Korean J. Math. **27** (2019), No. 2, pp. 375–415
https://doi.org/10.11568/kjm.2019.27.2.375

MATRIX OPERATORS ON FUNCTION-VALUED FUNCTION SPACES

Sing-Cheong Ong, Jitti Rakbud, and
Titarii Wootijirattikal*

Abstract. We study spaces of continuous-function-valued functions that have the property that composition with evaluation functionals induce weak* to norm continuous maps to ℓ^p space ($p \in (1, \infty)$). Versions of Hölder's inequality and Riesz representation theorem are proved to hold on these spaces. We prove a version of Dixmier's theorem for spaces of function-valued matrix operators on these spaces, and an analogue of the trace formula for operators on Hilbert spaces. When the function space is taken to be the complex field, the spaces are just the ℓ^p spaces and the well-known classical theorems follow from our results.

1. Introduction

Operators on the ℓ^p, $1 < p < \infty$ sequence spaces have matrix representations. This is mainly a consequence of the fact that the complex field $\mathbb{C}$ acts on itself as operators (self dual). It is natural to investigate how much this can be pushed toward the self action of algebras. We take an initial step by considering the closest allies of $\mathbb{C}$, namely commutative C^*-algebras. We show that several of the beautiful theorems for sequence spaces can be extended to this setting. As one would

Received December 17, 2018. Revised March 18, 2019. Accepted March 19, 2019.
2010 Mathematics Subject Classification: 46E40, 46L30.
Key words and phrases: C^*-algebra, function space, operators, M-ideal, dual space.
∗ Corresponding author.

guess that the most obvious way of extension is to replace the absolute value on $\mathbb{C}$ by the norm on $\mathcal{A}$, then the extension of many theorems for sequence spaces will follow almost immediately. The topology induced on $\mathbb{C}$ by the absolute value has several analogues in $\mathcal{A}$. Here we use the weak* topology coupled with evaluation functionals. With a fixed commutative C^* algebra (or algebra of continuous functions on a compact Hausdorff space), $\mathcal{A}$, we study $\mathcal{A}$-valued spaces of functions that give rise to ℓ^p functions when composed with evaluation functionals, and the composition is weak* to norm continuous. We show that Hölder's inequality [7, p.140], Riesz representation theorem [7, p.160] have analogues on these spaces. We then consider operators on these spaces that are defined by matrices with entries from $\mathcal{A}$. We identify a subset of $\mathcal{A}$-matrix operators that act like the compact operators on ℓ^p. Then a version of Dixmier's decomposition theorem of linear functionals on bounded operators on Hilbert space [3] is proved for $\mathcal{A}$-valued matrix operators, acting on $\mathcal{A}$-valued function spaces. We will give two proofs of this results. The first one is using very elementary methods, as is the rest of the paper, except the second proof. The second proof is based on a deep theorem of Alfsen and Effros. This work extends results in [6]

Dixmier's theorem states that for each bounded linear functional f on the algebra, $\mathcal{B}(H)$, of bounded linear operators on a Hilbert space there are unique bounded linear functionals g and h on $\mathcal{B}(H)$ such that (1) $g|_{\mathcal{K}(H)} = f|_{\mathcal{K}(H)}$ on the ideal, $\mathcal{K}(H)$, of compact operators on H, (2) $h \in (\mathcal{K}(H))^{\perp}$, the annihilator of $\mathcal{K}(H)$, (3) $f = g + h$ and (4) $\|f\| = \|g\| + \|h\|$. This result has lead to a vast literature. culminating in the pinnacle of the introduction of the notion of M-ideals by Alfsen and Effros [1]. A subspace Y of a Banach space X is called an M-ideal [1] if each f in the dual space, $X^{\#}$, of X has a decomposition $f = g+h$ with $h \in Y^{\perp}$ and $\|f\| = \|g\|+\|h\|$. Thus Dixmier's theorem states that $\mathcal{K}(H)$ is an M-ideal of $\mathcal{B}(H)$. Much of the extensions of Dixmier's results is along the line of operators on vector-valued sequence spaces, in which each sequence gives rise to an ℓ^p sequence of norms. Our approach is much weaker, using only point evaluation. It's worth mentioning that our work is completely elementary, and that it is our hope that it can inspire aspiring analysts.

We organize our work as follows. In Section 2, we introduce the spaces to be dealt with in the paper. We prove a version of Riesz representation theorem our class of function spaces in Section 3. In Section 4, we study properties of function-valued matrix operators. A subspace of the $\mathcal{A}$-matrix operators as a candidate for the M-ideal is identified in Section 5. An analogue of Dixmier's theorem is proved in Section 6, in which we also give an alternate proof using a big theorem of Alfsen and Effros [1].

2. Function-valued function spaces

Fix a nonempty set S. The collection $\mathcal{F}(S) := \mathcal{F}$ of all finite subsets of S is directed by set inclusion

$$F \prec G \Leftrightarrow F \subseteq G \qquad F,\ G \in \mathcal{F}(S).$$

A function $\mathbf{x}$ from S to a normed space X is *summable* if there is a vector $x \in X$ such that the *net*, $\left\{\sum_{s\in F} \mathbf{x}(s)\right\}_{F\in\mathcal{F}(S)}$ of *finite partial sums*, norm converges (with respect to this partial ordering of $\mathcal{F}$) to x in X. If no such a vector exists in X, $\mathbf{x}$ is not summable. If $\mathbf{x}$ is summable to $x \in X$ we will write

$$\sum_{s\in S} \mathbf{x}(s) = x = \lim_{F\in\mathcal{F}(s)} \left[\sum_{s\in F} \mathbf{x}(s)\right].$$

That is $\sum_{s\in S} \mathbf{x}(s) = x$ if and only if for all $\epsilon > 0$ there is a finite subset $F_\epsilon \in \mathcal{F}(S)$ such that

$$\left\| x - \sum_{s\in G} \mathbf{x}(s) \right\|_X < \epsilon \qquad \forall\ F_\epsilon \subseteq G \in \mathcal{F}(S).$$

If $\sum_{s\in S} \mathbf{x}(s) = x$, the vector x is called the *sum* of the function $\mathbf{x}$. If no such $x \in X$ exists, the sum $\sum_{s\in S} \mathbf{x}(s)$ is *divergent.*

For $1 \le p < \infty$, let $\ell^p(S)$ denote the space of all complex-valued functions $x : S \to \mathbb{C}$ such that

$$\|x\|_p^p := \sum_{s\in S} |x(s)|^p < \infty.$$

A routine adaptation of the proofs for ℓ^p sequence spaces shows that $\ell^p(S)$ is a Banach space with the norm $\|\cdot\|_p$ defined above. Note that while ℓ^p sequence spaces are separable, $\ell^p(S)$ is not separable, unless S is countable, and if $S = \mathbb{N}$, then $\ell^p(S) = \ell^p$.

A fixed commutative C^*-algebra $\mathcal{A}$, with identity 1 may be regarded as, $\mathcal{A} = C(\Omega)$, the algebra of all continuous complex-valued functions on a compact Hausdorff space Ω [5, p. 270, Th 4.4.3]. Spaces of sequences in a Banach space have been extensively studied in the literature, and so have been operators acting on them. The general idea is to replace the scalars by vectors and absolute value by norm. When the space is replaced by a commutative C^*-algebra $\mathcal{A}$, we may consider weaker versions of convergence than convergence in norm. We will consider a very weak version that still retain most of the important and interesting properties of ℓ^p $(1 < p < \infty)$ sequence spaces and matrix operators that act on them.

Consider the space $\ell^p(S, \mathcal{A})$ of all functions $\mathbf{x} : S \to \mathcal{A}$ such that for each $\omega \in \Omega$ the function $\mathbf{x}_\omega : S \to \mathbb{C}$ defined by

$$\mathbf{x}_\omega(s) = [\mathbf{x}(s)](\omega) \qquad \forall\, s \in S$$

is in $\ell^p(S)$, and the map $\omega \mapsto \mathbf{x}_\omega$ is norm continuous from Ω to $\ell^p(S)$. Then a routine verification reveals that $\ell^p(S, \mathcal{A})$ is a Banach space with the norm

$$\|\mathbf{x}\| = \sup_{\omega \in \Omega} \|\mathbf{x}_\omega\|_p = \sup_{\omega \in \Omega} \left[\sum_{s \in S} |[\mathbf{x}(s)](\omega)|^p \right]^{1/p} \qquad \mathbf{x} \in \ell^p(S, \mathcal{A}).$$

Note that the notation has been used to study space of functions $\mathbf{x}$ such that $\sum_{s \in S} \|\mathbf{x}(s)\|_{\mathcal{A}}^p < \infty$. This more relaxed definition captures a wider class of functions. For example with $\Omega = [0, 1]$ (the closed unit interval with usual topology) and $S = \mathbb{N}$ (the set of positive integers), the sequence $\{\mathbf{x}_n\}_{n=1}^\infty$ where $\mathbf{x}_n$ is the piecewise linear continuous function whose value at the midpoint of the interval $\left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]$ is $\frac{1}{\sqrt{n}}$ and is 0 outside the interval is in $\ell^2(S, \mathcal{A})$, but $\{\|\mathbf{x}_n\|\}_{n=1}^\infty$ is not in ℓ^2. We will see that the space $\ell^p(S, \mathcal{A})$ enjoys many nice and interesting properties of the ℓ^p spaces.

For a function $\mathbf{x} \in \mathcal{A}^S$ (the space of all functions from S to $\mathcal{A}$) and a subset $G \subseteq S$, the function which agrees with $\mathbf{x}$ on G and take the value 0 on $S \setminus G$ will be denoted by $\mathbf{x}_G$ (i.e., $\mathbf{x}_G(s) = \mathbf{x}(s)$ for $s \in G$ and $\mathbf{x}(s) = 0$ for $s \in S \setminus G$).

We list the following properties for convenience of reference. They all follow directly from the definitions, and we omit their routine verifications.

Proposition 1. *The following are true for each* $\mathbf{x} \in \ell^p(S, \mathcal{A})$.

1. $\|\mathbf{x}(s)\|_{\mathcal{A}} \le \|\mathbf{x}\|$ *for all* $s \in S$, *and* $\|\mathbf{x}_G\| \le \|\mathbf{x}_H\|$ *for all* $G \subseteq H \subseteq S$.
2. $\sup_{\omega \in \Omega} \sum_{s \in S} |(\mathbf{x}(s))(\omega)|^p = \|\mathbf{x}\|^p$.

Each $a \in \mathcal{A}$ is a function on Ω and so are $|a| = \sqrt{\bar{a}a} \in \mathcal{A}$ and $|a|^{\alpha} \in \mathcal{A}$ for all $\alpha \geq 0$.

Proposition 2. *The following conditions on a function $\mathbf{x} \in \mathcal{A}^{S}$ are equivalent.*

1. $\mathbf{x} \in \ell^{p}(S, \mathcal{A})$;
2. *For each $\epsilon > 0$ there is an $F_{\epsilon} \in \mathcal{F}(S)$ such that*

$$\left\| \mathbf{x}_{S\setminus F_\epsilon} \right\| = \sup_{\omega\in\Omega} \left\| \left(\mathbf{x}_{S\setminus F_\epsilon}\right)_{\omega} \right\|_{\ell^p(S)} = \sup_{\omega\in\Omega} \left\| (\mathbf{x}_{\omega})_{S\setminus F_\epsilon} \right\|_{\ell^p(S)}$$

$$= \sup_{\omega\in\Omega} \left[\sum_{s\in S\setminus F_\epsilon} |(\mathbf{x}(s))(\omega)|^{p} \right]^{1/p} < \epsilon;$$

3. *The function $|\mathbf{x}|^{p}$ defined by $|\mathbf{x}|^{p}(s) = |\mathbf{x}(s)|^{p} = [(\mathbf{x}(s))^{*}(\mathbf{x}(s))]^{p/2}$, for all $s \in S$, is summable in $\mathcal{A}$.*

Proof. [(1)⇒(2)] Assume $\mathbf{x} \in \ell^{p}(S, \mathcal{A})$, and let $\epsilon > 0$. By continuity of the map $\omega \to \mathbf{x}_{\omega}$, for each $\omega \in \Omega$ there is an open set $\mathcal{O}_{\omega} \subseteq \Omega$ such that

$$\left\| \mathbf{x}_{\omega} - \mathbf{x}_{\omega'} \right\|_{\ell^p(S)} < \frac{\epsilon}{3} \qquad \forall\, \omega' \in \mathcal{O}_{\omega}.$$

By compactness of Ω, there exist $\omega_1, \omega_2, \cdots, \omega_n \in \Omega$ such that $\Omega \subseteq \bigcup_{j=1}^{n} \mathcal{O}_{\omega_j}$. The convergence of each of

$$\left\| \mathbf{x}_{\omega_j} \right\|_{\ell^p(S)}^{p} = \sum_{s\in S} \left|(\mathbf{x}(s))(\omega_j)\right|^{p},\ 1 \leq j \leq n,$$

implies the existence of an $F_j \in \mathcal{F}(S)$ such that

$$\sum_{s\in S\setminus F_j} \left|(\mathbf{x}(s))(\omega_j)\right|^{p} < \left(\frac{\epsilon}{2}\right)^{p} \qquad \forall\, 1 \leq j \leq n. \qquad (\dagger)$$

Let $F_{\epsilon} = \bigcup_{j=1}^{n} F_j$. Then $F_{\epsilon} \in \mathcal{F}(S)$. We show that F_{ϵ} satisfies the requirements. Let $w \in \Omega$. Then $\omega \in \mathcal{O}_{\omega_j}$ for some j, and $F_j \subseteq F_{\epsilon}$.

Thus, by (†),

$$\begin{aligned}\left\|(\mathbf{x}_\omega)_{S\setminus F_\epsilon}\right\|_{\ell^p(S)} &\le \left\|(\mathbf{x}_\omega)_{S\setminus F_\epsilon} - (\mathbf{x}_{\omega_j})_{S\setminus F_\epsilon}\right\|_{\ell^p(S)} + \left\|(\mathbf{x}_{\omega_j})_{S\setminus F_\epsilon}\right\|_{\ell^p(S)} \\ &= \left\|(\mathbf{x}_\omega - \mathbf{x}_{\omega_j})_{S\setminus F_\epsilon}\right\|_{\ell^p(S)} + \left[\sum_{s\in S\setminus F_\epsilon} |(\mathbf{x}(s))(\omega)|^p\right]^{1/p} \\ &< \left\|\mathbf{x}_\omega - \mathbf{x}_{\omega_j}\right\|_{\ell^p(S)} + \frac{\epsilon}{2} < \frac{\epsilon}{3} + \frac{\epsilon}{2} = \frac{5\epsilon}{6} < \epsilon.\end{aligned}$$

Thus,

$$\left\|\mathbf{x}_{S\setminus F_\epsilon}\right\| = \sup_{\omega\in\Omega}\left\|\left(\mathbf{x}_{S\setminus F_\epsilon}\right)_\omega\right\|_{\ell^p(S)} = \left\|(\mathbf{x}_\omega)_{S\setminus F_\epsilon}\right\|_{\ell^p(S)} \le \frac{5\epsilon}{6} < \epsilon.$$

[(2) ⇒ (3)] Suppose $\mathbf{x}$ satisfies the condition (2). Let $\epsilon > 0$. Choose an $F_\epsilon \in \mathcal{F}(S)$ that satisfy the assumption for $\mathbf{x}$ with the ϵ replaced by $\epsilon^{1/p}$. Let $H \in \mathcal{F}(S \setminus F_\epsilon)$. Then

$$\begin{aligned}\left\|\sum_{s\in H} |\mathbf{x}|^p(s)\right\|_{\mathcal{A}} &= \sup_{\omega\in\Omega}\left|\left[\sum_{s\in H}(|\mathbf{x}|^p(s))\right](\omega)\right| = \sup_{\omega\in\Omega}\left[\sum_{s\in H}|\mathbf{x}(s)|^p(\omega)\right] \\ &= \sup_{\omega\in\Omega}\left[\sum_{s\in H}|(\mathbf{x}(s))(\omega)|^p\right] \le \sup_{\omega\in\Omega}\left[\sum_{s\in S\setminus F_\epsilon}|(\mathbf{x}(s))(\omega)|^p\right] \\ &= \left\|\mathbf{x}_{S\setminus F_\epsilon}\right\|^p < \epsilon.\end{aligned}$$

This shows that for all $\epsilon > 0$ there is an $F_\epsilon \in \mathcal{F}(S)$ for which all finite sums over subsets of the complement of F_ϵ have norm $< \epsilon$, therefore $\sum_{s\in S} |\mathbf{x}|^p(s)$ converges in $\mathcal{A}$.

[(3) ⇒(1)] Suppose $|\mathbf{x}|^p$ is summable in $\mathcal{A}$. Then, since $|\mathbf{x}|(s) \ge 0$ for all $s \in S$,

$$\left\|\sum_{s\in F} |\mathbf{x}|^p(s)\right\|_{\mathcal{A}} \le \left\|\sum_{s\in S} |\mathbf{x}|^p(s)\right\|_{\mathcal{A}} \qquad \forall\ F \in \mathcal{F}(S).$$

Hence, for each $\omega \in \Omega$,

$$\sum_{s\in S} |\mathbf{x}_\omega(s)|^p = \sup_{F\in\mathcal{F}(S)} \sum_{s\in F} |\mathbf{x}_\omega(s)|^p = \sup_{F\in\mathcal{F}(S)} \sum_{s\in F} |[\mathbf{x}(s)](\omega)|^p$$

$$= \sup_{F\in\mathcal{F}(S)} \sum_{s\in F} [|\mathbf{x}|^p(s)](\omega) = \sup_{F\in\mathcal{F}(S)} \left[\sum_{s\in F} |\mathbf{x}|^p(s)\right](\omega) \le \left[\sum_{s\in S} |\mathbf{x}|^p(s)\right](\omega.)$$

Thus $\mathbf{x}_\omega \in \ell^p(S)$ for all $\omega \in \Omega$.

To see that the map $\omega \mapsto \mathbf{x}_\omega$ is continuous, let $\{\omega_\alpha\}$ be a net in Ω that converges to ω, and let $\epsilon > 0$. The convergence of $\mathbf{y} := \sum_{s\in S} |\mathbf{x}|^p(s)$ in $\mathcal{A}$ implies that there is an $F_\epsilon \in \mathcal{F}(S)$ such that

$$\left\|\sum_{s\in S\setminus G} |\mathbf{x}|^p(s)\right\|_{\mathcal{A}} = \sup_{\omega\in\Omega} \sum_{s\in S\setminus G} |[\mathbf{x}(s)](\omega)|^p < \left(\frac{\epsilon}{4}\right)^p \quad \forall\ F_\epsilon \subseteq G \in \mathcal{F}(S). \quad (\dagger)$$

The finiteness of F_ϵ, and continuity of each $\mathbf{x}(s), s \in F_\epsilon$, together with the convergence of $\omega_\alpha \to \omega$, imply that there is an α_0 such that

$$\sum_{s\in F_\epsilon} |(\mathbf{x}(s))(\omega) - (\mathbf{x}(s))(\omega_\alpha)|^p < \frac{\epsilon^p}{2} \qquad \forall\ \alpha \succeq \alpha_0.$$

Let $\alpha \succeq \alpha_0$. Then, by Minkowski's inequality for $\ell^p(S)$ and ($\dagger$),

$$\left\|\mathbf{x}_\omega - \mathbf{x}_{\omega_\alpha}\right\|^p_{\ell^p(S)} = \sum_{s\in S} \left|\mathbf{x}_\omega(s) - \mathbf{x}_{\omega_\alpha}(s)\right|^p$$

$$= \sum_{s\in F_\epsilon} \left|\mathbf{x}_\omega(s) - \mathbf{x}_{\omega_\alpha}(s)\right|^p + \sum_{s\in S\setminus F_\epsilon} \left|\mathbf{x}_\omega(s) - \mathbf{x}_{\omega_\alpha}(s)\right|^p$$

$$= \sum_{s\in F_\epsilon} |(\mathbf{x}(s))(\omega) - (\mathbf{x}(s))(\omega_\alpha)|^p + \sum_{s\in S\setminus F_\epsilon} |(\mathbf{x}(s))(\omega) - (\mathbf{x}(s))(\omega_\alpha)|^p$$

$$< \frac{\epsilon^p}{2} + \left(\left[\sum_{s\in S\setminus F_\epsilon} |(\mathbf{x}(s))(\omega)|^p\right]^{1/p} + \left[\sum_{s\in S\setminus F_\epsilon} |(\mathbf{x}(s))(\omega_\alpha)|^p\right]^{1/p}\right)^p$$

$$< \frac{\epsilon^p}{2} + \left(\frac{\epsilon}{4} + \frac{\epsilon}{4}\right)^p < \epsilon^p.$$

Therefore $\left\|\mathbf{x}_{\omega_\alpha} - \mathbf{x}_\omega\right\|_{\ell^p(S)} \to 0$. $\square$

Theorem 3. *The space $\ell^p(S,\mathcal{A})$ is a Banach space under the usual sum and scalar multiplication of functions and the norm defined above.*

That $\ell^p(S,\mathcal{A})$ is a Banach space follows from the general [9, Theorem 4.2 (p. 381)]. We give a direct proof of the completeness of the space.

Proof. Let $\{\mathbf{x}_n\}$ be a Cauchy sequence in $\ell^p(S,\mathcal{A})$. By Proposition 1 (2), for each $s\in S$, the sequence $\{\mathbf{x}_n(s)\}$ is a Cauchy sequence in $\mathcal{A}$. Since $\mathcal{A}$ is complete, there is an element, say $\mathbf{x}(s)$, of $\mathcal{A}$ to which $\{\mathbf{x}_n(s)\}$ converges. We show that the function $\mathbf{x}: s\mapsto \mathbf{x}(s)$ in $\mathcal{A}^S$ is, in fact, in $\ell^p(S,\mathcal{A})$ and that $\|\mathbf{x}_n-\mathbf{x}\|\to 0$ in $\ell^p(S,\mathcal{A})$.

For each $\omega\in\Omega$, since $\{(\mathbf{x}_n)_\omega\}$ is a sequence in $\ell^p(S)$, and, for all $n,m\in\mathbb{N}$,

$$\|(\mathbf{x}_n)_\omega-(\mathbf{x}_m)_\omega\|_{\ell^p(S)}=\|(\mathbf{x}_n-\mathbf{x}_m)_\omega\|_{\ell^p(S)}\le\|\mathbf{x}_n-\mathbf{x}_m\|_{\ell^p(S,\mathcal{A})}$$

it is a Cauchy sequence in $\ell^p(S)$. The completeness of $\ell^p(S)$ implies the convergence of $\{(\mathbf{x}_n)_\omega\}$. But since $(\mathbf{x}_n)(s)\to\mathbf{x}(s)$ in $\mathcal{A}$ for each $s\in S$, uniqueness of limit implies that $\|(\mathbf{x}_n)_\omega-\mathbf{x}_\omega\|_{\ell^p(S)}\to 0$ and $\mathbf{x}_\omega\in\ell^p(S)$, for each $\omega\in\Omega$.

To see that $\mathbf{x}\in\ell^p(S,\mathcal{A})$, we show that the map $\omega\to\mathbf{x}_\omega$ from Ω to $\ell^p(S)$ is continuous. To that end, let $\{\omega_\alpha\}$ be a net in Ω that converges to $\omega\in\Omega$, and let $\epsilon>0$. By the Cauchy assumption on $\{\mathbf{x}_n\}$ in $\ell^p(S,\mathcal{A})$, there is an $N\in\mathbb{N}$ such that

$$\|\mathbf{x}_n-\mathbf{x}\|_{\ell^p(S,\mathcal{A})}<\frac{\epsilon}{6}\qquad\forall\ n,\ m\ge N.$$

Since $\mathbf{x}_N\in\ell^p(S,\mathcal{A})$, there is an α_0 such that

$$\left\|(\mathbf{x}_N)_{\omega_\alpha}-(\mathbf{x}_N)_\omega\right\|_{\ell^p(S)}<\frac{\epsilon}{6}\qquad\forall\ \alpha\succeq\alpha_0.$$

Let $\alpha\succeq\alpha_0$. Since $\|(\mathbf{x}_n)_\alpha-\mathbf{x}_\alpha\|_{\ell^p(S)}\to 0$, there is a $k\ge N$ such that

$$\|(\mathbf{x}_k)_\omega-\mathbf{x}_\omega\|_{\ell^p(S)}<\frac{\epsilon}{6}.$$

Similarly, since $\left\|(\mathbf{x}_n)_{\omega_\alpha}-\mathbf{x}_{\omega_\alpha}\right\|_{\ell^p(S)}\to 0$, there is an $l\ge N$ such that

$$\left\|(\mathbf{x}_l)_{\omega_\alpha}-\mathbf{x}_{\omega_\alpha}\right\|_{\ell^p(S)}<\frac{\epsilon}{6}.$$

Then, since $k, l \geq N$,

$$\begin{aligned}\left\|\mathbf{x}_{\omega_\alpha} - \mathbf{x}_\omega\right\|_{\ell^p(S)} \leq& \left\|\mathbf{x}_{\omega_\alpha} - (\mathbf{x}_k)_{\omega_\alpha}\right\|_{\ell^p(S)} + \left\|(\mathbf{x}_k)_{\omega_\alpha} - (\mathbf{x}_N)_{\omega_\alpha}\right\|_{\ell^p(S)} \\ &+ \left\|(\mathbf{x}_N)_{\omega_\alpha} - (\mathbf{x}_N)_\omega\right\|_{\ell^p(S)} + \|(\mathbf{x}_N)_\omega - (\mathbf{x}_l)_\omega\|_{\ell^p(S)} \\ &+ \|(\mathbf{x}_l)_\omega - \mathbf{x}_\omega\|_{\ell^p(S)} \\ \leq& \frac{\epsilon}{6} + \|\mathbf{x}_k - \mathbf{x}_N\|_{\ell^p(S,\mathcal{A})} + \frac{\epsilon}{6} + \|\mathbf{x}_N - \mathbf{x}_l\|_{\ell^p(S,\mathcal{A})} + \frac{\epsilon}{6} < \epsilon.\end{aligned}$$

Thus $\mathbf{x} \in \ell^p(S, \mathcal{A})$. To see that $\|\mathbf{x}_n - \mathbf{x}\|_{\ell^p(S,\mathcal{A})} \to 0$, let $\eta > 0$. Then the Cauchy assumption on $\{\mathbf{x}_n\}$ implies that there is an N such that

$$\|\mathbf{x}_n - \mathbf{x}_m\|_{\ell^p(S,\mathcal{A})} < \frac{\eta}{6} \qquad \forall\ n,\ m \geq N.$$

Since

$$\|\mathbf{x}_N - \mathbf{x}\|_{\ell^p(S,\mathcal{A})} = \sup_{\omega\in\Omega} \|(\mathbf{x}_N)_\omega - (\mathbf{x})_\omega\|_{\ell^p(S)},$$

there is an $\omega_0 \in \Omega$ such that

$$\|\mathbf{x}_N - \mathbf{x}\|_{\ell^p(S,\mathcal{A})} < \left\|(\mathbf{x}_N)_{\omega_{\alpha_0}} - \mathbf{x}_{\omega_{\alpha_0}}\right\|_{\ell^p(S)} + \frac{\eta}{6}.$$

Since

$$\lim_{n\to\infty} \left\|(\mathbf{x}_n)_{\omega_{\alpha_0}} - \mathbf{x}_{\omega_{\alpha_0}}\right\|_{\ell^p(S)} = 0$$

there is a $k \geq N$ such that

$$\left\|(\mathbf{x}_k)_{\omega_{\alpha_0}} - \mathbf{x}_{\omega_{\alpha_0}}\right\|_{\ell^p(S)} < \frac{\eta}{6}.$$

Let $n \geq N$.

$$\begin{aligned}\|\mathbf{x}_n - \mathbf{x}\|_{\ell^p(S,\mathcal{A})} <& \|\mathbf{x}_n - \mathbf{x}_N\|_{\ell^p(S,\mathcal{A})} + \|\mathbf{x}_N - \mathbf{x}\|_{\ell^p(S,\mathcal{A})} \\ <& \frac{\eta}{6} + \left\|(\mathbf{x}_N)_{\omega_{\alpha_0}} - \mathbf{x}_{\omega_{\alpha_0}}\right\|_{\ell^p(S)} + \frac{\eta}{6} \\ \leq& \frac{\eta}{3} + \left\|(\mathbf{x}_N)_{\omega_{\alpha_0}} - (\mathbf{x}_k)_{\omega_{\alpha_0}}\right\|_{\ell^p(S)} + \left\|(\mathbf{x}_k)_{\omega_{\alpha_0}} - \mathbf{x}_{\omega_{\alpha_0}}\right\|_{\ell^p(S)} \\ <& \frac{\eta}{3} + \|\mathbf{x}_N - \mathbf{x}_k\|_{\ell^p(S,\mathcal{A})} + \frac{\eta}{6} < \eta.\end{aligned}$$

which finishes the proof of completeness of $\ell^p(S, \mathcal{A})$. □

3. Riesz representation for linear maps into $\mathcal{A}$

Riesz representation theorem for bounded linear functionals on sequence spaces has the following analogue for bounded linear maps from $\ell^p(S,\mathcal{A})$ to $\mathcal{A}$. Note that the second part is just Hölder's inequality.

THEOREM 4. *1. Let $\mathbf{y}\in\mathcal{A}^S$, $1<p<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\mathbf{y}\in\ell^q(S,\mathcal{A}) \quad \textit{iff} \quad \sum_{s\in S}\mathbf{y}(s)\mathbf{x}(s) \ \textit{converges in } \mathcal{A} \textit{ for each } \mathbf{x}\in\ell^p(S,\mathcal{A}).$$

2. For each $\mathbf{y}\in\ell^q(S,\mathcal{A})$

$$\left\|\sum_{s\in S}\mathbf{y}(s)\mathbf{x}(s)\right\|_{\mathcal{A}} \le \|\mathbf{y}\|_{\ell^q(S,\mathcal{A})}\|\mathbf{x}\|_{\ell^p(S,\mathcal{A})} \qquad \forall\ \mathbf{x}\in\ell^p(S,\mathcal{A})$$

and

$$\|\mathbf{y}\|_{\ell^q(S,\mathcal{A})} = \sup_{\substack{\mathbf{x}\in\ell^p(S,\mathcal{A})\\ \|\mathbf{x}\|\le 1}}\left\|\sum_{s\in S}\mathbf{y}(s)\mathbf{x}(s)\right\|_{\mathcal{A}}.$$

Proof. (1) [$\Rightarrow$] Assume $\mathbf{y}\in\ell^q(S,\mathcal{A})$. Let $\mathbf{x}\in\ell^p(S,\mathcal{A})$ and $1>\epsilon>0$. Then there exists an $F_\epsilon\in\mathcal{F}(S)$ such that

$$\left\|\mathbf{x}_{S\setminus G}\right\|_{\ell^p(S,\mathcal{A})} = \sup_{\omega\in\Omega}\left[\sum_{s\in S\setminus G}|\mathbf{x}_\omega(s)|^p\right]^{1/p} < \epsilon, \qquad \text{and}$$

$$\left\|\mathbf{y}_{S\setminus G}\right\|_{\ell^q(S,\mathcal{A})} = \sup_{\omega\in\Omega}\left[\sum_{s\in S\setminus G}|\mathbf{y}_\omega(s)|^q\right]^{1/q} < \epsilon \qquad \forall\ F_\epsilon\subseteq G\in\mathcal{F}(S).$$

Let $H\in\mathcal{F}(S\setminus F_\epsilon)$ and $\omega\in\Omega$.

$$\begin{aligned}\left\|\sum_{s\in H}\mathbf{x}(s)\mathbf{y}(s)\right\|_{\mathcal{A}} &= \sup_{\omega\in\Omega}\left|\sum_{s\in H}(\mathbf{x}(s))(\omega)(\mathbf{y}(s))(\omega)\right| \\ &\le \sup_{\omega\in\Omega}\|(\mathbf{x}_H)_\omega\|_{\ell^p(S)}\|(\mathbf{y}_H)_\omega\|_{\ell^q(S)} \le \epsilon^2 < \epsilon.\end{aligned}$$

This shows that the net, $\left\{\sum_{s\in F}\mathbf{x}(s)\mathbf{y}(s)\right\}_{F\in\mathcal{F}(S)}$, of finite partial sums, is a Cauchy net in the C^*-algebra $\mathcal{A}$. Hence it converges in $\mathcal{A}$.

Conversely, suppose $\mathbf{y} \in \mathcal{A}^S$ "multiplies every $\mathbf{x} \in \ell^p(S, \mathcal{A})$ to a summable function". Then

$$T\mathbf{x} = \sum_{s \in S} \mathbf{x}(s)\mathbf{y}(s) \qquad \mathbf{x} \in \ell^p(S, \mathcal{A})$$

is a well-defined map from $\ell^p(S, \mathcal{A})$ to $\mathcal{A}$. We will use the uniform boundedness principle to prove that T is a bounded linear transformation from $\ell^p(S, \mathcal{A})$ to $\mathcal{A}$. The linearity of T is routinely verified. For each $F \in \mathcal{F}(S)$, define

$$T_F\mathbf{x} = \sum_{s \in F} \mathbf{x}(s)\mathbf{y}(s) \qquad \mathbf{x} \in \ell^p(S, \mathcal{A}).$$

For each fixed $\mathbf{x} \in \ell^p(S, \mathcal{A})$, we have

$$\begin{aligned} \|T_F\mathbf{x}\| &= \left\| \sum_{s \in F} \mathbf{x}(s)\mathbf{y}(s) \right\|_{\mathcal{A}} = \sup_{\omega \in \Omega} \left| \left[\sum_{s \in F} \mathbf{x}(s)\mathbf{y}(s) \right] (\omega) \right| \\ &= \sup_{\omega \in \Omega} \left| \sum_{s \in F} (\mathbf{x}(s))(\omega)(\mathbf{y}(s))(\omega) \right| \\ &\leq \sup_{\omega \in \Omega} \left[\sum_{s \in F} |(\mathbf{x}(s))(\omega)|^p \right]^{1/p} \left[\sum_{s \in F} |(\mathbf{y}(s))(\omega)|^q \right]^{1/q} \\ &\leq \sup_{\omega \in \Omega} \left[\sum_{s \in F} |(\mathbf{y}(s))(\omega)|^q \right]^{1/q} \|\mathbf{x}\|_{\ell^p(S, \mathcal{A})}. \end{aligned}$$

Thus each T_F is abounded linear transformation from $\ell^p(S, \mathcal{A})$ to $\mathcal{A}$. Furthermore, from the convergence of the sum $\sum_{s \in S} \mathbf{x}(s)\mathbf{y}(s)$ in $\mathcal{A}$, it follows that the collection $\left\{ \sum_{s \in F} \mathbf{x}(s)(\mathbf{y}(s) \right\}_{F \in \mathcal{F}(S)}$ of finite partial sums is bounded (though convergent nets may not be bounded in general), and hence

$$\|T_F\mathbf{x}\|_{\mathcal{A}} \leq \sup_{G \in \mathcal{F}(S)} \left\| \sum_{s \in G} \mathbf{x}(s)\mathbf{y}(s) \right\|_{\mathcal{A}} < \infty \qquad \forall\ F \in \mathcal{F}(S).$$

The completeness of both $\ell^p(S, \mathcal{A})$ and $\mathcal{A}$ imply, by the uniform boundedness principle, that $\sup_{F \in \mathcal{F}(S)} \|T_F\| < \infty$. Then for each $\mathbf{x} \in \ell^p(S, \mathcal{A})$

we have

$$\begin{aligned}\|T\mathbf{x}\|_{\mathcal{A}} &= \sup_{\omega\in\Omega}|(T\mathbf{x})(\omega)| = \sup_{\omega\in\Omega}\left|\lim_{F\in\mathcal{F}(S)}\sum_{s\in F}(\mathbf{x}(s))(\omega)(\mathbf{y}(s))(\omega)\right| \\ &= \sup_{\omega\in\Omega}\lim_{F\in\mathcal{F}(S)}\left|\left[\sum_{s\in F}\mathbf{x}(s)\mathbf{y}(s)\right](\omega)\right| = \sup_{\omega\in\Omega}\lim_{F\in\mathcal{F}(S)}|(T_F\mathbf{x})(w)| \\ &\le \lim_{F\in\mathcal{F}(S)}\|T_F\mathbf{x}\|_{\mathcal{A}} \le \sup_{F\in\mathcal{F}(S)}\|T_F\|\,\|\mathbf{x}\|.\end{aligned}$$

Therefore T is bounded with $\|T\| \le \sup_{F\in\mathcal{F}(S)}\|T_F\|$.

To see that $\mathbf{y}\in\ell^q(S,\mathcal{A})$, fix an $\omega\in\Omega$ and $F\in\mathcal{F}(S)$. Since each $z\in\ell^p(S)$ gives rise to a $\mathbf{z}$ defined by $\mathbf{z}(s) = z(s)1$, where 1 is the identity element in $\mathcal{A}$, we have $\ell^p(S)\hookrightarrow\ell^p(S,\mathcal{A})$. Thus, by the scalar version of the Riesz representation theorem,

$$\begin{aligned}\|(\mathbf{y}_F)_\omega\|_{\ell^q(S)} &= \left[\sum_{s\in F}|(\mathbf{y}(s))(\omega)|^q\right]^{1/q} \le \sup_{\substack{z\in\ell^p(S)\\ \|z\|\le 1}}\left|\sum_{s\in F}z(\omega)(\mathbf{y}(s))(\omega)\right| \\ &\le \sup_{\substack{\mathbf{x}\in\ell^p(S,\mathcal{A})\\ \|\mathbf{x}\|\le 1}}\left|\sum_{s\in F}(\mathbf{x}(s))(\omega)(\mathbf{y}(s))(\omega)\right| \le \sup_{\substack{\mathbf{x}\in\ell^p(S,\mathcal{A})\\ \|\mathbf{x}\|\le 1}}\|T_F\mathbf{x}\|_{\mathcal{A}} \\ &= \sup_{\substack{\mathbf{x}\in\ell^p(S,\mathcal{A})\\ \|\mathbf{x}\|\le 1}}\|T(\mathbf{x}_F)\|_{\mathcal{A}} \le \|T\|.\end{aligned}$$

Hence

$$\|\mathbf{y}_\omega\|_{\ell^q(S)} = \sup_{F\in\mathcal{F}(S)}\|(\mathbf{y}_F)_\omega\|_{\ell^q(S)} \le \|T\|.$$

Since this holds for all $\omega\in\Omega$, $\mathbf{y}\in\ell^q(S,\mathcal{A})$ with $\|\mathbf{y}\|_{\ell^q(S,\mathcal{A})}\le\|T\|$.

(2) For $\mathbf{x}\in\ell^p(S,\mathcal{A})$ and $\mathbf{y}\in\ell^q(S,\mathcal{A})$, we have, for each $\omega\in\Omega$,

$$\begin{aligned}\left\|\sum_{s\in S}\mathbf{x}(s)\mathbf{y}(s)\right\|_{\mathcal{A}} &= \sup_{\omega\in\Omega}\left|\sum_{s\in S}(\mathbf{x}(s))(w)(\mathbf{y}(s))(\omega)\right| \\ &\le \sup_{\omega\in\Omega}\left[\sum_{s\in S}|(\mathbf{x}(s))(\omega)|^p\right]^{1/p}\left[\sum_{s\in S}|(\mathbf{y}(s))(\omega)|^q\right]^{1/q} \\ &= \sup_{\omega\in\Omega}\|\mathbf{x}_\omega\|_{\ell^p(S)}\|\mathbf{y}_\omega\|_{\ell^q(S)} \le \|\mathbf{x}\|_{\ell^p(S,\mathcal{A})}\|\mathbf{y}\|_{\ell^q(S,\mathcal{A})}.\end{aligned}$$

It then follows that

$$\sup_{\substack{\mathbf{x}\in\ell^p(S,\mathcal{A})\\ \|\mathbf{x}\|\le 1}} \left\|\sum_{s\in S}\mathbf{x}(s)\mathbf{y}(s)\right\|_{\mathcal{A}} \le \|\mathbf{y}\|_{\ell^q(S,\mathcal{A})}. \qquad (*)$$

To prove the opposite inequality, let $\epsilon > 0$ be given. Then by definition of $\|\mathbf{y}\|_{\ell^q(S,\mathcal{A})}$, there is an $\omega \in \Omega$ such that

$$\|\mathbf{y}_w\|_{\ell^q(S)} = \left[\sum_{s\in S}|(\mathbf{y}(s))(\omega)|^q\right]^{1/q} > \|\mathbf{y}\|_{\ell^q(S,\mathcal{A})} - \frac{\epsilon}{2}.$$

Since

$$\|\mathbf{y}_\omega\|_{\ell^q(S)} = \sup_{\substack{z\in\ell^p(S)\\ \|z\|\le 1}} \left|\sum_{s\in S} z(s)\mathbf{y}_\omega(s)\right|$$

there exists $z \in \ell^q(S)$ such that

$$\|z\|_{\ell^q(S)} \le 1 \qquad \text{and} \qquad \left|\sum_{s\in S} z(s)(\mathbf{y}(s))(\omega)\right| > \|\mathbf{y}_\omega\|_{\ell^q(S)} - \frac{\epsilon}{2}.$$

Let $\mathbf{x}(s) = z(s)1$, with 1 being the identity of $\mathcal{A}$. Then we have $\mathbf{x} \in \ell^p(S,\mathcal{A})$ with $\|\mathbf{x}\|_{\ell^p(S,\mathcal{A})} = \|z\|_{\ell^p(S)}$. Furthermore,

$$\begin{aligned}\left\|\sum_{s\in S}\mathbf{x}(s)\mathbf{y}(s)\right\|_{\mathcal{A}} &\ge \left|\sum_{s\in S}(\mathbf{x}(s))(\omega)(\mathbf{y}(s))(\omega)\right| = \left|\sum_{s\in S} z(s)(\mathbf{y}(s))(\omega)\right| \\ &> \|\mathbf{y}_\omega\|_{\ell^q(S)} - \frac{\epsilon}{2} > \|\mathbf{y}\|_{\ell^q(S,\mathcal{A})} - \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary,

$$\left\|\sum_{s\in S}\mathbf{x}(s)\mathbf{y}(s)\right\|_{\mathcal{A}} \ge \|\mathbf{y}\|_{\ell^q(S,\mathcal{A})}. \qquad (**)$$

Combining the two inequalities (*) and (**), equality in (2) holds. □

4. $\mathcal{A}$-matrix operators

We will fix a $1 < p < \infty$, a commutative C^*-algebra $\mathcal{A} = C(\Omega)$, and $\mathcal{X} = \ell^p(S, \mathcal{A})$. A function $A : S \times S \to \mathcal{A}$ (commutative C^*-algebra) is said to define an *operator* on $\mathcal{X} := \ell^p(S, \mathcal{A})$ if for each $\mathbf{x} \in \mathcal{X}$

$$(A\mathbf{x})(s) := \sum_{t \in S} [A(s,t)]\mathbf{x}(t) \quad \text{converges in } \mathcal{A}, \ \forall \ s \in S, \qquad \text{and}$$

$$\text{the function } A\mathbf{x} : s \mapsto (A\mathbf{x})(s), \text{ as defined, is in } \mathcal{X}.$$

A uniform boundedness argument shows that whenever A defines an operator, the operator is bounded [9]. Such an operator will be called an $\mathcal{A}$-*matrix operator* or simply a *matrix operator*.

For each $\mathcal{A}$-matrix operator A and each subset $G \subseteq S$, denote by $A_{\underline{G}}$ the function $\chi_{G \times S} A$ (regarding A as a matrix, the "horizontal G band" of A), i.e.,

$$A_{\underline{G}}(s,t) = \begin{cases} A(s,t) & \text{for all } s \in G, \text{ and all } t \in S \\ 0 & \text{otherwise.} \end{cases}$$

Denote by $A_{G|}$ the function $\chi_{S \times G} A$ (the "vertical G strip" of A), i.e.,

$$A_{G|}(s,t) = \begin{cases} A(s,t) & \text{for all } s \in S, \text{ and all } t \in G \\ 0 & \text{otherwise.} \end{cases}$$

For $G, H \subseteq S$, the function $\chi_{G \times H} A = (A_{\underline{G}})_{H|}$ will be denoted by $A_{\underline{(G \times H)|}}$ (the $G \times H$ "corner" of A), i.e.,

$$A_{\underline{(G \times H)|}}(s,t) = (A_{\underline{G}})_{H|}(s,t) = \begin{cases} A(s,t) & \text{if } (s,t) \in G \times H \\ 0 & \text{otherwise.} \end{cases}$$

Thus $A_{\underline{G}} = A_{\underline{(G \times S)|}}$ and $A_{H|} = A_{\underline{(S \times H)|}}$.

We will also use $A_{\underline{G|}}$ to denote the function $A_{\underline{(G \times G)|}} = \chi_{G \times G} A$, i.e.,

$$\begin{aligned} A_{\underline{G|}}(s,t) &= A_{\underline{(G \times G)|}}(s,t) = \left(A_{\underline{G}}\right)_{G|}(s,t) = \left(A_{G|}\right)_{\underline{G}}(s,t) \\ &= \begin{cases} A(s,t) & \text{if } s,t \in G \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We now establish some useful properties of $\mathcal{A}$-matrix operators and their "rows", "columns", and "corners".

Proposition 5. *Let $\mathcal{M}$ denote the space of all $\mathcal{A}$-matrix operators on $\ell^p(S,\mathcal{A})$, and $A \in \mathcal{M}$. Then we have the following.*

1. *The map $\mathbf{x} \mapsto A\mathbf{x}$ is a bounded linear operator on $\ell^p(S,\mathcal{A})$.*
2. *$\|A(s,t)\|_{\mathcal{A}} \le \|A\|$ for all $s,t \in S$.*
3. *For each $G, H \subseteq S$*

$$\left\|A_{\underline{G\times H|}}\right\| \le \max\left\{\left\|A_{\underline{G}}\right\|, \left\|A_{H|}\right\|\right\} \le \|A\|.$$

4. *For each $s \in S$, the function $A_{\underline{\{s\}}}$ is an $\mathcal{A}$-matrix operator on $\ell^p(S,\mathcal{A})$ and the function $A(s,\cdot)$ belongs to $\ell^q(S,\mathcal{A})$ with*

$$\left\|A_{\underline{\{s\}}}\right\| = \|A(s,\cdot)\|_{\ell^q(S,\mathcal{A})} \le \|A\|.$$

Proof. (1) For each $s \in S$, since the function $A(s,\cdot)$ "multiplies" each function in $\ell^p(S,\mathcal{A})$ to an $\mathcal{A}$-summable function in $\mathcal{A}^S$, by Theorem 4, it is in $\ell^q(S,\mathcal{A})$, and hence (4) follows.

For each $F \in \mathcal{F}(S)$, the map $T_F : \mathbf{x} \mapsto A_{\underline{F}}\mathbf{x}$ on $\ell^p(S,\mathcal{A})$ is clearly linear Boundedness follows from

$$\|T_F\| = \left\|\sum_{s\in F} T_{\{s\}}\right\| \le \sum_{s\in F}\left\|T_{\{s\}}\right\| = \left[\sum_{s\in F}\|A(s,\cdot)\|^p_{\ell^q(S,\mathcal{A})}\right]^{1/p}.$$

Furthermore, for a fixed $\mathbf{x} \in \ell^p(S,\mathcal{A})$,

$$\|T_F(\mathbf{x})\|_{\ell^p(S,\mathcal{A})} = \|(A\mathbf{x})_F\|_{\ell^p(S,\mathcal{A})} \le \|A\mathbf{x}\|_{\ell^p(S,\mathcal{A})} < \infty \qquad \forall\, F \in \mathcal{F}(S).$$

Thus, by uniform boundedness principle, there is $M > 0$ such that $\|T_F\| \le M$ for all $F \in \mathcal{F}(S)$. Thus, for each $\mathbf{x} \in \ell^p(S,\mathcal{A})$,

$$\begin{aligned}\|A\mathbf{x}\|_{\ell^p(S,\mathcal{A})} &= \sup_{F\in\mathcal{F}(S)}\|(A\mathbf{x})_F\|_{\ell^p(S,\mathcal{A})} = \sup_{F\in\mathcal{F}(S)}\|T_F(\mathbf{x})\|_{\ell^p(S,\mathcal{A})}\\ &\le \sup_{F\in\mathcal{F}(S)}\|T_F\|\,\|\mathbf{x}\|_{\ell^p(S,\mathcal{A})} \le M\,\|\mathbf{x}\|_{\ell^p(S,\mathcal{A})} \qquad \forall\,\mathbf{x}\in\ell^p(S,\mathcal{A}).\end{aligned}$$

So $\mathbf{x} \mapsto A\mathbf{x}$ is bounded.

The following observation proves (3)

$$\begin{aligned}\left\|(A_{\underline{G\times H|}})\mathbf{x}\right\|_{\ell^p(S,\mathcal{A})} &= \left\|(A_{\underline{G}})(\mathbf{x}_H)\right\|_{\ell^p(S,\mathcal{A})} = \|(A(\mathbf{x}_H))_G\| \le \|A(\mathbf{x}_H)\|\\ &\le \|A\|\,\|\mathbf{x}_H\|\,\ell^p(S,\mathcal{A}) \le \|A\|\,\|x\|_{\ell^p(S,\mathcal{A})} \qquad \forall\,\mathbf{x}\in\ell^p(S,\mathcal{A}).\end{aligned}$$

Note that (2) follows from (1) and (3).

□

We now identify the space of $\mathcal{A}$-matrix operators that is the C^*-matrix analogue of operators on the ℓ^p space.

Proposition 6. *The space $\mathcal{M}$ of all $A \in \mathcal{A}^{S\times S}$ that are $\mathcal{A}$-matrix operators on $\mathcal{X} = \ell^p(S,\mathcal{A})$, $1 < p < \infty$, is a Banach subalgebra of the space of bounded linear operators on $\mathcal{X}$, under the composition of functions on $\mathcal{X}$ and the operator norm.*

A more general version of this result is proved in [9]. We give a direct proof of this version for completeness.

Proof. That each $A \in \mathcal{M}$ defines a bounded linear operator $T_A : \mathbf{x} \to A\mathbf{x}$ on $\mathcal{X}$ follows from Proposition 5 (1).

For a fixed $s \in S$, since $\sum_{t\in S}(A(s,t))(\mathbf{x}(t))$ converges for each $\mathbf{x} \in \ell^p(S,\mathcal{A})$, by (1) of Theorem 1 the function $A(s,\cdot)$ belongs to $\ell^q(S,\mathcal{A})$. For each $B \in \mathcal{M}$ and $t \in S$, since the function $\mathbf{e}_t$ defined by $\mathbf{e}_t(t) = 1$, and $\mathbf{e}_t(u) = 0$ for all $u \in S\setminus\{t\}$, is in $\ell^p(S,\mathcal{A})$, the function $B(\cdot,t)$ given by

$$B(s,t) = \sum_{u\in S}(B(s,u))(\mathbf{e}_t(u)) = (B\mathbf{e}_t)(s) \qquad \forall\ s \in S$$

also belongs to $\ell^p(S,\mathcal{A})$. Thus, for each $(s,t) \in S\times S$, the sum

$$\sum_{u\in S}(A(s,u))(B(u,t))$$

converges in $\mathcal{A}$ by (1) of Theorem 1. Thus we may define the function AB on $S\times S$ as follows:

$$(AB)(s,t) = \sum_{u\in S}(A(s,u))(B(u,t)) \qquad \forall\ (s,t) \in S\times S.$$

We show that $AB \in \mathcal{M}$. For this it suffices to show that $(AB)\mathbf{x}$ exists and belongs to $\ell^p(S,\mathcal{A})$ for each $\mathbf{x} \in \ell^p(S,\mathcal{A})$. The existence of $(AB)\mathbf{x}$ is just the usual associativity extended to the infinite sums, and we omit its routine verification. Since $B\mathbf{x} \in \ell^p(S,\mathcal{A})$, we have $(AB)\mathbf{x} = A(B\mathbf{x})$

by the extended associativity. For $F \in \mathcal{F}(S)$ we have

$$\sum_{t\in F}\left[\sum_{u\in S}(A(s,u)B(u,t))\right]\mathbf{x}(t) = \sum_{t\in F}\left[\sum_{u\in S}(A(s,u)B(u,t))\mathbf{x}(t)\right]$$

$$= \sum_{t\in F}\left[\sum_{u\in S}A(s,u)(B(u,t)\mathbf{x}(t))\right] = \sum_{u\in S}\left[\sum_{t\in F}A(s,u)(B(u,t)\mathbf{x}(t))\right]$$

$$= \sum_{u\in S}A(s,u)\left[\sum_{t\in F}B(u,t)\mathbf{x}(t)\right] = \sum_{u\in S}A(s,u)\left[\sum_{t\in S}B(u,t)\mathbf{x}_F(t)\right]$$

$$= \sum_{u\in S}A(s,u)(B\mathbf{x}_F)(u) = (A(B\mathbf{x}_F))(s).$$

Since $B\mathbf{x} \in \ell^p(S,\mathcal{A})$; and A, B both define bounded linear operators on $\ell^p(S,\mathcal{A})$; and $\lim_{F\in\mathcal{F}(S)}\|B\mathbf{x} - B\mathbf{x}_F\| \to 0$; we have

$$\sum_{t\in S}\left[\sum_{u\in S}A(s,u)B(u,t)\right]\mathbf{x}(t) = \lim_{F\in\mathcal{F}(S)}\sum_{t\in F}\left[\sum_{u\in S}A(s,u)B(u,t)\right]\mathbf{x}(t)$$

$$= \lim_{s\in F}(A(B\mathbf{x}_F))(s) = (A(B\mathbf{x}))(s).$$

This shows that $((AB)\mathbf{x})(s) = (A(B\mathbf{x}))(s)$ for all $s \in S$, and the sum

$$\sum_{t\in S}\left[\sum_{u\in S}A(s,u)B(u,t)\right]\mathbf{x}(t) \qquad \text{converges in } \mathcal{A}, \text{ for each } s \in S.$$

That is $(AB)\mathbf{x}$ exists for each $\mathbf{x} \in \ell^p(S,\mathcal{A})$. It follows also that

$$\|(AB)\mathbf{x}\| = \sup_{s\in S}\|((AB)\mathbf{x})(s)\| = \sup_{s\in S}\|(A(B\mathbf{x}))(s)\| = \|A(B\mathbf{x})\|$$

$$\leq \|A\|\,\|B\|\,\|x\|, \quad \text{i.e., } (AB)\mathbf{x} \in \ell^p(S,\mathcal{A}).$$

This shows that $\mathcal{M}$ is closed under the multiplication of matrix operators, and $\|AB\| \leq \|A\|\,\|B\|$ for all $A, B \in \mathcal{M}$.

To see completeness of $\mathcal{M}$, let $\{A_n\} \subseteq \mathcal{M}$ be a Cauchy sequence. For each $s, t \in S$, the sequence $\{A_n(s,t)\}$ is a Cauchy sequence in the complete space $\mathcal{A}$, and hence has a limit $A(s,t) \in \mathcal{A}$. For the function $A \in \mathcal{A}^{S\times S}$ to be in $\mathcal{M}$, we show that for each $s \in S$, the sum $(A\mathbf{x})(s) =: \sum_{t\in S}A(s,t)\mathbf{x}(t)$ converges for each $\mathbf{x} \in \ell^p(S,\mathcal{A})$ and that $A\mathbf{x} \in \ell^p(S,\mathcal{A})$.

Fix $s \in S$ and $\mathbf{x} \in [\ell^p(S,\mathcal{A})]_1$. Since the function $f_s(t) = A(s,t),\ t \in S,$ is the limit of the sequence $\{A_n(s,\cdot)\}$ in $\ell^q(S,\mathcal{A})$, f_s belongs to

$\ell^q(S,\mathcal{A})$. Hence we have the guaranteed convergence of

$$\sum_{t\in S} A(s,t)\mathbf{x}(t) \qquad \forall\, \mathbf{x} \in \ell^p(S,\mathcal{A}).$$

To see that $A\mathbf{x}$ is in fact in $\ell^p(S,\mathcal{A})$, let $\epsilon > 0$. The Cauchy assumption on $\{A_n\}$ implies the existence of an N such that

$$\|A_n - A_k\| < \frac{\epsilon}{3} \qquad \forall\, n,\ k \geq N.$$

Since $A_N\mathbf{x} \in \ell^p(S,\mathcal{A})$, there is an $F \in \mathcal{F}(S)$ such that

$$\|(A_N\mathbf{x})_G\|_{\ell^p(S,\mathcal{A})} < \frac{\epsilon}{3} \qquad \forall\, G \in \mathcal{F}(S \setminus F).$$

Let $G \in \mathcal{F}(S\setminus F)$. Since

$$(A_n - A)_{\underline{G}} = \sum_{s\in G} (A_n - A)_{\underline{\{s\}}}$$

and each $\left\|(A_n - A)_{\underline{\{s\}}}\right\|_{\mathcal{M}} = \|(A_n - A)(s,\ \cdot)\|_{\ell^q(S,\mathcal{A})} \to 0$ as $n \to \infty$, by Proposition 5 (4), we have

$$\lim_{n\to\infty} \left\|(A_n - A)_{\underline{G}}\right\| = \lim_{n\to\infty} \left\|\sum_{s\in G}(A_n - A)_{\underline{\{s\}}}\right\| = 0. \qquad (\dagger)$$

This together with the finiteness of G gives rise to a $k \geq N$ such that

$$\left\|(A_k - A)_{\underline{G}}\right\| < \frac{\epsilon}{3}.$$

Thus

$$\begin{aligned}
\|(A\mathbf{x})_G\|_{\ell^p(S,\mathcal{A})} &\leq \|(A\mathbf{x})_G - (A_k\mathbf{x})_G\|_{\ell^p(S,\mathcal{A})} \\
&\qquad + \|(A_k\mathbf{x})_G - (A_N\mathbf{x})_G\|_{\ell^p(S,\mathcal{A})} + \|(A_N\mathbf{x})_G\|_{\ell^p(S,\mathcal{A})} \\
&< \left\|(A - A_k)_{\underline{G}}\mathbf{x}_G\right\|_{\ell^p(S,\mathcal{A})} + \|A_k - A_N\| + \frac{\epsilon}{3} \\
&< \left\|(A - A_k)_{\underline{G}}\right\| \|\mathbf{x}\|_{\ell^p(S,\mathcal{A})} + \frac{2\epsilon}{3} < \epsilon.
\end{aligned}$$

Since $G \in \mathcal{F}(S\setminus F)$ is arbitrary,

$$\left\|(A\mathbf{x})_{S\setminus F}\right\|_{\ell^p(S,\mathcal{A})} = \sup_{G\in\mathcal{F}(S\setminus F)} \|(A\mathbf{x})_G\|_{\ell^p(S,\mathcal{A})} \leq \epsilon.$$

This shows that for each $\epsilon > 0$ there is an $F \in \mathcal{F}(S)$ such that

$$\left\|(A\mathbf{x})_{S\setminus F}\right\|_{\ell^p(S,\mathcal{A})} \leq \epsilon.$$

Thus $A\mathbf{x} \in \ell^p(S,\mathcal{A})$ by Proposition 2 part (2). Since $\mathbf{x} \in [\ell^p(S,\mathcal{A})]_1$ is arbitrary, $A \in \mathcal{M}$.

To prove the convergence of $\{A_n\}$ to A in the operator norm, from the basic definitions, let $\eta > 0$. Then by the Cauchy assumption on $\{A_n\}$ there is an integer N such that

$$\|A_n - A_m\| < \frac{\eta}{4} \qquad \forall\, n,\ m \geq N.$$

Let $n \geq N$. Since $A - A_n \in \mathcal{M}$, there is an $\mathbf{x} \in [\ell^p(S,\mathcal{A})]_1$ such that

$$\|A - A_n\| < \|(A - A_n)\mathbf{x}\|_{\ell^p(S,\mathcal{A})} + \frac{\eta}{4}.$$

Since $(A - A_n)\mathbf{x} \in \ell^p(S,\mathcal{A})$, by Proposition 2 part (2), there is an $F \in \mathcal{F}(S)$ such that

$$\|(A - A_n)\mathbf{x}\|_{\ell^p(S,\mathcal{A})} - \frac{\eta}{4} < \|[(A - A_n)\mathbf{x}]_G\|_{\ell^p(S,\mathcal{A})} = \left\|(A - A_n)_{\underline{G}}\mathbf{x}\right\|_{\ell^p(S,\mathcal{A})} \qquad \forall\ F \subseteq G \in \mathcal{F}(S).$$

The argument for (†) also shows that the finiteness of $F \in \mathcal{F}(S)$ implies the existence of a $k \in \mathbb{N}$ such that

$$\left\|(A - A_k)_{\underline{F}}\right\| < \frac{\eta}{4}.$$

Thus

$$\begin{aligned}\|A - A_n\| &= \|(A - A_n)\mathbf{x}\|_{\ell^p(S,\mathcal{A})} + \frac{\eta}{4} < \left\|(A - A_n)_{\underline{F}}\mathbf{x}\right\|_{\ell^p(S,\mathcal{A})} + \frac{\eta}{2}\\ &\leq \left\|(A - A_k)_{\underline{F}}\right\| + \left\|(A_k - A_n)_{\underline{F}}\right\| + \frac{\eta}{2}\\ &< \frac{\eta}{4} + \|A_k - A_n\| + \frac{\eta}{2} < \eta.\end{aligned}$$

Hence $\|A - A_n\| \to 0$, and completing the proof. □

We have the following $\mathcal{A}$-matrix operators analogue of block matrix operators on a sequence space.

Proposition 7. *For each matrix operator A in $\mathcal{M}$, the following are true.*

1. For $G, H \subseteq S$

$$\left\|A_{\underline{G}}\right\| \leq \|A\|, \quad \left\|A_{G|}\right\| \leq \|A\|, \quad \text{and} \quad \left\|A_{\underline{(G\times H)|}}\right\| \leq \|A\|.$$

2. If $G \subseteq H \subseteq S$, then

$$\left\|A_{\underline{G}}\right\| \leq \left\|A_{\underline{H}}\right\| \qquad \text{and} \qquad \left\|A_{G|}\right\| \leq \left\|A_{H|}\right\|.$$

3. If $H \subseteq S \setminus G$ and $B \in \mathcal{M}$, then

$$\left\|(A_{\underline{G}})_{H|} + (B_{\underline{H}})_{G|}\right\| = \max\left\{\left\|(A_{\underline{G}})_{H|}\right\|, \left\|(B_{\underline{H}})_{G|}\right\|\right\}.$$

These all follow straightforwardly from the fact that the norm on $\ell^p(S)$ and $\ell^p(S, \mathcal{A})$ are monotone, i.e., $\|\mathbf{x}_G\| \leq \|\mathbf{x}_H\|$ for all $G \subseteq H \subseteq S$ and all $\mathbf{x} \in \ell^p(S, \mathcal{A})$. We omit the routine verifications.

5. The subclass $\mathcal{K}$ of $\mathcal{M}$

We now identify a subclass of $\mathcal{M}$ that is an analogue of the compact operators in the bounded operators on the ℓ^p sequence space. The following definition is an analogue to an equivalent formulation of compact operators on sequence spaces.

Denote by $\mathcal{K}$ the space of all $A \in \mathcal{M}$ that are (operator) norm limits of the net $\left\{A_{\underline{F|}}\right\}_{F\in\mathcal{F}(S)}$, i.e.,

$$\mathcal{K} = \left\{K \in \mathcal{M} : \lim_{F\in\mathcal{F}(S)} \left\|K - K_{\underline{F|}}\right\| = 0\right\}.$$

Proposition 8. *$\mathcal{K}$ is a closed subspace of $\mathcal{M}$.*

Proof. Let $\{K_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{K}$ such that $\|K_n - A\| \to 0$ for some $A \in \mathcal{M}$. Let $\epsilon > 0$ be given. Then there is an $N \in \mathbb{N}$ such that

$$\|K_n - A\| < \frac{\epsilon}{4} \qquad \forall\, n \geq N.$$

Now, since $K_N \in \mathcal{K}$, there is an $F_\epsilon \in \mathcal{F}(S)$ such that

$$\left\|(K_N)_{\underline{F|}} - K_N\right\| < \frac{\epsilon}{4} \qquad \forall\, F_\epsilon \subseteq F \in \mathcal{F}(S).$$

Let $F_\epsilon \subseteq F \in \mathcal{F}(S)$. Then

$$\begin{aligned}\left\|A_{\underline{F}\rfloor} - A\right\| &\le \left\|A_{\underline{F}\rfloor} - (K_N)_{\underline{F}\rfloor}\right\| + \left\|(K_N)_{\underline{F}\rfloor} - K_N\right\| + \|K_N - A\| \\ &\le \|A - K_N\| + \frac{\epsilon}{2} < \epsilon.\end{aligned}$$

Thus $A \in \mathcal{K}$. Hence $\mathcal{K}$ is closed in $\mathcal{M}$. □

The following lemma provides a variety of ways to construct elements of $\mathcal{K}$. The first two are just operator matrices with finite number of nonzero rows or columns are in $\mathcal{K}$. The third one is a combination of the first two. Fourth and fifth are about sums of disjoint horizontal and vertical bands. That is, with the respective correspondence, $\mathcal{M}$ and $\mathcal{K}$ have all the desirable properties for Dixmier's decomposition theorem to hold for $\mathcal{B}(\ell^2)$ (all bounded operators) and $\mathcal{K}(\ell^2)$ (all compact operators).

Lemma 9. *1. Let $A \in \mathcal{M}$ be such that $A = A_{\underline{F}}$ for some $F \in \mathcal{F}(S)$. Then $A \in \mathcal{K}$.*

2. Let $A \in \mathcal{M}$ satisfy $A = A_{F|}$ for some $F \in \mathcal{F}(S)$. Then $A \in \mathcal{K}$.

3. Let $A \in \mathcal{M}$ and $F \in \mathcal{F}(S)$. Then $K = A - A_{\underline{(S\setminus F)|}} \in \mathcal{K}$.

4. Let $\{A_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{M}$ and $\{F_n\}_{n\in\mathbb{N}}$ be a pairwise disjoint sequence in $\mathcal{F}(S)$ such that $A_j = (A_j)_{\underline{F_j}}$ for all $j \in \mathbb{N}$. Then for each ℓ^p sequence $\{\xi_n\}_{n\in\mathbb{N}}$ in $\mathbb{C}$,

$$\textit{the sum} \quad B := \sum_{n=1}^{\infty} \xi_n A_n \quad \textit{converges in } \mathcal{M}, \qquad \textit{and} \qquad B \in \mathcal{K}.$$

5. Let $A \in \mathcal{M}$. Suppose $\{F_n\}_{n\in\mathbb{N}}$ be a pairwise disjoint sequence in $\mathcal{F}(S)$ and $A_j = (A)_{(F_j)|}$ for all $j \in \mathbb{N}$. Then for each ℓ^q sequence $\{\xi_n\}_{n\in\mathbb{N}}$ in $\mathbb{C}$,

$$\textit{the sum} \quad B := \sum_{n=1}^{\infty} \xi_n A_n \quad \textit{converges in } \mathcal{M}, \qquad \textit{and} \qquad B \in \mathcal{K}.$$

Proof. (1) Let $\epsilon > 0$. Membership in $\mathcal{M}$ ensures, by (1) of Theorem 1, that for each $s \in F$, the function $A(s,\cdot)$ is in $\ell^q(S,\mathcal{A})$. Thus, by the finiteness of F, there is a $G_\epsilon \in \mathcal{F}(S)$ such that

$$\left\|[A(s,\cdot)]_{(S\setminus G)}\right\|_{\ell^q(S,\mathcal{A})} < \frac{\epsilon}{(\mathrm{Card}(F)+1)^{1/p}} \quad \forall\, s \in F, \forall\, G \in \mathcal{F}(S \setminus G_\epsilon).$$

Let $F_\epsilon = G_\epsilon \cup F$. We show that

$$\left\| A - A_{\underline{G}|} \right\| < \epsilon \qquad \text{for all } F_\epsilon \subseteq G \in \mathcal{F}(S).$$

For if $\mathbf{x} \in [\mathcal{X}]_1$ and $F_\epsilon \subseteq G \in \mathcal{F}(S)$, since $F \subseteq G$ and $A = A_{\underline{F}}$, we have

$$\begin{aligned}
\left\| (A - A_{\underline{G}|})\mathbf{x} \right\| &= \left\| (A_{\underline{(F \times (S\setminus G))}|})\mathbf{x}_{(S\setminus G)} \right\| \\
&\leq \left[\sum_{s \in F} \left(\left\| [A(s,\cdot)]_{(S\setminus G)} \right\|_{\ell^p(S,\mathcal{A})} \|\mathbf{x}\|_{\ell^p(S)} \right)^p \right]^{1/p} \\
&\leq \left[\sum_{s \in F} \left\| [A(s,\cdot)]_{(S\setminus G)} \right\|^p_{\ell^p(S,\mathcal{A})} \right]^{1/p} \\
&< \left[(\mathrm{Card}(F)) \left(\frac{\epsilon}{(\mathrm{Card}(F)+1)^{1/p}} \right)^p \right]^{1/p} < \epsilon.
\end{aligned}$$

(2) For each $t \in F$, $A(\cdot, t) = A\mathbf{e}_t \in \ell^p(S, \mathcal{A})$, since the function $\mathbf{e}_t = \mathbf{e}_{\{t\}}$ defined by $\mathbf{e}_t(s) = 1$ for $s = t$ and $\mathbf{e}_s(t) = 0$ for $t \in S \setminus \{t\}$, is a member of $\ell^p(S, \mathcal{A})$. Let $\epsilon > 0$. Since F is finite, memberships in $\ell^p(S, \mathcal{A})$ imply that there is a $G_\epsilon \in \mathcal{F}(S)$ such that

$$\left\| (A\mathbf{e}_t)_{(S\setminus G_\epsilon)} \right\| < \frac{\epsilon}{1 + \mathrm{Card}(F)} \qquad \forall \;\; t \in F.$$

Let $F_\epsilon = G_\epsilon \cup F$. Let $F_\epsilon \subseteq G \in \mathcal{F}(S)$. Then

$$\begin{aligned}
\left\| A - A_{\underline{G}|} \right\| &= \left\| A_{\underline{((S\setminus G)\times F)}|} \right\| = \left\| \sum_{t\in F} A_{\underline{((S\setminus G)\times \{t\})}|} \right\| \leq \sum_{t \in F} \left\| A_{\underline{((S\setminus G)\times\{t\})}|} \right\| \\
&= \sum_{t\in F} \left\| (A\mathbf{e}_t)_{(S\setminus G)} \right\| \leq \sum_{t \in F} \left\| (A\mathbf{e}_t)_{(S\setminus G_\epsilon)} \right\| < \sum_{t\in F} \frac{\epsilon}{1 + \mathrm{Card}(F)} < \epsilon.
\end{aligned}$$

(3) It suffices to observe that, as functions on $S \times S$,

$$K = A - A_{\underline{(S\setminus F)}|} = A_{F|} + A_{\underline{F}} - A_{\underline{F}|} \qquad \text{which is a sum with all summands in } \mathcal{K}.$$

(4) From part (1), each $A_j \in \mathcal{K}$, thus for each $k \in \mathbb{N}$,

$$B_k = \sum_{j=1}^{k} \xi_j A_j \in \mathcal{K}.$$

It suffices to show that the sequence $\{B_k\}_{k\in\mathbb{N}}$ of partial sums is a Cauchy sequence in $\mathcal{K}$ and hence converges to some $B \in \mathcal{K}$. By assumption, there is an M such that $\|A_j\| \le M$ for all $j \in \mathbb{N}$. For $\epsilon > 0$, since $\{\xi_j\} \in \ell^p$, there is an $N \in \mathbb{N}$ such that

$$\left[\sum_{j=N+1}^{\infty} |\xi_j|^p\right]^{1/p} < \frac{\epsilon}{(1+M)}.$$

For each $n > N$, $\mathbf{x} \in [\mathcal{X}]_1$, we have, from disjointness of $\{F_j\}$,

$$\left\|\left[\sum_{j=N+1}^{n} \xi_j A_j\right]\mathbf{x}\right\|_{\ell^p(S,\mathcal{A})} = \sup_{\omega\in\Omega}\left\|\sum_{j=N+1}^{n} \xi_j[(A_j\mathbf{x})_{F_j}](\omega)\right\|_{\ell^p(S)}$$

$$= \sup_{\omega\in\Omega}\left[\sum_{j=N+1}^{n} \left\|\xi_j[(A_j\mathbf{x})_{F_j}](\omega)\right\|_{\ell^p(S)}^p\right]^{1/p}$$

$$= \sup_{\omega\in\Omega}\left[\sum_{j=N+1}^{n} |\xi_j|^p \left\|[(A_j\mathbf{x})_{F_j}](\omega)\right\|_{\ell^p(S)}^p\right]^{1/p}$$

$$\le \left[\sum_{j=N+1}^{n} |\xi_j|^p \|A_j\|^p \|\mathbf{x}\|^p\right]^{1/p} = \left[\sum_{j=N+1}^{n} |\xi_j|^p M^p\right]^{1/p} < \epsilon.$$

(5) From part (2) each $A_j \in \mathcal{K}$. We show that the sequence of partial sums in $\mathcal{K}$ is a Cauchy sequence. Let $\epsilon > 0$. Since $\{\xi_n\}$ is an ℓ^q sequence, there is an $N \in \mathbb{N}$ such that

$$\sum_{j=N+1}^{\infty} |\xi_j|^q < \left[\frac{\epsilon}{\|A\|+1}\right]^q.$$

Let $\mathbf{x} \in [\mathcal{X}]_1$, and $n > N+1$. For each $j \in \mathbb{N}$, denote by $\mathbf{x}_j = \mathbf{x}_{F_j}$. Then the $\mathbf{x}_j$'s have pairwise disjoint supports by assumption on the F_j's.

Since $A_j = A_{(F_j)|}$, $A_j\mathbf{x} = A_j\mathbf{x}_j = A\mathbf{x}_j$.

Note that for functions $x, y \in \ell^p(S)$ with disjoint supports (i.e., for all $s \in S$, at most one of $x(s)$ and $y(s)$ is nonzero),

$$\|x+y\|_{\ell^p(S)}^p = \|x\|_{\ell^p(S)}^p + \|y\|_{\ell^p(S)}^p.$$

Thus for $\xi, \eta \in \mathbb{C}$, we have (from $a^p + b^p \leq (a+b)^p$ for $a, b \geq 0$)

$$\|\xi x + \eta y\|_{\ell^p(S)} = \left[|\xi|^p \|x\|_{\ell^p(S)}^p + |\eta|^p \|y\|_{\ell^p(S)}^p \right]^{1/p} \leq |\xi| \|x\|_{\ell^p(S)} + |\eta| \|y\|_{\ell^p(S)} .$$

Inductively, both of the preceding extend to any finite sum of such disjointly supported functions, in particular the sum

$$\sum_{j=N+1}^{n} \xi_j (\mathbf{x}_\omega)_j) = \sum_{j=N+1}^{n} \xi_j (\mathbf{x}_j)_\omega).$$

$$\begin{aligned}
&\left\| \left[\sum_{j=N+1}^{n} \xi_j A_j \right] \mathbf{x} \right\|_{\ell^p(S,\mathcal{A})} = \left\| \sum_{j=N+1}^{n} \xi_j A_j \mathbf{x}_j \right\|_{\ell^p(S,\mathcal{A})} = \left\| \sum_{j=N+1}^{n} A_j (\xi_j \mathbf{x}_j) \right\|_{\ell^p(S,\mathcal{A})} \\
&\leq \|A\| \left\| \sum_{j=N+1}^{n} \xi_j \mathbf{x}_j \right\|_{\ell^p(S,\mathcal{A})} = \|A\| \left[\sup_{\omega \in \Omega} \left\| \sum_{j=N+1}^{n} \xi_j (\mathbf{x}_j)_\omega \right\|_{\ell^p(S)} \right] \\
&\leq \|A\| \left[\sup_{\omega \in \Omega} \sum_{j=N+1}^{n} \left(|\xi_j| \, \|(\mathbf{x}_j)_\omega\|_{\ell^p(S)} \right) \right] \\
&\leq \|A\| \sup_{\omega \in \Omega} \left[\sum_{j=N+1}^{n} |\xi|^q \right]^{1/q} \left[\sum_{j=1}^{n} \|(\mathbf{x}_j)_\omega\|_{\ell^p(S)}^p \right]^{1/p} && \text{by Hölder's inequality} \\
&\leq \|A\| \left[\frac{\epsilon}{\|A\| + 1} \right] \left[\sup_{\omega \in \Omega} \left\| \sum_{j=N+1}^{n} (\mathbf{x}_j)_\omega \right\|_{\ell^p(S)} \right] && \text{from the above comments} \\
&< \frac{\|A\| \, \epsilon}{\|A\| + 1} \sup_{\omega \in \Omega} \|\mathbf{x}_\omega\|_{\ell^p(S)} \leq \frac{\|a\| \, \epsilon}{\|A\| + 1} \|\mathbf{x}\| < \epsilon.
\end{aligned}$$

Thus

$$\left\| \sum_{j=N+1}^{n} \xi_j A_j \right\| \leq \epsilon \qquad \forall\, n > N.$$

□

6. Functionals and decompositions

With Propositions 7 and 8, and Lemma 9, we have sufficient tools to establish the following results leading up to the analogue of the decomposition theorem of Dixmier. The statements of results and proofs are just appropriate modifications of proofs of Propositions 15, 16, and Theorem 17 of [8]. we include the proofs for completeness.

PROPOSITION 10. *For each $f \in \mathcal{K}^{\#}$, there is a unique function $\widetilde{f} : S \times S :\to \mathcal{A}^{\#}$ such that*

$$\sum_{s\in S}\sum_{t\in S}(\widetilde{f}(s,t))(A(s,t)) = f(A) \qquad \text{for all } A \in \mathcal{K}.$$

Furthermore,

1. *both*

$$\widehat{f}(A) = \sum_{s\in S}\sum_{t\in S}(\widetilde{f}(s,t))(A(s,t)) \qquad \text{and}$$

$$g(A) = \sum_{t\in S}\sum_{s\in S}(\widetilde{f}(s,t))(A(s,t)) \qquad \text{converge for all } A \in \mathcal{M};$$

2. $\widehat{f},\ g \in \mathcal{M}^{\#}$;

3. $\left\|\widehat{f}\right\|_{\mathcal{M}^{\#}} = \|f\|_{\mathcal{K}^{\#}} = \|g\|_{\mathcal{M}^{\#}}$; *and*

4. $\widehat{f}(A) = g(A)$ *for all* $A \in \mathcal{M}$.

Note that $\widehat{f}$ is the unique Hahn-Banach extension of $f \in \mathcal{K}^{\#}$ to all of $\mathcal{M}$. Thus we may regard $\mathcal{K}^{\#}$ as a subspace of $\mathcal{M}^{\#}$. When $\mathcal{A} = \mathbb{C}$ and $p = 2$ the equality in (4) is the well known fact that trace of AB is equal to the trace of BA for a trace class operator A and a bounded operator B on ℓ^2.

Proof. For each $(s,t) \in S\times S$ and each $a \in \mathcal{A}$, define $E_{s,t,a} : S\times S \to \mathcal{A}$ by

$$E_{s,t,a}(u,v) = \begin{cases} a & \text{if } (s,t) = (u,v) \\ 0 & \text{otherwise.} \end{cases} \qquad u,v \in S$$

Then $E_{s,t,a} \in \mathcal{K}$ with $\left\|E_{s,t,a}\right\| = \|a\|$. Let $f \in \mathcal{K}^{\#}$. Define

$$[\widetilde{f}(s,t)](a) = f(E_{s,t,a}) \qquad \forall\ s,t \in S, \quad a \in \mathcal{A}.$$

Then, for each $s, t \in S$, $\widetilde{f}(s,t)$ maps $\mathcal{A}$ to $\mathbb{C}$. A routine verification yields linearity of $\widetilde{f}(s,t)$. To see that it is bounded, note that

$$\begin{aligned}\left\|\widetilde{f}(s,t)\right\| &= \sup_{a\in[\mathcal{A}]_1} \left|[\widetilde{f}(s,t)](a)\right| = \sup_{a\in[\mathcal{A}]_1} \left|f(E_{s,t,a})\right| \\ &\le \sup_{a\in[\mathcal{A}]_1} \|f\| \left\|E_{s,t,a}\right\| = \sup_{a\in[\mathcal{A}]_1} \|f\| \|a\| = \|f\|.\end{aligned}$$

Thus $\widetilde{f}(s,t) \in \mathcal{A}^{\#}$ with $\left\|\widetilde{f}(s,t)\right\| \le \|f\|$. Therefore f induces a map $\widetilde{f}$ from $S \times S$ to the dual $\mathcal{A}^{\#}$ of $\mathcal{A}$. We show that

$$f(A) = \sum_{s\in S}\sum_{t\in S}[\widetilde{f}(s,t)](A(s,t)) \qquad \forall\ A \in \mathcal{K}$$

and that

$$\widehat{f}(B) := \sum_{s\in S}\sum_{t\in S}[\widetilde{f}(s,t)](B(s,t))$$

converges for each $B \in \mathcal{M}$, and $\widehat{f}$ defines a bounded linear functional on $\mathcal{M}$. Furthermore the order of the double sum is interchangeable. Let $A \in \mathcal{M}$ and $s \in S$. For each $F \in \mathcal{F}(S)$,

$$f\left(\left[A_{\underline{\{s\}}}\right]_{F|}\right) = f\left(\sum_{t\in F} E_{s,t,(A(s,t))}\right) = \sum_{t\in F} f\left(E_{s,t,(A(s,t))}\right) = \sum_{t\in F}[\widetilde{f}(s,t)](A(s,t)).$$

Let $\epsilon > 0$. By Lemma 9 (1), $A_{\underline{\{s\}}} \in \mathcal{K}$, and

$$\lim_{F\in\mathcal{F}(S)} \left\|\left[A_{\underline{\{s\}}}\right]_{F|} - A_{\underline{\{s\}}}\right\| = \lim_{F\in\mathcal{F}(S)} \left\|\left[A_{\underline{\{s\}}}\right]_{\underline{F}|} - A_{\underline{\{s\}}}\right\| = 0,$$

there is, by Cauchy criterion, an $F_\epsilon \in \mathcal{F}(S)$ such that

$$\left\|\left[A_{\underline{\{s\}}}\right]_{G|} - \left[A_{\underline{\{s\}}}\right]_{H|}\right\| < \frac{\epsilon}{\|f\|+1} \qquad \forall\ F_\epsilon \subseteq G,\ H \in \mathcal{F}(S).$$

Thus, for $F_\epsilon \subseteq G,\ H \in \mathcal{F}(S)$,

$$\begin{aligned}&\left|\sum_{t\in G}[\widetilde{f}(s,t)](A(s,t)) - \left[\sum_{t\in H}[\widetilde{f}(s,t)](A(s,t))\right]\right| \\ &= \left|f\left(\left[A_{\underline{\{s\}}}\right]_{G|}\right) - f\left(\left[A_{\underline{\{s\}}}\right]_{H|}\right)\right| = \left|f\left(\left[A_{\underline{\{s\}}}\right]_{G|} - \left[A_{\underline{\{s\}}}\right]_{H|}\right)\right| \\ &\le \|f\| \left\|\left[A_{\underline{\{s\}}}\right]_{G|} - \left[A_{\underline{\{s\}}}\right]_{H|}\right\| \le \|f\| \frac{\epsilon}{\|f\|+1} < \epsilon.\end{aligned}$$

Hence, by Cauchy criterion again,

$$\lim_{F\in\mathcal{F}(S)}\sum_{t\in F}[\widetilde{f}(s,t)](A(s,t)) = \lim_{F\in\mathcal{F}(S)} f\left(\left[A_{\underline{\{s\}}}\right]_{F|}\right) \qquad \text{exists.}$$

By continuity of f

$$\begin{aligned} f\left(A_{\underline{\{s\}}}\right) =& f\left(\lim_{F\in\mathcal{F}(S)}\left[A_{\underline{\{s\}}}\right]_{F|}\right) = \lim_{F\in\mathcal{F}(S)}\left[f\left(\left[A_{\underline{\{s\}}}\right]_{F|}\right)\right] \\ &= \lim_{F\in\mathcal{F}(S)}\left[\sum_{t\in F}[\widetilde{f}(s,t)]((A(s,t))\right] = \sum_{t\in S}[\widetilde{f}(s,t)]((A(s,t)). \end{aligned}$$

That is all the inner sums converge for all $A\in\mathcal{M}$. If $A\in\mathcal{K}$, since

$$A - A_{\underline{F}} = (A - A_{\underline{F}|})_{\underline{F}} \quad \text{for all } F\in\mathcal{F}(S),$$

$$\lim_{F\in\mathcal{F}(S)}\left\|A - A_{\underline{F}}\right\| = \lim_{F\in\mathcal{F}(S)}\left\|\left(A - A_{\underline{F}|}\right)_{\underline{F}}\right\| \leq \lim_{F\in\mathcal{F}(S)}\left\|A - A_{\underline{F}|}\right\| = 0.$$

Thus, by continuity of f again, we have

$$\begin{aligned} f(A) =& f\left(\lim_{F\in\mathcal{F}(S)} A_{\underline{F}}\right) = \lim_{F\in\mathcal{F}(S)} f(A_{\underline{F}}) = \lim_{F\in\mathcal{F}(S)} f\left(\sum_{s\in F} A_{\underline{\{s\}}}\right) \\ &= \lim_{F\in\mathcal{F}(S)}\left[\sum_{s\in F} f(A_{\underline{\{s\}}})\right] = \lim_{F\in\mathcal{F}(S)}\left[\sum_{s\in F}\left(\sum_{t\in S}[\widetilde{f}(s,t)](A(s,t))\right)\right] \\ &= \sum_{s\in F}\left(\sum_{t\in S}[\widetilde{f}(s,t)](A(s,t))\right). \qquad \forall\ A\in\mathcal{K} \qquad (\dagger) \end{aligned}$$

The following proof by contradiction for convergence of the sum for $\widehat{f}$ is just details of adapting the different convergence behavior of $\sum_{n=1}^{\infty} n^{-p}$ for $1\leq p<\infty$ to the the situation. So we assume that the double sum for $\widehat{f}$ diverges for some $A\in\mathcal{M}$. By Cauchy criterion, there is an $\epsilon>0$ such that for all $F\in\mathcal{F}(S)$, there is a $G\in\mathcal{F}(S\setminus F)$ such that

$$\left|\sum_{s\in G}\sum_{t\in S}[\widetilde{f}(s,t)](A(s,t))\right| > \epsilon.$$

Inductively, there is a pairwise disjoint sequence $\{F_n : n \in \mathbb{N}\} \subseteq \mathcal{F}(S)$ such that

$$\left| \sum_{s \in F_n} \sum_{t \in S} [\widetilde{f}(s,t)](A(s,t)) \right| > \epsilon \qquad \forall \;\; n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, let

$$\alpha_n = \operatorname{sgn} \left(\sum_{s \in F_n} \sum_{t \in S} [\widetilde{f}(s,t)](A(s,t) \right)$$

where, for each $z \in \mathbb{C}$, $\operatorname{sgn}(z) \in \mathbb{C}$ satisfies $z(\operatorname{sgn}(z)) = |z|$ and $\operatorname{sgn}(0) = 1$. Since $p > 1$, $\left\{ \frac{\alpha_n}{n} \right\}$ is an ℓ^p sequence. By Lemma 9(4)

$$B := \sum_{n=1}^{\infty} \left[\frac{\alpha_n}{n} \right] A_{\underline{F_n}} \in \mathcal{K}.$$

On the other hand, by continuity of f,

$$\begin{aligned} f(B) =& f \left(\sum_{n=1}^{\infty} \left[\frac{\alpha_n}{n} \right] A_{\underline{F_n}} \right) = \sum_{n=1}^{\infty} \left(\left[\frac{\alpha_n}{n} \right] f \left(A_{\underline{F_n}} \right) \right) \\ =& \sum_{n=1}^{\infty} \left(\left[\frac{\alpha_n}{n} \right] \left[\sum_{s \in F_n} \sum_{t \in S} [\widetilde{f}(s,t)](A(s,t)) \right] \right) \\ =& \sum_{n=1}^{\infty} \left(\frac{1}{n} \left| \sum_{s \in F_n} \sum_{t \in S} [\widetilde{f}(s,t)](A(s,t)) \right| \right) > \sum_{n=1}^{\infty} \frac{\epsilon}{n} = \infty \end{aligned}$$

a contradiction. Therefore the double sum in (†) converges for all $A \in \mathcal{M}$.

Similar arguments can be used to show that, for each $t \in S$, the inner sum of $g(A)$ converges to $f(A_{\{t\}|})$ for all $A \in \mathcal{M}$. Suppose that $g(A)$ diverges for some $A \in \mathcal{M}$. Then there are an $\epsilon > 0$ and a pairwise disjoint sequence $\{G_n\} \subseteq \mathcal{F}(S)$ such that the absolute value of the outer sum over each G_n, corresponding to the matrix operator $A_{G_n|}$, is greater than ϵ. By Lemma 9 parts (2) and (5), we can form a matrix operator B in $\mathcal{K}$ for which $g(B)$ diverges.

Let $A \in \mathcal{M}$ and $t \in S$. For each $F \in \mathcal{F}(S)$,

$$f\left(\left[A_{\{t\}|}\right]_{\underline{F}}\right) = f\left(\sum_{s\in F} E_{s,t,(A(s,t))}\right) = \sum_{s\in F} f\left(E_{s,t,(A(s,t))}\right) = \sum_{s\in F}[\widetilde{f}(s,t)](A(s,t)).$$

Let $\epsilon > 0$. Since $A_{\{t\}|} \in \mathcal{K}$, by Lemma 9 (2), and

$$\lim_{F\in\mathcal{F}(S)} \left\| \left[A_{\{t\}|}\right]_{\underline{F}} - A_{\{t\}|} \right\| = \lim_{F\in\mathcal{F}(S)} \left\| \left[A_{\{t\}|}\right]_{\underline{F|}} - A_{\{t\}|} \right\| = 0,$$

there is, by Cauchy criterion, an $F_\epsilon \in \mathcal{F}(S)$ such that

$$\left\| \left[A_{\{t\}|}\right]_{\underline{G}} - \left[A_{\{t\}|}\right]_{\underline{H}} \right\| < \frac{\epsilon}{\|f\|+1} \qquad \forall\ \ F_\epsilon \subseteq G,\ H \in \mathcal{F}(S).$$

Thus, for $F_\epsilon \subseteq G,\ H \in \mathcal{F}(S)$,

$$\begin{aligned}
&\left| \sum_{s\in G}[\widetilde{f}(s,t)](A(s,t)) - \left[\sum_{s\in H}[\widetilde{f}(s,t)](A(s,t))\right] \right| \\
&= \left| f\left(\left[A_{\{t\}|}\right]_{\underline{G}}\right) - f\left(\left[A_{\{t\}|}\right]_{\underline{H}}\right) \right| = \left| f\left(\left[A_{\{t\}|}\right]_{\underline{G}} - \left[A_{\{t\}|}\right]_{\underline{H}}\right) \right| \\
&\le \|f\| \left\| \left[A_{\{t\}|}\right]_{\underline{G}} - \left[A_{\{t\}|}\right]_{\underline{H}} \right\| \le \|f\| \frac{\epsilon}{\|f\|+1} < \epsilon.
\end{aligned}$$

Hence, by Cauchy criterion again,

$$\lim_{F\in\mathcal{F}(S)} \sum_{s\in F}[\widetilde{f}(s,t)](A(s,t)) = \lim_{F\in\mathcal{F}(S)} f\left(\left[A_{\{t\}|}\right]_{\underline{F}}\right) \qquad \text{exists.}$$

By continuity of f

$$\begin{aligned}
f\left(A_{\{t\}|}\right) &= f\left(\lim_{F\in\mathcal{F}(S)} \left[A_{\{t\}|}\right]_{\underline{F}}\right) = \lim_{F\in\mathcal{F}(S)} \left[f\left(\left[A_{\{t\}|}\right]_{\underline{F}}\right)\right] \\
&= \lim_{F\in\mathcal{F}(S)} \left[\sum_{s\in F}[\widetilde{f}(s,t)]((A(s,t))\right] = \sum_{s\in S}[\widetilde{f}(s,t)]((A(s,t)).
\end{aligned}$$

That is all the inner sums in the definition of g converge for all $A \in \mathcal{M}$. If $A \in \mathcal{K}$, since

$$A - A_{F|} = (A - A_{\underline{F|}})_{F|} \qquad \text{for all } F \in \mathcal{F}(S),$$

$$\lim_{F\in\mathcal{F}(S)} \left\|A - A_{F|}\right\| = \lim_{F\in\mathcal{F}(S)} \left\|\left(A - A_{\underline{F|}}\right)_{F|}\right\| \le \lim_{F\in\mathcal{F}(S)} \left\|A - A_{\underline{F|}}\right\| = 0.$$

Thus, by continuity of f again, we have

$$\begin{aligned} f(A) =& f\left(\lim_{F\in\mathcal{F}(S)} A_{F|}\right) = \lim_{F\in\mathcal{F}(S)} f(A_{F|}) = \lim_{F\in\mathcal{F}(S)} f\left(\sum_{t\in F} A_{\{t\}|}\right) \\ =& \lim_{F\in\mathcal{F}(S)}\left[\sum_{t\in F} f(A_{\{t\}|})\right] = \lim_{F\in\mathcal{F}(S)}\left[\sum_{t\in F}\left(\sum_{s\in S}[\widetilde{f}(s,t)](A(s,t))\right)\right] \\ =& \sum_{t\in S}\left(\sum_{s\in S}[\widetilde{f}(s,t)](A(s,t))\right) =: g(A) \qquad \forall\ A\in\mathcal{K}. \end{aligned} \tag{‡}$$

Suppose the double sum for $g(A)$ as in (‡) diverges for some $A\in\mathcal{M}$. By Cauchy criterion, there is an $\epsilon > 0$ such that for all $F\in\mathcal{F}(S)$, there is a $G\in\mathcal{F}(S\setminus F)$ such that

$$\left|\sum_{t\in G}\sum_{s\in S}[\widetilde{f}(s,t)](A(s,t))\right| > \epsilon.$$

Inductively, there is a pairwise disjoint sequence $\{F_n : n\in\mathbb{N}\}\subseteq\mathcal{F}(S)$ such that

$$\left|\sum_{t\in F_n}\sum_{s\in S}[\widetilde{f}(s,t)](A(s,t))\right| > \epsilon \qquad \forall\ n\in\mathbb{N}.$$

For each $n\in\mathbb{N}$, let

$$\beta_n = \operatorname{sgn}\left(\sum_{t\in F_n}\sum_{s\in S}[\widetilde{f}(s,t)](A(s,t)\right).$$

Since $p > 1$, $\left\{\frac{\beta_n}{n}\right\}$ is an ℓ^q sequence. By Lemma 9(5)

$$C := \sum_{n=1}^{\infty}\left[\frac{\beta_n}{n}\right] A_{(F_n)|} \in \mathcal{K}$$

and hence $f(C) \in \mathbb{C}$. On the other hand, by continuity of f,

$$\begin{aligned} f(C) =& f\left(\sum_{n=1}^{\infty}\left[\frac{\beta_n}{n}\right] A_{(F_n)|}\right) = \sum_{n=1}^{\infty}\left(\left[\frac{\beta_n}{n}\right] f\left(A_{(F_n)|}\right)\right) \\ =& \sum_{n=1}^{\infty}\left(\left[\frac{\beta_n}{n}\right]\left[\sum_{t\in F_n}\sum_{s\in S}[\widetilde{f}(s,t)](A(s,t))\right]\right) \\ =& \sum_{n=1}^{\infty}\left(\frac{1}{n}\left|\sum_{t\in F_n}\sum_{s\in S}[\widetilde{f}(s,t)](A(s,t))\right|\right) > \sum_{n=1}^{\infty}\frac{\epsilon}{n} = \infty \end{aligned}$$

a contradiction. Therefore the double sum in (‡) converges to $g(A)$ for all $A \in \mathcal{M}$.
Verification of linearity of $\widehat{f}$ and g is routine and omitted. To show that $\widetilde{f},\ g \in \mathcal{M}^{\#}$, we make repeated use of the uniform boundedness principle. For a fixed $F \in \mathcal{F}(S)$, since, for each $A \in \mathcal{M}$, $A_{\underline{F}}$ and $A_{F|}$ are in $\mathcal{K}$,

$$f_F(A) = \sum_{s\in F}\sum_{t\in S}[\widetilde{f}(s,t)](A(s,t)) = f\left(A_{\underline{F}}\right) \qquad A \in \mathcal{M}$$

is a bounded linear functional on $\mathcal{M}$ with norm

$$\|f_F\| \le \|f\|\,.$$

The convergence of the double sum $g(A)$, for each $A \in \mathcal{M}$, implies the existence of an $M_A > 0$ such that

$$|f_F(A)| \le M_A \qquad \forall\ \ F \in \mathcal{F}(S).$$

By the uniform boundedness principle, there is an $M > 0$ such that

$$\|f_F\| \le M \qquad \forall\ \ F \in \mathcal{F}(S).$$

Then, for $A \in \mathcal{M}$,

$$\left|\widehat{f}(A)\right| = \lim_{F\in\mathcal{F}(S)} |f_F(A)| \le \limsup_{F\in\mathcal{F}(S)}(\|f_F\|\,\|A\|) \le M\,\|A\|\,.$$

The boundedness of g is similarly verified.
It follows from (†) and (‡) that $f(A) = \widehat{f}(A) = g(A)$ for each $A \in \mathcal{K}$.
(3) For each $A \in \mathcal{M}$, since $A_{\underline{F}},\ A_{F|} \in \mathcal{K}$ for all $F \in \mathcal{F}$,

$$\left|g(A_{G|})\right| = \left|f(A_{G|})\right| \le \|f\|_{\mathcal{K}^{\#}}\left\|A_{G|}\right\| \le \|f\|_{\mathcal{K}^{\#}}\|A\|\,,$$

hence

$$|g(A)| = \lim_{G\in\mathcal{F}(S)} \left|g(A_{G|})\right| = \lim_{G\in\mathcal{F}(S)} \left|f(A_{G|})\right| \le \|f\|_{\mathcal{K}^{\#}} \|A\| .$$

That is

$$\|g\|_{\mathcal{M}^{\#}} \le \|f\|_{\mathcal{K}^{\#}} .$$

Since $g|_{\mathcal{K}} = f$,

$$\|g\|_{\mathcal{M}^{\#}} \ge \|f\|_{\mathcal{K}^{\#}} .$$

Similarly, since $\widehat{f}(A) = \lim_{G\in\mathcal{F}} f(A_{\underline{G}})$, $\left\|\widehat{f}\right\|_{\mathcal{M}^{\#}} = \left\|\widehat{f}\Big|_{\mathcal{K}}\right\|_{\mathcal{K}^{\#}} = \|f\|_{\mathcal{K}^{\#}}$.

(4) Define

$$g_G(A) = \sum_{t\in G}\sum_{s\in S}(\widetilde{f}(s,t))(A(s,t)) \quad \text{for } G \in \mathcal{F}(S), \text{ and } A \in \mathcal{M}.$$

From earlier observations

$$|g_G(A)| = \left|g(A_{G|})\right| \le \|g\| \left\|A_{G|}\right\| \le \|g\| \|A\| ,$$

hence $g_G \in \mathcal{M}^{\#}$. We show that $\{g_G : G \in \mathcal{F}\}$ is a Cauchy net in $\mathcal{K}^{\#}$. If not, then there are an $\epsilon > 0$ and pairwise disjoint sequences $\{G_n\}, \{H_n\} \subseteq \mathcal{F}$ and a sequence $\{A_n\} \subseteq \mathcal{M}$ such that

$$H_n \cap G_m = \emptyset \ \forall \ n, m \in \mathbb{N}, \quad \|A_n\| = 1, \quad \text{and} \quad (g_{G_n} - g_{H_n})(A_n) \ge \epsilon.$$

Let $B_n = (A_n)_{(G_n)|} - (A_n)_{(H_n)|}$. Then $A' = \sum_{n=1}^{\infty} \frac{1}{n} B_n \in \mathcal{K}$ converges in norm by Lemma 9(5), while

$$g(A') = \sum_{n=1}^{\infty} \frac{1}{n} g(B_n) = \sum_{n=1}^{\infty} \frac{1}{n}(g_{G_n} - g_{H_n})(A_n) \ge \sum_{n=1}^{\infty} \frac{\epsilon}{n} = \infty,$$

a contradiction. We conclude that $\{g_G : G \in \mathcal{F}\}$ is a Cauchy net in $\mathcal{K}^{\#}$, and hence by completeness of $\mathcal{K}^{\#}$, there is an $h \in \mathcal{K}^{\#}$ such that

$$\lim_{F\in\mathcal{F}(S)} \|g_F - h\| = 0.$$

Since for each $A \in \mathcal{K}$,

$$\lim_{G\in\mathcal{F}} \left\|A - A_{G|}\right\| = 0, \quad \text{and} \quad \lim_{G\in\mathcal{F}} g_G(A) = \lim_{G\in\mathcal{F}} g(A_{G|}) = g(A),$$

we have $g = h$ on $\mathcal{K}^{\#}$. That is $\lim_{G\in\mathcal{F}} \|g - g_G\|_{\mathcal{K}^{\#}} = 0$. A routine verification shows that, for all $k,\ l \in \mathcal{K}^{\#}$, $\widetilde{(k+l)} = \widetilde{k} + \widetilde{l}$ as maps from $S \times S$ to $\mathcal{A}^{\#}$, and $\widehat{(k+l)} = \widehat{k} + \widehat{l}$ as bounded linear functionals on $\mathcal{M}$. Thus

$$\begin{aligned}\lim_{G\in\mathcal{F}} \|\widehat{g} - \widehat{g}_G\|_{\mathcal{M}^{\#}} &= \lim_{G\in\mathcal{F}} \left\|\widehat{(g - g_G)}\right\|_{\mathcal{M}^{\#}} = \lim_{G\in\mathcal{F}} \left\| \widehat{(g - g_G)}\Big|_{\mathcal{K}} \right\|_{\mathcal{K}^{\#}} \\ &= \lim_{G\in\mathcal{F}} \|g - g_G\|_{\mathcal{K}^{\#}} = 0.\end{aligned}$$

For each $A \in \mathcal{M}$, since both of the double sums for $\widehat{f}$ and g converge,

$$\begin{aligned}(\widehat{f} - g_G)(A) =& \widehat{f}(A) - g_G(A) \\ =& \sum_{s\in S}\sum_{t\in S}(\widetilde{f}(s,t))(A(s,t)) - \sum_{s\in S}\sum_{t\in G}(\widetilde{f}(s,t))(A(s,t)) \\ =& \sum_{s\in S}\sum_{t\in S\setminus G}(\widetilde{f}(s,t))(A(s,t)) = \sum_{s\in S}\sum_{t\in S}(\widetilde{\widetilde{f}}(s,t))(A(s,t))\end{aligned}$$

where $\widetilde{\widetilde{f}}(s,t) = \widetilde{f}(s,t)$ for $t \in S \setminus G$ and 0 otherwise. From part (3), we have

$$\begin{aligned}\lim_{G\in\mathcal{F}} \left\|\widehat{f} - g_G\right\|_{\mathcal{M}^{\#}} &= \lim_{G\in\mathcal{F}} \left\|[f \widehat{- (g_G)}]\right\|_{\mathcal{M}^{\#}} = \lim_{G\in\mathcal{F}} \left\|[f \widehat{- (g_G)}]\big|_{\mathcal{K}}\right\|_{\mathcal{K}^{\#}} \\ &= \lim_{G\in\mathcal{F}} \left\|f - (g_G)\big|_{\mathcal{K}}\right\|_{\mathcal{K}^{\#}} = \lim_{G\in\mathcal{F}} \left\|g\big|_{\mathcal{K}} - (g_G)\big|_{\mathcal{K}}\right\|_{\mathcal{K}^{\#}} = 0.\end{aligned}$$

Therefore $\widehat{f}(A) = \lim_{G\in\mathcal{F}} g_G(A) = g(A)$ for all $A \in \mathcal{M}$, and hence

$$\begin{aligned}\sum_{s\in S}\sum_{t\in S}(\widetilde{f}(s,t))(A(s,t)) =& \widehat{f}(A) = g(A) \\ =& \sum_{t\in S}\sum_{s\in S}(\widetilde{f}(s,t)(A(s,t)) \quad \forall \ A \in \mathcal{M}.\end{aligned}$$

□

An immediate consequence of the preceding proof is the following.

Corollary 11. *Suppose a function $\Phi : S \times S \to \mathcal{A}^{\#}$ has the properties that, for each $A \in \mathcal{K}$,* *1. the sum*

$$\sum_{t\in S}[\Phi(s,t)](A(s,t))$$

converges in $\mathbb{C}$, for each $s \in S$; and

2. the sum

$$\sum_{s\in S}\sum_{t\in S}[\Phi(s,t)](A(s,t))$$

converges.

Then the function $f_{\Phi} : \mathcal{K} \to \mathbb{C}$ defined by

$$f_{\Phi}(A) = \sum_{s\in S}\sum_{t\in S}[\Phi(s,t)](A(s,t)) \qquad \forall \;\; A \in \mathcal{K}$$

is a bounded linear functional on $\mathcal{K}$. Furthermore, the double sum that defines f_{Φ} converges for all $A \in \mathcal{M}$, and

$$\begin{aligned}\sum_{s\in S}\sum_{t\in S}[\Phi(s,t)](A(s,t)) =& f_{\Phi}(A)\\ =& \sum_{t\in S}\sum_{s\in S}[\Phi(s,t)](A(s,t)) \qquad \forall \;\; A \in \mathcal{M}.\end{aligned}$$

By Corollary 1.1.12 of [2, p. 17], the space $\ell^p(S,\mathcal{A})$ is isometrically isomorphic to $\ell^p(S)\check{\otimes}\mathcal{A}$, the injective tensor product of $\ell^p(S)$ and $\mathcal{A}$.

Indeed, the map

$$\Phi : \sum_{j=1}^{k} x_j \otimes a_j \mapsto \sum_{j=1}^{k} x(\cdot)a_j \qquad k \in \mathbb{N},\; x_j \in \ell^p(S),\; a_j \in \mathcal{A},\; 1 \le j \le k$$

is isometric from the algebraic tensor product $\ell^p(S) \otimes \mathcal{A}$ into $\ell^p(S,\mathcal{A})$. For $\mathbf{x} \in \ell^p(S,\mathcal{A})$ and $\epsilon > 0$, there is an $F \in \mathcal{F}(S)$ such that

$$\|\widetilde{\varphi}(\mathbf{x}_G)\|_{\ell^p(S)} < \epsilon \qquad \forall \;\; G \in \mathcal{F}(S \setminus F),\; \varphi \in s(\mathcal{A}).$$

Since Φ maps

$$\sum_{t\in F} \chi_{\{t\}} \otimes \mathbf{x}(t) \in \ell^p(S) \otimes \mathcal{A}$$

to the function

$$\mathbf{y}(s) = \sum_{t\in F} \chi_{\{t\}}(s)\mathbf{x}(t) \qquad s \in S$$

we also have $\mathbf{y}_F = \mathbf{x}_F$ and $\mathbf{y}_{(S\setminus F)} = 0$, and hence

$$\|\mathbf{x}-\mathbf{y}\| = \left\|\mathbf{x}_{(S\setminus F)}\right\| = \sup_{G\in\mathcal{F}(S\setminus F)} \|\mathbf{x}_G\| = \sup_{G\in\mathcal{F}(S\setminus F)} \left[\sup_{\varphi\in s(\mathcal{A})} \|\widetilde{\varphi}(\mathbf{x}_G)\|_{\ell^p(S)}\right] \leq \epsilon.$$

Therefore $\Phi(\ell^p(S)\otimes\mathcal{A})$ is dense in $\ell^p(S,\mathcal{A})$ under the isometry Φ. Hence Φ extends to an isometric isomorphism from $\ell^p(S)\check{\otimes}\mathcal{A}$ onto $\ell^p(S,\mathcal{A})$.

This clearly leads to the question of whether the space $\mathcal{M}$ of matrix operators on $\ell^p(S,\mathcal{A})$ is naturally isomorphic to $\mathcal{B}(\ell^p(S))\check{\otimes}\mathcal{A}$ (the injective tensor product of bounded operators on $\ell^p(S)$ and $\mathcal{A}$). The following example shows that this is not the case, unlike $\ell^p(S,\mathcal{A})$.

With $\mathcal{A} = C([0,1])$, the matrix operator A with $A(n,n)$ the piecewise linear continuous function on $[0,1]$ with value 1 at 2^{-n} and 0 outside $(2^{-n-1}, 2^{-n+1})$ is a bounded matrix operator of norm 1 on $\ell^p(\mathbb{N}, C[0,1])$ but not in the injective tensor product $\mathcal{B}(\ell^p)\check{\otimes}(C[0,1])$ (with the usual identification of elements in the algebraic tensor product $\mathcal{B}(\ell^p)\otimes(C[0,1])$ as matrices over $\mathcal{A}$).

Suppose Φ is an isometric isomorphism from $\mathcal{B}(\ell^p)\check{\otimes}\mathcal{A}$ onto a subspace of $\mathcal{M}$ the space of $\mathcal{A}$-matrix operators on $\ell^p(S,\mathcal{A})$, that extends the natural identification

$$\Phi\left[\sum_{j=1}^{k} \alpha_j \otimes x_j\right](m,n) = \sum_{j=1}^{k} \alpha_j(m,n)x_j$$
$$\forall \ \ \alpha_j \in \mathcal{B}(\ell^p), \ \ x_j \in \mathcal{A}, \ 1\leq j\leq k; \quad \forall \quad m, \ n \in \mathbb{N}.$$

Each operator in $\mathcal{B}(\ell^p)$ is identified with its matrix representation with respect to the standard basis of ℓ^p. That is each algebraic tensor

$$B = \sum_{j=1}^{k} \alpha_j \otimes x_j \in \mathcal{B}(\ell^p)\otimes\mathcal{A}$$

is identified with an infinite matrix,

$$\Phi\left[\sum_{j=1}^{k} \alpha_j \otimes x_j\right]$$

with entries in the finite dimensional space generated by $\{x_j : 1\leq j\leq k\}$.

Let $\psi_0 \in [\mathcal{A}^{\#}]_1$ be arbitrary, and let φ_0 be the (m,n) entry functional on $\mathcal{B}(\ell^p)$, i.e., $\varphi_0(\beta) = \beta(m,n)$ for all $\beta \in \mathcal{B}(\ell^p)$. By definition of the injective cross norm, $\|\cdot\|_\lambda$, we have

$$\|B\|_\lambda = \left\| \sum_{j=1}^{k} \alpha_j \otimes x_j \right\|_\lambda = \sup \left\{ \left| \sum_{l=1}^{m} \varphi(\beta_l)\psi(y_l) \right| : \sum_{j=1}^{k} \alpha_j \otimes x_j = \sum_{l=1}^{m} \beta_l \otimes y_l, \quad \varphi \in [\mathcal{B}(\ell^p)^\#]_1, \quad \psi \in [\mathcal{A}^\#]_1 \right\} \geq |\psi_0(B(m,n)).|$$

The arbitrariness of $\psi_0 \in [\mathcal{A}^\#]_1$ implies

$$\|B(m,n)\|_{\mathcal{A}} \leq \|B\|_\lambda \qquad \text{uniformly for all } \ m, \ n \in \mathbb{N}.$$

Suppose otherwise: i.e., suppose $A \in \mathcal{B}(\ell^p)\check{\otimes}(C[0,1])$. Then there would be a sequence $\{B_k\}_{k\in\mathbb{N}} \subseteq \mathcal{B}(\ell^p) \otimes (C[0,1])$ converging to A in the injective cross norm $\|\cdot\|_\lambda$. In particular,

$$\|B_k(n,n) - A(n,n)\|_{\mathcal{A}} \to 0 \quad \text{as } k \to \infty, \qquad \text{uniformly for all } \ n \in \mathbb{N}.$$

For each k, all entries of B_k are in a same finite dimensional subspace, $\mathcal{V}_k$, of $\mathcal{A}$. Choose k such that $\|B_k - A\|_\lambda < \frac{1}{3}$. Then

$$\|A(n,n) - B_k(n,n)\|_{\mathcal{A}} \leq \|A - B_k\|_\lambda < \frac{1}{3} \qquad \forall \ n \in \mathbb{N}.$$

Thus, for distinct positive integers n and m, we have

$$\begin{aligned} &\|B_k(n,n) - B_k(m,m)\|_{\mathcal{A}} \\ \geq &\|A(n,n) - A(m,m)\|_{\mathcal{A}} - \|A(n,n) - B_k(n,n)\|_{\mathcal{A}} \\ &\qquad - \|A(m,m) - B_k(m,m)\|_{\mathcal{A}} \\ > &1 - \frac{1}{3} - \frac{1}{3} = \frac{1}{3}. \end{aligned}$$

Since the sequence $\{B_k(n,n)\}_{n\in\mathbb{N}}$ is in the finite dimensional subspace $\mathcal{V}_k$ of $\mathcal{A}$, and is bounded in norm by $\|B_k\|_\lambda$. There is a convergent subsequence of $\{B_k(n,n)\}_{n\in\mathbb{N}}$. But this contradicts the above inequality.

From this we conclude that the following theorem does not follow from Dixmier's theorem on sequence space operators and tensor product ([4, Th. VI.3.1, p. 282]).

THEOREM 12. *For each $f \in \mathcal{M}^\#$ there are $g \in \mathcal{K}^\# \hookrightarrow \mathcal{M}^\#$ and $h \in \mathcal{K}^\perp$ such that $f = f + g$ and $\|f\| = \|g\| + \|h\|$. $\mathcal{K}$; i.e., $\mathcal{K}$ is an M-ideal in $\mathcal{M}$.*

First we will give an elementary and direct proof using arguments similar to that used in [8]. Then we will give a proof using the three-ball characterization theorem of Alfsen and Effros [1] (also [4, Th. I.2.2, p. 18]).

Proof. Let $f \in \mathcal{M}^{\#}$. Then, by Proposition 10, the restriction of f to $\mathcal{K}$ induces a map $\widetilde{f} : S \times S \to \mathcal{A}^{\#}$ such that

$$f(A) = \sum_{s\in S}\sum_{t\in S}[\widetilde{f}(s,t)](A(s,t)) \qquad \forall \;\; A \in \mathcal{K}.$$

Moreover, the sum

$$g(B) = \widehat{f}(B) = \sum_{s\in S}\sum_{t\in S}[\widetilde{f}(s,t)](B(s,t)) \qquad \forall \;\; B \in \mathcal{M}$$

defines the unique Hahn-Banach extension of f to all of $\mathcal{M}$. Let $h = f - g$. Then h vanishes on $\mathcal{K}$ by construction. Because of the triangle inequality, it suffices to show that $\|f\| \geq \|g\| + \|h\|$. Let $\epsilon > 0$. There are $A \in [\mathcal{K}]_1$ and $B \in [\mathcal{M}]_1$ such that

$$g(A) > \|g\| - \frac{\epsilon}{8} \qquad \text{and} \qquad h(B) > \|h\| - \frac{\epsilon}{8}.$$

Since $A \in \mathcal{K}$ and

$$\lim_{F\in\mathcal{F}(S)} \left\| A - A_{\boxed{F}} \right\| = 0$$

there is an $F_1 \in \mathcal{F}(S)$ such that

$$\Re\left(g(A_{\boxed{F}})\right) > \Re\left(g(a)\right) - \frac{\epsilon}{8} = g(A) - \frac{\epsilon}{8} \qquad \forall \;\; F_1 \subseteq F \in \mathcal{F}(S).$$

Since

$$g(B) = \sum_{s\in S}\sum_{t\in S}[\widetilde{f}(s,t)](B(s,t))$$

converges, there is an $F_2 \in \mathcal{F}(S)$ such that $F_1 \subseteq F_2$ and

$$\left| \sum_{s\in S\setminus F}\sum_{t\in S}[\widetilde{f}(s,t)](B(s,t)) \right| < \frac{\epsilon}{8} \qquad \forall \;\; F_2 \subseteq F \in \mathcal{F}(S).$$

From this it follows that

$$\begin{aligned}
&\left|\sum_{s\in F_2}\sum_{t\in S}[\widetilde{f}(s,t)](B(s,t))\right| \\
&=\left|\sum_{s\in S}\sum_{t\in S}[\widetilde{f}(s,t)](B(s,t)) - \left[\sum_{s\in S\setminus F_2}\sum_{t\in S}[\widetilde{f}(s,t)](B(s,t))\right]\right| \\
&\geq\left|\sum_{s\in S}\sum_{t\in S}[\widetilde{f}(s,t)](B(s,t))\right| - \left|\sum_{s\in S\setminus F_2}\sum_{t\in S}[\widetilde{f}(s,t)](B(s,t))\right| \\
&> g(B)-\frac{\epsilon}{8}.
\end{aligned}$$

Since F_2 is a finite set, there is a $G_1 \in \mathcal{F}(S)$ such that $F_1 \subseteq F_2 \subseteq G_1$ and

$$\begin{aligned}
\left|\sum_{s\in F_2}\sum_{t\in G}[\widetilde{f}(s,t)](B(s,t))\right| &> \left|\sum_{s\in F_2}\sum_{t\in S}[\widetilde{f}(s,t)](B(s,t))\right| - \frac{\epsilon}{8} \\
&> g(B)-\frac{\epsilon}{4} \qquad \forall\ G_1 \subseteq G \in \mathcal{F}(S).
\end{aligned}$$

Using Corollary 11 with $\Phi(s,t) = \widetilde{f}(s,t)$ for $s \in S \setminus F_2$ and $t \in S$ and 0 otherwise, we have the convergence of

$$\sum_{s\in S\setminus F_2}\sum_{t\in S}[\widetilde{f}(s,t)](D(s,t)) = \sum_{t\in S}\sum_{s\in S\setminus F_2}[\widetilde{f}(s,t)](D(s,t)) \qquad \forall\ D \in \mathcal{M}.$$

With the B in place of D, there is a $G_2 \in \mathcal{F}(S)$ such that $G_2 \supseteq F_2 \cup G_1$, and, for all $G_2 \subseteq G \in \mathcal{F}(S)$,

$$\left|\sum_{t\in S\setminus G}\sum_{s\in S\setminus F_2}[\widetilde{f}(s,t)](B(s,t))\right| = \left|\sum_{s\in S\setminus F_2}\sum_{t\in S\setminus G}[\widetilde{f}(s,t)](B(s,t))\right| < \frac{\epsilon}{8}.$$

Let

$$A_1 = \left[A_{\underline{F_1}}\right]_{G_1|}, \qquad B_1 = \left[B_{\underline{(S\setminus F_2)}}\right]_{(S\setminus G_2)|}, \qquad \text{and} \qquad C = A_1 + B_1.$$

Then

$$|g(B_1)| = \left| \sum_{s \in S \setminus F_2} \sum_{t \in S \setminus G_2} [\widetilde{f}(s,t)](B(s,t)) \right| < \frac{\epsilon}{8}.$$

Since $F_1 \subseteq F_2$ and $G_1 \subseteq G_2$, it follows from Proposition 7 (3) that

$$\|C\| = \max\{\|A_1\|, \ \|B_1\|\} \leq 1.$$

Since $A_1 \in \mathcal{K}$ and $B - B_1 \in \mathcal{K}$, $h(B_1) = h(B)$ and $h(A_1) = 0$.

$$\begin{aligned} |f(C)| &= |f(A_1) + f(B_1)| = |g(A_1) + h(A_1) + g(B_1) + h(B_1)| \\ &\geq |g(A_1) + h(B)| - |g(B_1)| \geq \Re\left(g(A_1) + h(B)\right) - \frac{\epsilon}{4} \\ &> \Re(g(A)) - \frac{\epsilon}{8} + \|h\| - \frac{\epsilon}{8} - \frac{\epsilon}{8} > \|g\| - \frac{3\epsilon}{8} + \|h\| - \frac{\epsilon}{4} \\ &> \|g\| + \|h\| - \epsilon. \end{aligned}$$

□

To prove this result using the characterization theorem of Alfsen and Effros, we need to introduce some notation for convenience. For $X \in \mathcal{M}$ and $G \subseteq S$, let

$$X_{\not{G}\rfloor} := X - X_{\underline{G}\rfloor}, \ X^{G} := X_{\underline{G}\rfloor} + X_{\widetilde{G}\rfloor}, \text{ and } X^{G'} := X - X^{G} = X - X_{\underline{G}\rfloor} - X_{\widetilde{G}\rfloor},$$

where $\widetilde{G} = S \setminus G$ (the complement of G in S). (That is think of X as a rectangle, $X_{\not{G}\rfloor}$ is the G corner removed, X^{G} is the sum of the diagonal G corner and its complimentary $\widetilde{G}$ corner, and $X^{G'}$ is the two diagonal G and $\widetilde{G}$ corners removed.)

Note that by Proposition 7 (3)

$$\left\|X^{G}\right\| = \max\left\{\left\|X_{\underline{G}\rfloor}\right\|, \left\|X_{\widetilde{G}\rfloor}\right\|\right\}.$$

For the convenience of reference we state following version of the 3-ball characterization of M-ideals of Alfsen and Effros.

Theorem 13. *(Alfsen-Effros [1]) A subspace Y of a Banach space is an M-ideal iff for all $\epsilon > 0$, for all $x_1, x_2, x_3 \in [X]_1$, and for all $y \in [Y]_1$, there exists $z \in Y$ such that*

$$\left\|x + y_j - z\right\| < 1 + \epsilon, \qquad \textit{for } \ j = 1, 2, 3.$$

Now we are ready for the proof.

(Proof of $\mathcal{K}$ is an M-idel in $\mathcal{M}$ using the Alfsen-Effros theorem). To show that $\mathcal{K}$ is an M-ideal in $\mathcal{M}$, let $A_j \in [\mathcal{K}]_1$, $j = 1, 2, 3$; $B \in [\mathcal{M}]_1$, and let $\epsilon > 0$. Then by definition of $\mathcal{K}$, there exists $F \in \mathcal{F}(S)$ such that

$$\left\| \left(A_j\right)_{\not{F}\lrcorner} \right\| = \left\| A_j - \left(A_j\right)_{F\lrcorner} \right\| < \frac{\epsilon}{4} \qquad j = 1, 2, 3.$$

Thus, for $j = 1, 2, 3$,

$$\left\| \left(A_j\right)^{F'} \right\| \leq \left\| \left(A_j\right)_{\not{F}\lrcorner} \right\| < \frac{\epsilon}{4} \qquad \text{and} \qquad \left\| \left(A_j\right)_{\widetilde{F}\lrcorner} \right\| < \frac{\epsilon}{4}.$$

Let

$$A = B_{\widetilde{\not{F}}\lrcorner} + \frac{1}{3}\left[A_1 + A_2 + A_3\right]_{\not{F}\lrcorner}.$$

Then $A \in \mathcal{K}$. With $j = 1, 2, 3$ and $\{j', j''\} = \{1, 2, 3\} \setminus \{j\}$, we have, from Proposition 7,

$$\begin{aligned}
\left\| B + A_j - A \right\| &\leq \left\| \left(B + A_j - A\right)^{F} \right\| + \left\| \left(B + A_j - A\right)^{F'} \right\| \\
&= \max\left\{ \left\| \left(B + A_j - A\right)_{F\lrcorner} \right\|, \left\| \left(B + A_j - A\right)_{\widetilde{F}\lrcorner} \right\| \right\} \\
&\qquad + \left\| \frac{1}{3}\left(2A_j - A_{j'} - A_{j''}\right)^{F'} \right\| \\
&\leq \max\left\{ \left\| \left(A_j\right)_{F\lrcorner} \right\|, \left\| \left(B + \frac{1}{3}\left(2A_j - A_{j'} - A_{j''}\right)\right)_{\widetilde{F}\lrcorner} \right\| \right\} \\
&\qquad + \frac{1}{3}\left(2\left\| \left(A_j\right)^{F'} \right\| + \left\| \left(A_{j'}\right)^{F'} \right\| + \left\| \left(A_{j''}\right)^{F'} \right\| \right) \\
&\leq \max\left\{ 1, \left\| B_{\widetilde{F}\lrcorner} \right\| + \frac{1}{3}\left[2\left\| \left(A_j\right)_{\widetilde{F}\lrcorner} \right\| + \left\| \left(A_{j'}\right)_{\widetilde{F}\lrcorner} \right\| \right.\right. \\
&\qquad \left.\left. + \left\| \left(A_{j''}\right)_{\widetilde{F}\lrcorner} \right\| \right] \right\} + \frac{\epsilon}{3} \\
&\leq \max\left\{ 1, 1 + \frac{\epsilon}{3} \right\} + \frac{\epsilon}{3} < 1 + \epsilon.
\end{aligned}$$

□

We note that this proof relies heavily on a deep theorem of Alfsen and Effros [1] and also our preparatory results in preceding sections.

References

[1] E.M. Alfsen and E.G. Effros, *Structure in real Banach spaces*, Ann. Math. **96** (1972), 98–173.

[2] J Diestel, J.H. Fourie, and J. Swart, *The Metric Theory of Tensor Products: Grothendieck's Résumé Revisited*, MBK, no. 52, AMS, 2008.

[3] J. Dixmier, *Les fonctionnelles lin'earies sur l'ensemble des op'erateurs born'es dans espace de Hilbert*, Ann. Math. **51** (1950), 387–408.

[4] P. Harmand, D. Werner and W. Werner, *M-Ideals in Banach Spaces and Banach Algebras*, Lecture Notes in Mathematics, vol. 1547, Springer-Verlag, 1993.

[5] R.V. Kadison and J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras*, vol. I, Academic Press, New York, 1983.

[6] L. Livshits, S.-C. Ong, and S. Wang, *Schur algebras over function algebras*, Houston J. Math. **30** (2004), 1195–1217.

[7] H.L. Royden and P.M. Fitzpatrick, *Real Analysis*, 4 ed., Prentice Hall, 2010.

[8] T. Wootijirattikal, S.-C. Ong, and J. Rakbud, *Functional decompositions on vector-valued function spaces via operators*, J. Math. Anal. Appl. **389** (2012), 1173–1190.

[9] O. Wootijiruttikal, S.-C. Ong, P. Chaisuriya, and J. Rakbud, *Banach spaces of functions taking values in a* C^**-algebra*, Oper. Matrices **3** (2009), 373–396.

Sing-Cheong Ong
Department of Mathematics, Central Michigan University
Mount Pleasant, MI 48859, USA
E-mail: ong3pf@gmail.com

Jitti Rakbud
Department of Mathematics, Faculty of Science
Silpakorn University, Nakorn Pathom, 73000, Thailand
E-mail: RAKBUD_J@su.ac.th

Titarii Wootijirattikal
Department of Mathematics, Statistics and Computer
Faculty of Science, Ubon Ratchathani University
Ubon Ratchathani 34190, Thailand

Center of Excellence in Mathematics
Bangkok 10400, Thailand
E-mail: ma_pintto@yahoo.com