $a \in A$.

Proof. Suppose that μ is fuzzy semi-prime. We use contradiction. Assume that there exists $a \in A$ such that

$$\mu(a) < \bigwedge\{\mu(x) : x \in [a, a]\}$$

Put $\alpha = \wedge\{\mu(x) : x \in [a, a]\}$ and define a fuzzy subset θ of A by:

$$\theta(x) = \begin{cases} 1 & \textit{if } x = 0 \\ \alpha & \textit{if } x \in \langle a \rangle - \{0\} \\ 0 & \textit{otherwise} \end{cases}$$

for all $x \in A$. Then θ is a fuzzy ideal of A such that for each $x \in A$ we have:

$$[\theta, \theta](x) = \begin{cases} 1 & \textit{if } x = 0 \\ \alpha & \textit{if } x \in [a, a] - \{0\} \\ 0 & \textit{otherwise} \end{cases}$$

So that $[\theta, \theta] \leq \mu$. Since μ is fuzzy semi-prime it yields that $\theta \leq \mu$. This is a contradiction, because $\theta(a) > \mu(a)$. Therefore the inequality (1) holds for all $a \in A$. Conversely suppose that the inequality (1) holds for all $a \in A$. Let θ be any fuzzy ideal of A such that $[\theta, \theta] \leq \mu$. We show that $\theta \leq \mu$. Suppose not. Then there exists $a \in A$ such that $\theta(a) > \mu(a)$. For each $x \in [a, a]$, we can verify that $[\theta, \theta](x) \geq \theta(a)$. Since $[\theta, \theta] \leq \mu$, it yields that $\mu(x) \geq \theta(a)$ for all $x \in [a, a]$. So that

$$\bigwedge\{\mu(x) : x \in [a, a]\} \geq \theta(a) > \mu(a)$$

This is a contradiction. Therefore μ is fuzzy semi-prime. $\square$

4. The radical of fuzzy ideals

According to [21], the prime radical of an ideal I of A, denoted by $\sqrt{I}$ is the intersection of all prime ideals of A containing I. Here we define the prime radical of fuzzy ideals using their level ideals.

Definition 4.1. For a fuzzy ideal μ of A, its prime radical of μ denoted by $\sqrt{\mu}$ is defined as a fuzzy subset of A such that, for each $x \in A$:

$\sqrt{\mu}(x) = \alpha$ if and only if $x \in \sqrt{\mu_\alpha}$ and $x \notin \sqrt{\mu_\beta}$ for all $\beta > \alpha$ in L.

Lemma 4.2. *Let μ be a fuzzy ideal of A and $x \in A$. Then*

$$\sqrt{\mu}(x) = \bigvee\{\alpha \in L : x \in \sqrt{\mu_\alpha}\}$$

Lemma 4.3. *The following holds for all $\mu, \nu \in \mathcal{FI}(A)$:*

1. $\sqrt{(\mu_\alpha)} = (\sqrt{\mu})_\alpha$ *for all* $\alpha \in L$
2. $\mu \leq \sqrt{\mu}$
3. $\mu \leq \nu \Rightarrow \sqrt{\mu} \leq \sqrt{\nu}$

Lemma 4.4. *For any $\mu \in \mathcal{FI}(A)$, $\sqrt{\mu}$ is a fuzzy ideal of A.*

Proof. It is clear that $\sqrt{\mu}(0) = 1$. Let $\overrightarrow{a} \in A^n$, $\overrightarrow{b} \in A^m$ and $P(\overrightarrow{x}, \overrightarrow{y})$ be an ideal term in $\overrightarrow{y}$. Then consider:

$$\begin{aligned}(\sqrt{\mu})^m(\overrightarrow{b}) &= \bigwedge\{\sqrt{\mu}(b_i) : 1 \leq i \leq m\} \\ &= \bigwedge\{\bigvee\{\alpha_i \in L : b_i \in \sqrt{\mu_{\alpha_i}}\} : 1 \leq i \leq m\} \\ &= \bigvee\{\bigwedge\{\alpha_i \in L : 1 \leq i \leq m\} : b_i \in \sqrt{\mu_{\alpha_i}}\}\end{aligned}$$

If we put $\beta = \wedge\{\alpha_i \in L : 1 \leq i \leq m\}$, then we get $\mu_{\alpha_i} \subseteq \mu_\beta$ for all $1 \leq i \leq m$. This implies that $\sqrt{\mu_{\alpha_i}} \subseteq \sqrt{\mu_\beta}$ for all $1 \leq i \leq m$. Then we have the following:

$$\begin{aligned}(\sqrt{\mu})^m(\overrightarrow{b}) &= \bigvee\{\bigwedge\{\alpha_i \in L : 1 \leq i \leq m\} : b_i \in \sqrt{\mu_{\alpha_i}}\} \\ &\leq \bigvee\{\beta \in L : \overrightarrow{b} \in b_i \in \sqrt{\mu_\beta},\ \textit{for all } 1 \leq i \leq m\} \\ &= \bigvee\{\beta \in L : \overrightarrow{b} \in (\sqrt{\mu_\beta})^m\} \\ &\leq \bigvee\{\beta \in L : P(\overrightarrow{a}, \overrightarrow{b}) \in \sqrt{\mu_\beta}\} \\ &= \sqrt{\mu}(P(\overrightarrow{a}, \overrightarrow{b}))\end{aligned}$$

Therefore $\sqrt{\mu}$ is a fuzzy ideal of A. □

Lemma 4.5. *For any $\mu \in \mathcal{FI}(A)$, $\sqrt{\mu}$ is fuzzy semi-prime.*

Proof. The proof follows from (1) of Lemma (4.3) and Theorem (3.2). □

Lemma 4.6. *For any $\theta \in \mathcal{FI}(A)$, if θ is fuzzy semi-prime, then $\sqrt{\theta} = \theta$.*

Proof. Suppose that θ is fuzzy semi-prime. By Theorem (3.2), every level ideal θ_α is semi-prime. By the equivalency in (3.5) of [21], we get $\sqrt{\theta_\alpha} = \theta_\alpha$ for all $\alpha \in L$. This confirms that $\sqrt{\theta} = \theta$. □

Corollary 4.7. *For any $\mu \in \mathcal{FI}(A)$, $\sqrt{\sqrt{\mu}} = \sqrt{\mu}$.*

Lemma 4.8. *For any $\mu \in \mathcal{FI}(A)$, if θ is a fuzzy semi-prime ideal of A such that $\mu \leq \theta$, then $\sqrt{\mu} \leq \theta$.*

Proof. The proof is straight forward. □

Corollary 4.9. *For any $\mu \in \mathcal{FI}(A)$,*

$$\sqrt{\mu} = \cap\{\theta : \theta \text{ is a fuzzy semi-prime ideal of } A, \mu \leq \theta\}$$

Lemma 4.10. *For any $\mu, \nu \in \mathcal{FI}(A)$,*

$$\sqrt{[\mu, \nu]} = \sqrt{\mu \cap \nu} = \sqrt{\mu} \cap \sqrt{\nu}$$

Proof. For any $x \in A$, it is clear to see that:

$$\sqrt{[\mu, \nu]}(x) \leq \sqrt{\mu \cap \nu}(x) \leq \sqrt{\mu}(x) \wedge \sqrt{\nu}(x)$$

It is enough to show that $\sqrt{\mu}(x) \wedge \sqrt{\nu}(x) \leq \sqrt{[\mu, \nu]}(x)$. Let $\alpha \in L$ such that $\sqrt{\mu}(x) \wedge \sqrt{\nu}(x) = \alpha$. Then $x \in (\sqrt{\mu})_\alpha = \sqrt{\mu_\alpha}$ and $x \in (\sqrt{\nu})_\alpha = \sqrt{\nu_\alpha}$. So that $x \in P$ for all prime ideals P containing μ_α (respectively ν_α). Let Q be any prime ideal of A such that $[\mu, \nu]_\alpha \subseteq Q$. Since $[\mu, \nu]_\alpha = [\mu_\alpha, \nu_\alpha]$, we get that either $\mu_\alpha \subseteq Q$ or $\nu_\alpha \subseteq Q$. So that $x \in Q$. Thus $x \in \sqrt{[\mu, \nu]_\alpha}$ and hence $\sqrt{[\mu, \nu]}(x) \geq \alpha = \sqrt{\mu}(x) \wedge \sqrt{\nu}(x)$. □

Theorem 4.11. *If L is a chain and μ is a fuzzy ideal of A satisfying the sup property, then*

$$\sqrt{\mu} = \cap\{\theta : \theta \text{ is a fuzzy prime ideal of } A, \mu \leq \theta\}$$

Proof. Let $x \in A$ and $\alpha \in L$. Suppose that $\sqrt{\mu}(x) = \alpha$. Then $x \in \sqrt{\mu_\alpha}$ and $x \notin \sqrt{\mu_\beta}$ for all $\beta > \alpha$. So that $x \in P$ for all prime ideals P of A with $\mu_\alpha \subseteq P$. Let θ be any fuzzy prime ideal of A such that $\mu \leq \theta$. By Theorem (4.4) of [2], $Img(\theta) = \{1, \beta\}$, where $\beta \in L - \{1\}$ and the set $\theta_* = \{x \in A : \theta(x) = 1\}$ is a prime ideal of A.

Case(1) If $\beta \geq \alpha$, then it is clear that $\theta(x) \geq \alpha$.

Case(2) If $\beta < \alpha$, then we can verify that $\theta_\alpha = \theta_*$ (which is a prime ideal of A) and $\mu_\alpha \subseteq \theta_\alpha = \theta_*$. That is, θ_* is a prime ideal of A containing μ_α. So that $x \in \theta_*$ and hence $\theta(x) \geq \alpha$.

Therefore

$$\bigwedge\{\theta(x) : \theta \text{ is a fuzzy prime ideal of } A, \mu \le \theta\} \ge \alpha$$

To prove the other side of the inequality, Let

$$\alpha = \bigwedge\{\theta(x) : \theta \text{ is a fuzzy prime ideal of } A, \mu \le \theta\}$$

Then $\theta(x) \ge \alpha$ for all fuzzy prime ideals θ of A with $\mu \le \theta$. Let P be any prime ideal of A such that $\mu_\alpha \subseteq P$. We show that $x \in P$. If $x \in \mu_\alpha$, then it is clear. Assume that $x \notin \mu_\alpha$. Then $\mu(x) < \alpha$. Put $\beta = \vee\{\mu(y) : y \notin P\}$. Since μ has the sup-property, $\beta < \alpha$. Let us define a fuzzy subset θ_P of A as follows:

$$\theta_P(z) = \begin{cases} 1 & \text{if } z \in P \\ \beta & \text{otherwise} \end{cases}$$

for all $z \in A$. Then θ_P is a fuzzy prime ideal of A such that $\mu \le \theta_P$. Thus $\theta_P(x) \ge \alpha > \beta$ and hence $\theta_P(x) = 1$. So that $x \in P$, which implies that $x \in \sqrt{\mu_\alpha}$. This confirms that the equality holds. □

Theorem 4.12. *If the commutator* [,] *of ideals in* A *is finitary and* μ *has the sup-property, then*

$$\sqrt{\mu}(a) = \bigvee\{\bigwedge_{x\in(a)^{(n)}} \mu(x) : n \in Z_+\}$$

for all $a \in A$.

Proof. Let $\alpha \in L$ such that $\bigvee\{\bigwedge_{x\in(a)^{(n)}} \mu(x) : n \in Z_+\} = \alpha$. Then there exists $n \in Z_+$ such that $(a)^{(n)} \subseteq \mu_\alpha$. If P is any prime ideal containing μ_α, then $(a)^{(n)} \subseteq P$. So that $a \in P$. Thus $a \in \sqrt{\mu_\alpha}$ and hence $\sqrt{\mu}(a) \ge \alpha$. To prove the other side of the inequality, let $\beta = \sqrt{\mu}(a)$. Then it follows from Corollary (4.9) that, $\theta(a) \ge \beta$ for all fuzzy semi-prime ideals θ of A with $\mu \le \theta$. We need to show that

$$\bigvee\{\bigwedge_{x\in(a)^{(n)}} \mu(x) : n \in Z_+\} \ge \beta$$

Suppose not. Then

$$\bigwedge_{x\in(a)^{(n)}} \mu(x) < \beta \text{ for all } n \in Z_+$$

That is; for each $n \in Z_+$, $(a)^{(n)} \nsubseteq \mu_\beta$. Then the set

$$\mathfrak{F} = \{I \in \mathcal{I}(A) : \mu_\beta \subseteq I, (a)^{(n)} \nsubseteq I \textit{ for all } n \in Z_+\}$$

is a nonempty. Moreover, $\mathfrak{F}$ together with the usual inclusion order forms a poset satisfying the hypothesis of Zorn's Lemma (here we use the condition; [,] is finitary). So that $\mathfrak{F}$ has a maximal element, say M. Our aim is to show that M is semi-prime. Take $b \notin M$. Then $M \vee \langle b \rangle \notin \mathfrak{F}$. By the property of $\mathfrak{F}$, $(a)^{(n)} \subseteq M \vee \langle b \rangle$ for some $n \in Z_+$. Then

$$\begin{aligned} (a)^{(n+1)} &= [(a)^{(n)}, (a)^{(n)}] \\ &\subseteq [M \vee \langle b \rangle, M \vee \langle b \rangle] \\ &= [M, M] \vee [M, \langle b \rangle] \vee [b, b] \\ &\subseteq M \vee [b, b] \end{aligned}$$

So that $M \vee [b, b] \notin \mathfrak{F}$ and hence $[b, b] \nsubseteq M$. Therefore M is a semi-prime ideal of A such that $\mu_\beta \subseteq M$ such that $a \notin M$. Put $\alpha = \vee\{\mu(y) : y \in A - M\}$. Since μ has the sup-property, $\alpha < \beta$. Now define a fuzzy subset θ_M of A as follows:

$$\theta_M(z) = \begin{cases} 1 & \textit{if } z \in M \\ \alpha & \textit{otherwise} \end{cases}$$

for all $z \in A$. Then θ_M is fuzzy semi-prime ideal of A such that $\mu \leq \theta_M$. But $\theta_M(a) = \alpha < \beta$, which is a contradiction. Therefore the equality holds. □

THEOREM 4.13. *Let $\mu \in \mathcal{FI}(A)$. If the commutator [,] of ideals in A is associative and finitary, then for each $x \in A$:*

$$\sqrt{\mu}(x) = \bigvee\{\alpha \in L : \exists n \in Z_+ \textit{ such that } (x_\alpha)^{(n)} \leq \mu\}$$

where x_α is a fuzzy point of A defined as:

$$x_\alpha(z) = \begin{cases} \alpha & \textit{if } z = x \\ 0 & \textit{otherwise} \end{cases}$$

for all $z \in A$.

Proof. For each $\alpha > 0$ and $n \in Z_+$, we first show that $(x)^{(n)} \subseteq \mu_\alpha$ if and only if $(x_\alpha)^{(n)} \leq \mu$. It is clear that $(x)^{(n)} \subseteq \mu_\alpha$ if and only if

$\mu(z) \geq \alpha$ for all $z \in (x)^{(n)}$. On the other hand we can verify that:

$$(x_\alpha)^{(n)}(z) = \begin{cases} 1 & \textit{if } z = 0 \\ \alpha & \textit{if } z \in (x)^{(n)} - \{0\} \\ 0 & \textit{otherwise} \end{cases}$$

for all $z \in A$. Therefore $(x)^{(n)} \subseteq \mu_\alpha$ if and only if $(x_\alpha)^{(n)} \leq \mu$. Now consider the following:

$$\begin{aligned} \sqrt{\mu}(x) &= \bigvee\{\alpha \in L : x \in \sqrt{\mu_\alpha}\} \\ &= \bigvee\{\alpha \in L : \exists n \in Z_+, (x)^{(n)} \subseteq \mu_\alpha\} \\ &= \bigvee\{\alpha \in L : \exists n \in Z_+, (x_\alpha)^{(n)} \leq \mu\} \end{aligned}$$

□

THEOREM 4.14. *If the commutator* [,] *of ideals in* A *is finitary, then*

$$\sqrt{\mu} = \cup\{\eta \in L^A : \exists n \in Z_+ \textit{ such that } \eta^{(n)} \leq \mu\}$$

Proof. For each $x \in A$, let us define two sets H_x and G_x as follows:

$$\begin{aligned} H_x &= \{\alpha \in L : x \in \sqrt{\mu_\alpha}\} \\ G_x &= \{\eta(x) : \eta \in L^A \textit{ such that } \eta^{(n)} \leq \mu \textit{ for some } n \in Z_+\} \end{aligned}$$

Clearly both H_x and G_x are subsets of L. Our aim is to show that $H_x = G_x$ for all $x \in A$. Let $\alpha \in H_x$ (without loss of generality we can assume that $\alpha > 0$). Then $x \in \sqrt{\mu_\alpha}$. Since the commutator [,] of ideals in A is finitary, there exists $n \in Z_+$ such that $(x)^{(n)} \subseteq \mu_\alpha$. Thus $(x_\alpha)^{(n)} \leq \mu$. If we take η to be the fuzzy point x_α, then $\eta \in L^A$, with $\eta(x) = \alpha$ such that $\eta^{(n)} \leq \mu$ for some $n \in Z_+$. Therefor $\alpha \in G_x$. So that $H_x \subseteq G_x$. Also let $\alpha \in G_x$. Then there exists $\eta \in L^A$ such that $\alpha = \eta(x)$ and $\eta^{(n)} \leq \mu$ for some $n \in Z_+$. Consider the fuzzy point x_α. Since $\eta(x) = \alpha$, $x_\alpha \in \eta$. So that $(x_\alpha)^{(n)} \leq \eta^{(n)} \leq \mu$. Then $(x_\alpha)^{(n)} \leq \mu$, which implies that $(x)^{(n)} \subseteq \mu_\alpha$. That is, $x \in \sqrt{\mu_\alpha}$. So that $\alpha \in H_x$. Therefore $H_x = G_x$. □

THEOREM 4.15. *If the commutator* [,] *of ideals in* A *is finitary, then*

$$\sqrt{\mu} = \cup\{\eta \in \mathcal{FI}(A) : \exists n \in Z_+ \textit{ such that } \eta^{(n)} \leq \mu\}$$

Proof. For each $\eta \in L^A$ and $n \in Z_+$, it yields that $\eta^{(n)} = \langle\eta\rangle^{(n)}$. So that the proof of this theorem follows from Theorem (4.14). □

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