A FIXED POINT APPROACH TO THE STABILITY OF A QUADRATIC-CUBIC FUNCTIONAL EQUATION

Yang-Hi Lee

ABSTRACT. In this paper, we investigate the stability of the functional equation

$$f(x + ky) - kf(x + y) + kf(x - y) - f(x - ky) - f(ky)$$
$$+ \frac{k^3 + k^2 - 2k}{2}f(-y) - \frac{k^3 - k^2 - 2k}{2}f(y) = 0$$

by using the fixed point theory in the sense of L. Cǎdariu and V. Radu.

1. Introduction

Throughout this paper, let V and W be real vector spaces, Y a real Banach space, and k a fixed nonzero real number such that $|k| \neq 1$. In 1940, the stability problem for group homomorphisms was first raised by S. M. Ulam [15]. In the next year, D. H. Hyers [10] gave a partial solution to Ulam's question for the case of additive mappings. His result was generalized by T. Aoki [1], Th. M. Rassias [13], and P. Gǎvruta [9]. Gǎvruta' result has greatly influenced the study of the stability problem of the functional equation.

In 2003, L. Cǎdariu and V. Radu [3] proved the stability of *the quadratic functional equation:*

(1) $$f(x + y) + f(x - y) - 2f(x) - 2f(y) = 0$$

by using the fixed point method [4]. We call a solution of (1) *a quadratic mapping.* Notice that a mapping $f : V \to W$ is called a cubic mapping if f is a solution of the cubic functional equation

$$(2) \qquad f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) - 6f(y) = 0.$$

A mapping f is called a quadratic and cubic mapping if f is represented by sum of a quadratic mapping and a cubic mapping. A functional equation is called a quadratic-cubic functional equation provided that each solution of that equation is a quadratic-cubic mapping and every quadratic and cubic mapping is a solution of that equation. Many mathematicians investigated the stability of the quadratic-cubic functional equations [5, 6, 11, 12, 14, 16]. Now we consider the functional equation:

$$f(x + ky) - kf(x + y) + kf(x - y) - f(x - ky) - f(ky)$$

$$(3) \qquad + \frac{k^3 + k^2 - 2k}{2}f(-y) - \frac{k^3 - k^2 - 2k}{2}f(y) = 0.$$

The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax^3 + bx^2$ is a solution of this functional equation, where a and b are real constants.

Many mathematicians proved the stability of the quadratic-cubic functional equations by handling the odd part and the even part of the given function f, respectively. In this paper, instead of splitting the given function $f : V \to Y$ into two parts, we will prove the stability of the functional equation (3) at once by using the fixed point theory and we will show that the functional equation (3) is a quadratic-cubic functional equation when k is a rational number.

2. Main results

We recall the following Margolis and Diaz's fixed point theorem to prove the main theorem.

THEOREM 2.1. ([8]) *Suppose that a complete generalized metric space (X, d), which means that the metric d may assume infinite values, and a strictly contractive mapping $J : X \to X$ with the Lipschitz constant $0 < L < 1$ are given. Then, for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = +\infty, \ \forall n \in \mathbb{N} \cup \{0\},$$

or there exists a nonnegative integer k such that:

(1) $d(J^n x, J^{n+1} x) < +\infty$ for all $n \geq k$;

(2) the sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;

(3) y^* is the unique fixed point of J in $Y := \{y \in X, d(J^k x, y) < +\infty\}$;

(4) $d(y, y^*) \leq (1/(1 - L)) d(y, Jy)$ for all $y \in Y$.

For a given mapping $f : V \to W$, we use the following abbreviations

$$f_e(x) := \frac{f(x) + f(-x)}{2}, \quad f_e(x) := \frac{f(x) - f(-x)}{2},$$
$$Cf(x, y) := f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) - 6f(y),$$
$$Qf(x, y) := f(x + y) + f(x - y) - 2f(x) - 2f(y),$$
$$D_k f(x, y) := f(x + ky) - kf(x + y) + kf(x - y) - f(x - ky) - f(ky)$$
$$+ \frac{k^3 + k^2 - 2k}{2} f(-y) - \frac{k^3 - k^2 - 2k}{2} f(y)$$

for all $x, y \in V$.

We need the following particular case of Baker's theorem [2] to prove Corollary 2.3.

THEOREM 2.2. (Theorem 1 in [2]) *Suppose that V and W are vector spaces over $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ and $\alpha_0, \beta_0, \ldots, \alpha_m, \beta_m$ are scalar such that $\alpha_j \beta_l - \alpha_l \beta_j \neq 0$ whenever $0 \leq j < l \leq m$. If $f_l : V \to W$ for $0 \leq l \leq m$ and*

$$\sum_{l=0}^{m} f_l(\alpha_l x + \beta_l y) = 0$$

for all $x, y \in V$, then each f_l is a "generalized" polynomial mapping of "degree" at most $m - 1$.

We easily obtain the next result from Baker's Theorem.

COROLLARY 2.3. *If a mapping $f : V \to W$ satisfies the functional equation $D_k f(x, y) = 0$ for all $x, y \in X$, then f is a generalized polynomial mapping of degree at most 3.*

As we stated in the previous section, solutions of $Qf \equiv 0$ and $Cf \equiv 0$ are called a quadratic mapping and a cubic mapping, respectively. Suppose that $f, g : X \to Y$ are generalized polynomial mapping of degree at most 3. It is well known that if the equalities $f(kx) = k^2 f(x)$ and $g(kx) = k^3 g(x)$ hold for all $x \in X$ and a fixed $k \in \mathbb{Q} \backslash \{-1, 0, 1\}$, then f and g are a quadratic mapping and a cubic mapping, respectively.

Now I will show that the functional equation $D_k f(x, y) = 0$ is a quadratic-cubic functional equation when $k \in \mathbb{Q} \backslash \{-1, 0, 1\}$.

THEOREM 2.4. *Let $k \in \mathbb{Q}\backslash\{-1, 0, 1\}$. A mapping satisfies the functional equation $D_k f(x, y) = 0$ for all $x, y \in V$ if and only if f_e is quadratic and f_o is cubic.*

Proof. Assume that a mapping $f : X \to Y$ satisfies the functional equation $D_k f(x, y) = 0$ for all $x, y \in V$. The equalities $f_o(kx) = k^3 f_o(x)$ and $f_e(kx) = k^2 f_e(x)$ follow from the equalities

$$f_o(kx) - k^3 f_o(x) = D_k f_o(0, x), \quad f_e(kx) - k^2 f_e(x) = -D_k f_e(0, x)$$

for all $x \in V$. Since f_o and f_e are generalized polynomial mappings of degree at most 3, f_o is a cubic mapping and f_e is a quadratic mapping.

Conversely, assume that f_o is a cubic mapping and f_e is a quadratic mapping, i.e., f is quadratic-cubic mapping. Notice that the equalities $f_o(kx) = k^3 f_o(x)$, $f_o(x) = -f_o(-x)$, $f_e(kx) = k^2 f_e(x)$, $f_e(x) = f_e(-x)$, and $f(x) = f_o(x) + f_e(x)$ hold for all $x \in V$ and $k \in \mathbb{Q}$.

First the equalities $D_2 f(x, y) = 0$ and $D_3 f(x, y) = 0$ follow from the equalities

$$\begin{aligned}
D_2 f_o(x, y) &= C f_o(x, y) + C f_o(x - y, y), \\
D_2 f_e(x, y) &= Q f_e(x + y, y) - Q f_e(x - y, y), \\
D_3 f_o(x, y) &= C f_o(x - y, 2y), \\
D_3 f_e(x, y) &= Q f_e(x + y, 2y) - Q f_e(x - y, 2y)
\end{aligned}$$

for all $x, y \in X$. If the equalities $D_j f(x, y) = 0$ hold for all $j \in \mathbb{N}$ when $2 \le j \le n - 1$, then the equality $D_n f(x, y) = 0$ follows from the equalities

$$\begin{aligned}
D_n f_o(x, y) =&\, D_{n-1} f_o(x + y, y) + D_{n-1} f_o(x - y, y) \\
&- D_{n-2} f_o(x, y) + (n - 1) D_2 f_o(x, y), \\
D_n f_e(x, y) =&\, D_{n-1} f_e(x + y, y) + D_{n-1} f_e(x - y, y) \\
&- D_{n-2} f_e(x, y) + (n - 1) D_2 f_e(x, y)
\end{aligned}$$

for all $x, y \in X$. Using mathematical induction, we obtain

$$D_n f(x, y) = 0$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Using the equalities

$$\begin{aligned}
D_k f_o(x, y) =&\, f_o(x + ky) - k f_o(x + y) + k f_o(x - y) \\
&- f_o(x - ky) - 2(k^3 - k) f_o(y), \\
D_k f_e(x, y) =&\, f_e(x + ky) - k f_e(x + y) + k f_e(x - y) - f_e(x - ky)
\end{aligned}$$

for all $x, y \in X$ and $k \in \mathbb{Q}$, we get the desired equalities $D_{\frac{n}{m}} f(x, y) = 0$ and $D_{\frac{-n}{m}} f(x, y) = 0$ from the equalities

$$D_{\frac{n}{m}} f_e(x, y) = D_n f_e(x, \frac{y}{m}) - \frac{n}{m} D_m f_e(x, \frac{y}{m}),$$

$$D_{\frac{n}{m}} f_o(x, y) = D_n f_o(x, \frac{y}{m}) - \frac{n}{m} D_m f_o(x, \frac{y}{m}),$$

$$D_{\frac{-n}{m}} f_e(x, y) = D_n f_e(x, \frac{-y}{m}) - \frac{n}{m} D_m f_e(x, \frac{-y}{m}),$$

$$D_{\frac{-n}{m}} f_o(x, y) = D_n f_o(x, \frac{-y}{m}) - \frac{n}{m} D_m f_o(x, \frac{-y}{m})$$

for all $x, y \in X$ and $n, m \in \mathbb{N}$. $\square$

Now we can prove some stability results of the functional equation (3) by using the fixed point theory.

THEOREM 2.5. *Let* $\varphi : V^2 \to [0, \infty)$ *be a given function and* $|k| > 1$. *Suppose that the mapping* $f : V \to Y$ *satisfies the inequality*

$$(4) \qquad \|D_k f(x, y)\| \le \varphi(x, y)$$

for all $x, y \in V$ *and* $f(0) = 0$. *If there exists a constant* $0 < L < 1$ *such that* φ *has the property*

$$(5) \qquad \varphi(kx, ky) \le k^2 L \varphi(x, y)$$

for all $x, y \in V$, *then there exists a unique mapping* $F : V \to Y$ *such that*

$$(6) \qquad \|f(x) - F(x)\| \le \frac{(|k| + 1)(\varphi(0, -x) + \varphi(0, x))}{2 \cdot |k|^3 (1 - L)}$$

for all $x \in V$ *and* $D_k F(x, y) = 0$ *for all* $x, y \in V$. *In particular,* F *is represented by*

$$(7) \qquad F(x) = \lim_{n \to \infty} \left(\frac{f(k^n x) + f(-k^n x)}{2 \cdot k^{2n}} + \frac{f(k^n x) - f(-k^n x)}{2 \cdot k^{3n}} \right)$$

for all $x \in V$.

Proof. Let S be the set of all mappings $g : V \to Y$ with $g(0) = 0$ and introduce a generalized metric on S by

$$d(g, h) = \inf\{K \in \mathbb{R}^+ \,|\, \|g(x) - h(x)\| \le K(\varphi(0, -x) + \varphi(0, x)) \,\forall x \in V\}.$$

It is easy to show that (S, d) is a generalized complete metric space. Now we consider the mapping $J : S \to S$, which is defined by

$$Jg(x) := \frac{g(kx) - g(-kx)}{2 \cdot k^3} + \frac{g(kx) + g(-kx)}{2 \cdot k^2}$$

for all $x \in V$. Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq K$. From the definition of d, we have

$$\|Jg(x) - Jh(x)\| \leq \frac{|k + 1|}{2 \cdot |k|^3}\|g(kx) - h(kx)\| + \frac{|k - 1|}{2 \cdot |k|^3}\|g(-kx) - h(-kx)\|$$

$$\leq \frac{1}{k^2}K\left(\varphi(0, -kx) + \varphi(0, kx)\right)$$

$$\leq KL(\varphi(0, -x) + \varphi(0, x))$$

for all $x \in V$, which implies that

$$d(Jg, Jh) \leq Ld(g, h)$$

for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L. Moreover, by (4), we see that

$$\|f(x) - Jf(x)\| = \frac{1}{2 \cdot |k|^3}\|(k + 1)D_k f(0, -x) + (k - 1)D_k f(0, x)\|$$

$$\leq \frac{|k| + 1}{2 \cdot |k|^3}(\varphi(0, -x) + \varphi(0, x))$$

for all $x \in V$. It means that $d(f, Jf) \leq \frac{|k|+1}{2 \cdot |k|^3} < \infty$ by the definition of d. Therefore according to Theorem 2.1, the sequence $\{J^n f\}$ converges to the unique fixed point $F : V \to Y$ of J in the set $T = \{g \in S | d(f, g) < \infty\}$, which is represented by (7) for all $x \in V$. Notice that

$$d(f, F) \leq \frac{1}{1 - L}d(f, Jf) \leq \frac{|k| + 1}{2 \cdot |k|^3(1 - L)}$$

which implies (6). By the definitions of F, together with (4) and (5), we have that

$$
\begin{aligned}
\|D_k f(x,y)\| &= \lim_{n\to\infty} \left\| \frac{D_k(k^n x, k^n y) - D_k f(-k^n x, -, k^n y)}{2 \cdot k^{3n}} \right. \\
&\qquad\qquad \left. + \frac{D_k f(k^n x, k^n y) + D_k f(-k^n x, -k^n y)}{2 \cdot k^{2n}} \right\| \\
&\leq \lim_{n\to\infty} \frac{|k|^n + 1}{2 \cdot |k|^{3n}} \left(\varphi(k^n x, k^n y) + \varphi(-k^n x, -k^n y) \right) \\
&\leq \lim_{n\to\infty} \frac{(|k|^n + 1)L^n}{2 \cdot |k|^n} \left(\varphi(x,y) + \varphi(-x,-y) \right) \\
&= 0
\end{aligned}
$$

for all $x, y \in V$. So F satisfies $D_k F(x,y) = 0$ for all $x, y \in V$. Notice that if F is a solution of the functional equation (3), then the equality $F(x) - JF(x) = \frac{(k+1)D_k F(0,-x) + (k-1)D_k F(0,x)}{2 \cdot k^3}$ implies that F is a fixed point of J. Hence F is unique mapping satisfying (6). $\qquad\square$

THEOREM 2.6. *Let $\varphi : V^2 \to [0, \infty)$ be a given function and $|k| < 1$. Suppose that the mapping $f : V \to Y$ satisfies the inequality (4) for all $x, y \in V$ and $f(0) = 0$. If there exists a constant $0 < L < 1$ such that φ has the property*

(8)
$$
\varphi(kx, ky) \leq |k|^3 L \varphi(x,y)
$$

for all $x, y \in V$, then there exists a unique mapping $F : V \to Y$ satisfying (6) for all $x \in V$ and $D_k F(x,y) = 0$ for all $x, y \in V$. In particular, F is represented by (7) for all $x \in V$.

Proof. Let the set (S, d) and the mapping $J : S \to S$ be as in the proof of Theorem 2.5. Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq K$. From the definition of d, we have

$$
\begin{aligned}
\|Jg(x) - Jh(x)\| &\leq \frac{1}{|k|^3} K \left(\varphi(0, -kx) + \varphi(0, kx) \right) \\
&\leq KL(\varphi(0, -x) + \varphi(0, x))
\end{aligned}
$$

for all $x \in V$, which implies that

$$
d(Jg, Jh) \leq Ld(g, h)
$$

for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L. Moreover, by (4), we see that

$$\|f(x) - Jf(x)\| = \frac{1}{2 \cdot |k|^3}\|(k+1)D_k f(0, -x) + (k-1)D_k f(0, x)\|$$

$$\leq \frac{|k|+1}{2 \cdot |k|^3}(\varphi(0, -x) + \varphi(0, x))$$

for all $x \in V$. It means that $d(f, Jf) \leq \frac{|k|+1}{2 \cdot |k|^3} < \infty$ by the definition of d. Therefore according to Theorem 2.1, the sequence $\{J^n f\}$ converges to the unique fixed point $F : V \to Y$ of J in the set $T = \{g \in S | d(f, g) < \infty\}$, which is represented by (7) for all $x \in V$. Notice that

$$d(f, F) \leq \frac{1}{1-L}d(f, Jf) \leq \frac{|k|+1}{2 \cdot |k|^3(1-L)}$$

which implies (6). By the definitions of F, together with (4) and (8), we have that

$$\|D_k f(x, y)\| \leq \lim_{n \to \infty} \frac{|k|^n + 1}{2 \cdot |k|^{3n}}(\varphi(k^n x, k^n y) + \varphi(-k^n x, -k^n y))$$

$$\leq \lim_{n \to \infty} \frac{(|k|^n + 1)L^n}{2}(\varphi(x, y) + \varphi(-x, -y))$$

$$= 0$$

for all $x, y \in V$. So F satisfies $D_k F(x, y) = 0$ for all $x, y \in V$. Notice that if F is a solution of the functional equation (3), then the equality $F(x) - JF(x) = \frac{(k+1)D_k F(0, -x) + (k-1)D_k F(0, x)}{2 \cdot k^3}$ implies that F is a fixed point of J. Hence F is unique mapping satisfying (6). $\qquad \square$

We continue our investigation with the next result.

THEOREM 2.7. *Let $\varphi : V^2 \to [0, \infty)$ and k be a real number such that $|k| > 1$. Suppose that $f : V \to Y$ satisfies the inequality (4) for all $x, y \in V$ and $f(0) = 0$. If there exists $0 < L < 1$ such that the mapping φ has the property*

$$(9) \qquad\qquad\qquad L\varphi(kx, ky) \geq |k|^3\varphi(x, y)$$

for all $x, y \in V$, then there exists a unique mapping $F : V \to Y$ such that

$$(10) \qquad \|f(x) - F(x)\| \leq \frac{L}{|k|^3(1-L)}(\varphi(0, -x) + \varphi(0, x))$$

for all $x \in V$ and $D_k F(x, y) = 0$ for all $x, y \in V$. In particular, F is represented by

(11)
$$F(x) = \lim_{n \to \infty} \left(\frac{k^{3n}}{2} \left(f \left(\frac{x}{k^n} \right) - f \left(-\frac{x}{k^n} \right) \right) + \frac{k^{2n}}{2} \left(f \left(\frac{x}{k^n} \right) + f \left(-\frac{x}{k^n} \right) \right) \right)$$

for all $x \in V$.

Proof. Let the set (S, d) be as in the proof of Theorem 2.5. Now we consider the mapping $J : S \to S$ defined by

$$Jg(x) := \frac{k^3}{2} \left(g \left(\frac{x}{k} \right) - g \left(-\frac{x}{k} \right) \right) + \frac{k^2}{2} \left(g \left(\frac{x}{k} \right) + g \left(-\frac{x}{k} \right) \right)$$

for all $g \in S$ and $x \in V$. Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq K$. From the definition of d, we have

$$\|Jg(x) - Jh(x)\|$$
$$\leq \frac{|k^3 + k^2|}{2} \left\| g \left(\frac{x}{k} \right) - h \left(\frac{x}{k} \right) \right\| + \frac{|k^3 - k^2|}{2} \left\| g \left(-\frac{x}{k} \right) - h \left(-\frac{x}{k} \right) \right\|$$
$$\leq |k|^3 K \left(\varphi \left(0, -\frac{x}{k} \right) + \varphi \left(0, \frac{x}{k} \right) \right)$$
$$\leq LK \left(\varphi(0, -x) + \varphi(0, x) \right)$$

for all $x \in V$. So
$$d(Jg, Jh) \leq Ld(g, h)$$

for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L. Also we see that

$$\|f(x) - Jf(x)\| = \left\| -D_k f \left(0, \frac{-x}{k} \right) \right\| \leq \varphi \left(0, -\frac{x}{k} \right) \leq \frac{L}{|k|^3} \varphi(0, -x)$$

for all $x \in V$, which implies that $d(f, Jf) \leq \frac{L}{|k|^3} < \infty$. Therefore according to Theorem 2.1, the sequence $\{J^n f\}$ converges to the unique fixed point F of J in the set $T := \{g \in S | d(f, g) < \infty\}$, which is represented by (11) for all $x \in V$.

Notice that

$$d(f, F) \leq \frac{1}{1 - L} d(f, Jf) \leq \frac{L}{|k|^3 (1 - L)}$$

which implies (10). From the definition of $F(x)$, (4), and (9), we have

$$
\begin{aligned}
\|D_kF(x,y)\| &= \lim_{n\to\infty} \left\| \frac{k^{3n}}{2} \left(D_kf\left(\frac{x}{k^n},\frac{y}{k^n}\right) - D_kf\left(-\frac{x}{k^n},-\frac{y}{k^n}\right)\right)\right. \\
&\quad \left. + \frac{k^{2n}}{2}\left(D_kf\left(\frac{x}{k^n},\frac{y}{k^n}\right) + D_kf\left(-\frac{x}{k^n},-\frac{y}{k^n}\right)\right)\right\| \\
&\leq \lim_{n\to\infty} \frac{|k|^{3n}+|k|^{2n}}{2}\left(\varphi\left(\frac{x}{k^n},\frac{y}{k^n}\right)+\varphi\left(-\frac{x}{k^n},-\frac{y}{k^n}\right)\right) \\
&\leq \lim_{n\to\infty} \frac{(|k|^n+1)L^n}{2\cdot|k|^n}\left(\varphi(x,y)+\varphi(-x,-y)\right) \\
&= 0
\end{aligned}
$$

for all $x,y \in V$. So F satisfies $D_kF(x,y)=0$ for all $x,y \in V$. Notice that if F is a solution of the functional equation (3), then the equality

$$
F(x) - JF(x) = -D_kF\left(0,\frac{-x}{k}\right)
$$

implies that F is a fixed point of J.   □

THEOREM 2.8. *Let $\varphi : V^2 \to [0,\infty)$ and k be a real number such that $|k| < 1$. Suppose that $f : V \to Y$ satisfies the inequality (4) for all $x,y \in V$ and $f(0) = 0$. If there exists $0 < L < 1$ such that the mapping φ has the property*

(12) $$L\varphi(kx,ky) \geq k^2\varphi(x,y)$$

for all $x,y \in V$, then there exists a unique mapping $F : V \to Y$ such that

(13) $$\|f(x) - F(x)\| \leq \frac{L}{|k|^2(1-L)}\left(\varphi(0,-x)+\varphi(0,x)\right)$$

for all $x \in V$ and $D_kF(x,y)=0$ for all $x,y \in V$. In particular, F is represented by (11) for all $x \in V$.

Proof. Let the set (S,d) and the mapping $J : S \to S$ be as in the proof of Theorem 2.7. Let $g,h \in S$ and let $K \in [0,\infty]$ be an arbitrary constant with $d(g,h) \leq K$. From the definition of d, we have

$$
\begin{aligned}
\|Jg(x) - Jh(x)\| &\leq k^2 K\left(\varphi\left(0,-\frac{x}{k}\right)+\varphi\left(0,\frac{x}{k}\right)\right) \\
&\leq LK\left(\varphi(0,-x)+\varphi(0,x)\right)
\end{aligned}
$$

for all $x \in V$. So

$$
d(Jg,Jh) \leq Ld(g,h)
$$

for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L. Also we see that

$$\|f(x) - Jf(x)\| = \left\| -D_k f\left(0, \frac{-x}{k}\right) \right\| \le \varphi\left(0, -\frac{x}{k}\right) \le \frac{L}{|k|^2}\varphi(0, -x)$$

for all $x \in V$, which implies that $d(f, Jf) \le \frac{L}{|k|^2} < \infty$. Therefore according to Theorem 2.1, the sequence $\{J^n f\}$ converges to the unique fixed point F of J in the set $T := \{g \in S | d(f, g) < \infty\}$, which is represented by (11) for all $x \in V$.

Notice that

$$d(f, F) \le \frac{1}{1 - L}d(f, Jf) \le \frac{L}{|k|^2(1 - L)}$$

which implies (13). From the definition of $F(x)$, (4), and (12), we have

$$\begin{aligned}
\|D_k F(x, y)\| &\le \lim_{n\to\infty} \frac{|k|^{3n} + |k|^{2n}}{2}\left(\varphi\left(\frac{x}{k^n}, \frac{y}{k^n}\right) + \varphi\left(-\frac{x}{k^n}, -\frac{y}{k^n}\right)\right) \\
&\le \lim_{n\to\infty} \frac{(|k|^n + 1)L^n}{2}\left(\varphi(x, y) + \varphi(-x, -y)\right) \\
&= 0
\end{aligned}$$

for all $x, y \in V$. So F satisfies $D_k F(x, y) = 0$ for all $x, y \in V$. Notice that if F is a solution of the functional equation (3), then the equality

$$F(x) - JF(x) = -D_k F\left(0, \frac{-x}{k}\right)$$

implies that F is a fixed point of J. $\qquad\square$

COROLLARY 2.9. *Let X be a normed space, Y a Banach space, and $|k| > 1$. Suppose that the mapping $f : X \to Y$ satisfies the inequality*

$$\|D_k f(x, y)\| \le \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$, where $\theta > 0$ and $p \in [0, 2) \cup (3, \infty)$. Then there exists a unique mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \le \begin{cases} \frac{2\theta\|x\|^p}{|k|^p - |k|^3} & \text{if } p > 3, \\[2mm] \frac{\theta(|k|+1)\|x\|^p}{|k|(|k|^2 - |k|^p)} & \text{if } 0 \le p < 2 \end{cases}$$

for all $x \in X$ and $D_k F(x, y) = 0$ for all $x, y \in X$.

Proof. This corollary follows from Theorem 2.5 and Theorem 2.7, by putting $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$, $L := |k|^{2-p} < 1$ when $p < 2$, and $L := |k|^{p-3} < 1$ when $p > 3$. $\qquad\square$

COROLLARY 2.10. *Let X be a normed space, Y a Banach space, and $|k| < 1$. Suppose that the mapping $f : X \to Y$ satisfies the inequality*

$$\|D_k f(x,y)\| \le \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$, where $\theta > 0$ and $p \in [0, 2) \cup (3, \infty)$. Then there exists a unique mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \le \begin{cases} \frac{(|k|+1)\theta\|x\|^p}{|k|^3 - |k|^p} & \text{if } p > 3, \\[2ex] \frac{2\theta\|x\|^p}{|k|^p - |k|^2} & \text{if } 0 \le p < 2 \end{cases}$$

for all $x \in X$ and $D_k F(x,y) = 0$ for all $x, y \in X$.

Proof. This corollary follows from Theorem 2.6 and Theorem 2.8, by putting $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$, $L := |k|^{2-p} < 1$ when $p < 2$, and $L := |k|^{p-3} < 1$ when $p > 3$. $\qquad\square$

COROLLARY 2.11. *Let X be a normed space and Y a Banach space. Suppose that the mapping $f : X \to Y$ satisfies $f(0) = 0$ and the inequality*

$$\|D_k f(x,y)\| \le \theta$$

for all $x, y \in X$, where $\theta > 0$. Then there exists a unique quadratic and cubic mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \le \begin{cases} \frac{\theta\|x\|^p}{1 - |k|^2} & \text{if } |k| < 1, \\[2ex] \frac{\theta(|k|+1)}{2|k|(|k|^2 - 1)} & \text{if } |k| > 1 \end{cases}$$

for all $x \in X$.

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.

[2] J. Baker, *A general functional equation and its stability*, Proc. Natl. Acad. Sci. **133** (6) (2005), 1657–1664.

[3] L. Cădariu and V. Radu, *Fixed points and the stability of quadratic functional equations*, An. Univ. Timisoara Ser. Mat.-Inform. **41** (2003), 25–48.

[4] L. Cădariu and V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach in Iteration Theory*, Grazer Mathematische Berichte, Karl-Franzens-Universitäet, Graz, Graz, Austria **346** (2004), 43–52.

[5] I.-S. Chang and Y.-S. Jung, *Stability of a functional equation deriving from cubic and quadratic functions*, J. Math. Anal. Appl. **283** (2003), 491–500.

[6] Y.-J. Cho, M. Eshaghi Gordji, and S. Zolfaghari, *Solutions and Stability of Generalized Mixed Type QC Functional Equations in Random Normed Spaces*, J. Inequal. Appl. **2010** (2010), Art. ID 403101.

[7] S. Czerwik, *On the stability of the quadratic mapping in the normed space*, Abh. Math. Sem. Hamburg, **62** (1992), 59–64.

[8] J. B. Diaz and B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74** (1968), 305–309.

[9] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. and Appl. **184** (1994), 431–436.

[10] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA, **27** (1941), 222–224.

[11] C.-J. Lee and Y.-H. Lee, *On the stability of a mixed type quadratic and cubic functional equation*, J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. **19** (2012), 383–396.

[12] Y.-H. Lee and S.-M. Jung, *Fuzzy stability of the cubic and quadratic functional equation*, Appl. Math. Sci. (Ruse) **10** (2016), 2671–2686.

[13] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[14] W. Towanlong and P. Nakmahachalasint, *A Mixed-Type Quadratic and Cubic Functional Equation and Its Stability*, Thai J. Math. **8** (4) (2012), 61–71.

[15] S.M. Ulam, Problems in Modern Mathematics, Wiley, New York, 1964.

[16] Z. Wang and W. X. Zhang, *Fuzzy stability of quadratic-cubic functional equations*, Acta Math. Sin. (Engl. Ser.) **27** (2011), 2191–2204.

Yang-Hi Lee
Department of Mathematics Education
Gongju National University of Education
Gongju 32553, Republic of Korea
E-mail: yanghi2@hanmail.net