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# THE CHROMATIC POLYNOMIAL FOR CYCLE GRAPHS

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Abstract. Let $P(G, \lambda)$ denote the number of proper vertex colorings of $G$ with $\lambda$ colors. The chromatic polynomial $P(C_n, \lambda)$ for the cycle graph $C_n$ is well-known as

$$P(C_n, \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1)$$

for all positive integers $n \geq 1$. Also its inductive proof is widely well-known by the *deletion-contraction recurrence*. In this paper, we give this inductive proof again and three other proofs of this formula of the chromatic polynomial for the cycle graph $C_n$.

## 1. Introduction

The number of proper colorings of a graph with finite colors was introduced only for planar graphs by George David Birkhoff [1] in 1912, in an attempt to prove the four color theorem, where the formula for this number was later called by the chromatic polynomial. In 1932, Hassler Whitney [3] generalized Birkhoff's formula from the planar graphs to general graphs. In 1968, Ronald Cedric Read [2] introduced the concept of chromatically equivalent graphs and asked which polynomials are the chromatic polynomials of some graph, that remains open.

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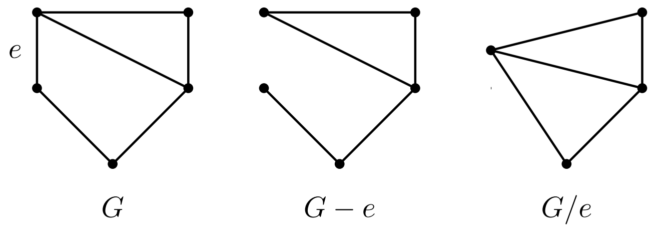



FIGURE 1. $G$ , $G-e$ and $G/e$

**Chromatic polynomial.** For a graph $G$, a *coloring* means almost always a *(proper) vertex coloring*, which is a labeling of vertices of $G$ with colors such that no two adjacent vertices have the same colors. Let $P(G,\lambda)$ denote the number of (proper) vertex colorings of $G$ with $\lambda$ colors and $\chi(G)$ the least number $\lambda$ satisfying $P(G,\lambda)>0$, where $P(G,\lambda)$ and $\chi(G)$ are called a *chromatic polynomial* and *chromatic number* of $G$, respectively.

In fact, it is clear that the number of $\lambda$-colorings is a polynomial in $\lambda$ from a deletion-contraction recurrence.

PROPOSITION 1 (Deletion-contraction recurrence). *For a given a graph $G$ and an edge $e$ in $G$, we have*

$$P(G,\lambda)=P(G-e,\lambda)-P(G/e,\lambda), \tag{1}$$

*where $G-e$ is a graph obtained by deletion the edge $e$ and $G/e$ is a graph obtained by contraction the edge $e$.*

EXAMPLE. The chromatic polynomials of graphs in Figure 1 are

$$\begin{aligned} P(G,\lambda) &= \lambda(\lambda-1)^2(\lambda-2), \\ P(G-e,\lambda) &= \lambda^2(\lambda-1)(\lambda-2), \textit{and} \\ P(G/e,\lambda) &= \lambda(\lambda-1)(\lambda-2). \end{aligned}$$

It is confirmed that (1) is true for the graph $G$ and the edge $e$ in Figure 1.

**Cycle graph.** A *cycle graph $C_n$* is a graph that consists of a single cycle of length $n$, which could be drown by a $n$-polygonal graph in a plane. The chromatic polynomial for cycle graph $C_n$ is well-known as follows.

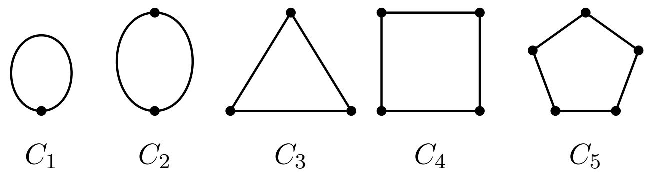



FIGURE 2. $C_n$ $(1 \leq n \leq 5)$

THEOREM 2. *For a positive integer* $n \geq 1$, *the chromatic polynomial for cycle graph* $C_n$ *is*

$$P(C_n, \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1) \tag{2}$$

EXAMPLE. For an integer $n \leq 3$, it is easily checked that the chromatic polynomials of $C_n$ are from (2) as follows.

$$P(C_1, \lambda) = (\lambda - 1) + (-1)(\lambda - 1) = 0,$$
$$P(C_2, \lambda) = (\lambda - 1)^2 + (-1)^2(\lambda - 1) = \lambda(\lambda - 1),$$
$$P(C_3, \lambda) = (\lambda - 1)^3 + (-1)^3(\lambda - 1) = \lambda(\lambda - 1)(\lambda - 2).$$

As shown in Figure 2, the cycle graph $C_1$ is a graph with one vertex and one loop and $C_1$ cannot be colored, that means $P(C_1, \lambda) = 0$. The cycle graph $C_2$ is a graph with two vertices, where two edges between two vertices, and $C_2$ can have colorings by assigning two vertices with different colors, that means $P(C_2, \lambda) = \lambda(\lambda - 1)$. The cycle graph $C_3$ is drawn by a triangle and $C_3$ can have colorings by assigning all three vertices with different colors, that means $P(C_3, \lambda) = \lambda(\lambda - 1)(\lambda - 2)$.

## 2. Four proofs of Theorem 2

In this section, we show the formula (2) in four different ways.

**2.1. Inductive proof.** This inductive proof is widely well-known. A *path graph* $P_n$ is a connected graph in which $n-1$ edges connect $n$ vertices of vertex degree at most 2, which could be drawn on a single straight line. The chromatic polynomial for path graph $P_n$ is easily obtained by coloring all vertices $v_1, \ldots, v_n$ where $v_i$ and $v_{i+1}$ have different colors for $i = 1, \ldots, n - 1$.

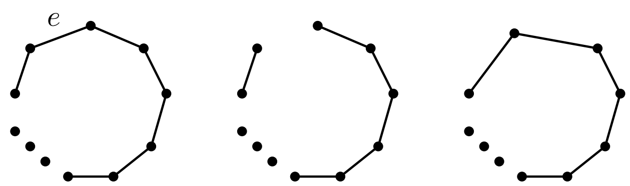



FIGURE 3. $C_{n+1}$ , $P_{n+1}$ and $C_n$

LEMMA 3. *For a positive integer $n \geq 1$, the chromatic polynomial for path graph $P_n$ is*

$$P(P_n, \lambda) = \lambda(\lambda - 1)^{n-1}. \tag{3}$$

We use an induction on the number $n$ of vertices by the deletion-contraction recurrence and the above lemma for path graph: It is already shown that (2) is true for $n \leq 3$ by the example in Section 1. Assume that (2) is true for a positive integer $n$. Using (1) and (3), we have

$$\begin{aligned} P(C_{n+1}, \lambda) &= P(C_{n+1} - e, \lambda) - P(C_{n+1}/e, \lambda) && \text{by (1)} \\ &= P(P_{n+1}, \lambda) - P(C_n, \lambda) \\ &= \lambda(\lambda - 1)^n - ((\lambda - 1)^n + (-1)^n(\lambda - 1)) && \text{by (3)} \\ &= (\lambda - 1)^{n+1} + (-1)^{n+1}(\lambda - 1). \end{aligned}$$

Thus, (2) is true for all positive integers $n \geq 1$.

**2.2. Proof by inclusion-exclusion principle.** The *inclusion-exclusion principle* is a technique of counting the size of the union of finite sets.

PROPOSITION 4 (Inclusion-exclusion principle). *Let $A_1, A_2, \ldots, A_n$ be subsets of a finite set $U$. Then number of elements excluding their union is as follows*

$$\begin{aligned} \left| \bigcap_{i=1}^{n} \overline{A_i} \right| &= \sum_{I \subset [n]} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right| \\ &= |U| - \sum_{i=1}^{n} |A_i| + \sum_{i<j} |A_i \cap A_j| - \cdots + (-1)^n |A_1 \cap \cdots \cap A_n| \end{aligned}$$

*where $\overline{A}$ is the complement of $A$ in $U$.*

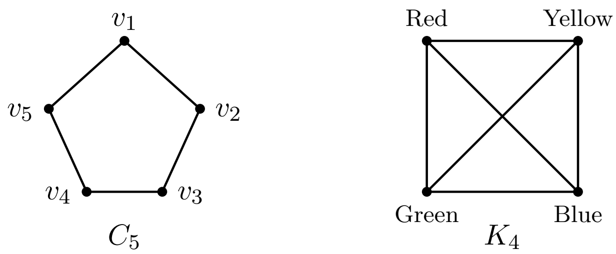



FIGURE 4. A cycle graph $C_5$ and a graph $K_4$ with names of colors

Considering every condition to assign different colors to two adjacent vertices, for each edge $e$, we define a finite sets of arbitrary (including improper) colorings to assign same color to two adjacent vertices by the edge $e$.

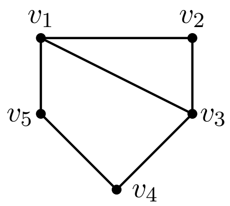
Let $A_i$ be a set of colorings such that two vertices $v_i$ and $v_{i+1}$ are of same color, where $v_{n+1}$ is regarded as $v_1$. Applying the inclusion-exclusion principle, we can write the following

$$\begin{aligned} P(C_n, \lambda) &= |U| - \sum_{i=1}^{n} |A_i| + \sum_{i<j} |A_i \cap A_j| + \cdots + (-1)^n |A_1 \cap \cdots \cap A_n| \\ &= \lambda^n - \binom{n}{1}\lambda^{n-1} + \binom{n}{2}\lambda^{n-2} + \cdots + (-1)^n\lambda \\ &= (\lambda-1)^n - (-1)^n + (-1)^n\lambda \\ &= (\lambda-1)^n + (-1)^n(\lambda-1). \end{aligned}$$

Thus, (2) is true for all positive integers $n \geq 1$.

**2.3. Algebric proof.** Let us consider a case of $n = 5$ and $\lambda = 4$, that is, to assign the vertices of $C_5$ in four colors: red, blue, yellow, and green. Also let us consider a complete graph $K_4$ with vertex names red, blue, yellow, and green, see Figure 4.

When red-blue-red-yellow-green is assigned in order from the vertex $v_1$ to the vertex $v_5$ in $C_5$, it is corresponding to a closed walk of length 5 in $K_4$ which begins and ends at red, that is, it is red-blue-red-yellow-green-red in $K_4$. By generalizing it, we have a correspondence between




$G$

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

FIGURE 5. A graph $G$ and its adjacency matrix $A$

$\lambda$-colorings of $C_n$ and closed walks of length $n$ in $K_\lambda$. By this correspondence, it is enough to count the number of closed walks of length $n$ in $K_\lambda$, instead of the number of $\lambda$-colorings of $C_n$.

For a graph $G$ with vertex set $\{v_1, \ldots, v_n\}$, the *adjacency matrix* of $G$ is an $n \times n$ square matrix $A$ such that its element $A_{ij}$ is one when there is an edge between two vertices $v_i$ and $v_j$, and zero when there is no edge between $v_i$ and $v_j$.

The following related to an adjacency matrix is well-known.

PROPOSITION 5. *Let $A$ be the adjacency matrix of the graph $G$ on $n$ vertices $v_1, \ldots, v_n$. Then the $(i, j)$th entry of the matrix $A^n$ is the number of the walk of length $n$ beginning at $v_i$ and ending at $v_j$.*

By Proposition 5, we can calculate the number of closed walk of length $n$ in the complete graph $K_\lambda$: Let $A$ be an adjacency matrix of $K_\lambda$. Then $A$ is a $\lambda \times \lambda$ matrix as follows

$$A = (a_{ij}) = \begin{pmatrix} 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix},$$

where $a_{ij} = 0$ if $i = j$, and otherwise $a_{ij} = 1$. So the number of closed walks of length $n$ in $K_\lambda$ is enumerated by $tr(A^n)$, which equals the sum of all eigenvalues of $A^n$. Also let all eigenvalues of the matrix $A$ be denoted by $u_1, \ldots, u_\lambda$, then all eigenvalues of the matrix $A^n$ are $u_1^n, \ldots, u_\lambda^n$.

$$A = \begin{pmatrix} 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} \lambda-1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix},$$

Since the matrix $A$ have $\lambda$ eigenvalues $u_1 = \lambda - 1$ and $u_2 = \cdots = u_\lambda = -1$, we have

$$tr(A^n) = \sum_{i=1}^{\lambda} u_i^n = (\lambda-1)^n + \underbrace{(-1)^n + \cdots + (-1)^n}_{\lambda - 1 \text{ times}}.$$

Thus, (2) is true for all positive integers $n \geq 1$.

**2.4. Bijective proof.** Let $X_n$ denote the set of $\lambda$-colorings of $C_n$ and $[\lambda - 1]^n$ be the set of $n$-tuples of positive integers less than $\lambda$, where $[\lambda - 1]$ means $\{1, \ldots, \lambda - 1\}$. We consider a mapping $\varphi$ from $\lambda$-colorings of $C_n$ in $X_n$ to $n$-tuples in $[\lambda - 1]^n$.

*A mapping $\varphi$ from $X_n$ to $[\lambda - 1]^n$.* The mapping $\varphi : X_n \to [\lambda - 1]^n$ is defined as follows: Let $\omega$ be a $\lambda$-coloring of $C_n$ in $X_n$, we write $\omega = (\omega_1, \ldots, \omega_n)$ where $\omega_i$ is the color of $v_i$ in $C_n$ and it is obvious that $\omega_i \neq \omega_{i+1}$ for $1 \leq i \leq \lambda$, where $\omega_{n+1}$ is regarded as $\omega_1$. An entry $\omega_i$ is called a *cyclic descent* of $C$ if $\omega_i > \omega_{i+1}$ for $1 \leq i \leq \lambda$. Then we define $\varphi(\omega) = \sigma = (\sigma_1, \ldots, \sigma_n)$ with

$$\sigma_i = \begin{cases} \omega_i - 1, & \text{if } \omega_i \text{ is a cyclic descent} \\ \omega_i, & \text{otherwise.} \end{cases}$$

Given a $\lambda$-coloring $\omega$, if $\omega_i = \lambda$ then $\omega_{i+1} < \lambda$, so $\omega_i = \lambda$ should be a cyclic descent. Thus we have $\sigma_i < \lambda$ for all $1 \leq i \leq n$ and $\varphi(\omega)$ belongs to $[\lambda - 1]^n$.

For example, in a case of $n = 9$ and $\lambda = 4$, $\omega = (1, 2, 1, 3, 2, 3, 1, 4, 2) \in X_9$ is given as an example of 4-colorings of $C_9$. Here $\omega_2 = 2$, $\omega_4 = 3$, $\omega_6 = 3$, $\omega_8 = 4$, and $\omega_9 = 2$ are cyclic descents of $\omega$. So we have

$$\varphi(\omega) = \sigma = (1, 1, 1, 2, 2, 2, 1, 3, 1) \in [3]^9.$$

*A mapping* $\psi$ *as the inverse of* $\varphi$. Let $Z_n$ be the set of $n$-tuples $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ in $[\lambda - 1]^n$ with

$$\sigma_1 = \sigma_2 = \cdots = \sigma_n$$

and it is obvious that the size of $Z_n$ is $\lambda - 1$.

We would like to describe a mapping $\psi : ([\lambda - 1]^n \setminus Z_n) \to X_n$ in order to satisfy $\varphi \circ \psi$ is the identity on $[\lambda - 1]^n \setminus Z_n$ as follows: Given a $\sigma \in [\lambda - 1]^n \setminus Z_n$, we define $\overline{\sigma} = (\overline{\sigma}_1, \ldots, \overline{\sigma}_n)$ with

$$\overline{\sigma}_i = \begin{cases} \sigma_i + 1, & \text{if } \sigma_i \text{ is a cyclic descent} \\ \sigma_i, & \text{otherwise.} \end{cases}$$

Since $\overline{\sigma}$ may have consecutive same entries, we define $\psi(\sigma) = \omega = (\omega_1, \ldots, \omega_n)$ from $\overline{\sigma}$ with $\omega_i = \overline{\sigma}_i + 1$ for any entry $\overline{\sigma}_i$ of $\overline{\sigma}$ with a finite positive even integer $\ell$ satisfying

$$\overline{\sigma}_i = \overline{\sigma}_{i+1} = \cdots = \overline{\sigma}_{i+\ell-1} \neq \overline{\sigma}_{i+\ell},$$

where $\overline{\sigma}_{n+k}$ is regarded as $\overline{\sigma}_k$ for $1 \leq k \leq n$, and $\omega_i = \overline{\sigma}_i$, otherwise. Thus $\omega$ has no consecutive same entries and $1 \leq \omega_i \leq \lambda$ for all $1 \leq i \leq n$, so $\psi(\sigma) = \omega$ belongs to $X_n$. Moreover, it is obvious that $\sigma_i \leq \omega_i \leq \sigma_i + 1$ for all $1 \leq i \leq n$ and if $\omega_i = \sigma_i + 1$ for some $1 \leq i \leq n$ then $\omega_i$ is a cyclic descent in $\omega$. Hence $\varphi(\omega) = \sigma$ and $\sigma \in [\lambda - 1]^n \setminus Z_n$ if and only if $\psi(\sigma) = \omega$.

In a previous example, $\sigma = (1, 1, 1, 2, 2, 2, 1, 3, 1)$ is denoted as an example of 9-tuples in $[3]^9$. Here $\sigma_6 = 2$, $\sigma_8 = 3$ are cyclic descents of $\sigma$ and we obtain $\overline{\sigma} = (1, 1, 1, 2, 2, 3, 1, 4, 1)$. And then there exist only three entries $\overline{\sigma}_2$, $\overline{\sigma}_4$, and $\overline{\sigma}_9$ in $\overline{\sigma}$ satisfying the following

$$\begin{aligned} k = 2 :&\quad \overline{\sigma}_2 = \overline{\sigma}_3 \neq \overline{\sigma}_4 \quad (\ell = 2), \\ k = 4 :&\quad \overline{\sigma}_4 = \overline{\sigma}_5 \neq \overline{\sigma}_6 \quad (\ell = 2), \text{ and} \\ k = 9 :&\quad \overline{\sigma}_9 = \overline{\sigma}_1 = \overline{\sigma}_2 = \overline{\sigma}_3 \neq \overline{\sigma}_4 \quad (\ell = 4), \end{aligned}$$

so we get $\omega_2 = \overline{\sigma}_2 + 1 = 2$, $\omega_4 = \overline{\sigma}_4 + 1 = 3$, $\omega_9 = \overline{\sigma}_9 + 1 = 2$, and

$$\psi(\sigma) = \omega = (1, 2, 1, 3, 2, 3, 1, 4, 2) \in X_9.$$

Let $Y_n$ be the set of $\lambda$-colorings $\omega$ in $X_n$ with $\varphi(\omega) \in Z_n$. Since two mapping $\varphi$ and $\psi$ are bijections between $X_n \setminus Y_n$ and $[\lambda - 1]^n \setminus Z_n$, the size of the set $X_n \setminus Y_n$ is same with the size of the $[\lambda - 1]^n \setminus Z_n$, which is equal to $(\lambda - 1)^n - (\lambda - 1)$.

When $n$ is even, for any $1 \leq i \leq \lambda - 1$, there exist only two $n$-tuples in $X_n$

$$\omega = (i+1, i, i+1, i, \ldots, i+1, i) \quad \text{and} \quad \omega = (i, i+1, i, i+1, \ldots, i, i+1)$$

satisfying $\varphi(\omega) = (i, i, \ldots, i) \in Z_n$. If $n$ is even, the size of $Y_n$ is equal to $2(\lambda - 1)$ and we obtain

$$\begin{aligned} P(C_n, \lambda) &= |X_n| = |X_n \setminus Y_n| + |Y_n| \\ &= [(\lambda - 1)^n - (\lambda - 1)] + 2(\lambda - 1). \end{aligned} \tag{4}$$

When $n$ is odd, there is no $n$-tuples satisfying $\varphi(\omega) \in Z_n$ and the set $Y_n$ is empty. If $n$ is odd, we obtain

$$\begin{aligned} P(C_n, \lambda) &= |X_n| = |X_n \setminus Y_n| + |Y_n| \\ &= [(\lambda - 1)^n - (\lambda - 1)] + 0. \end{aligned} \tag{5}$$

Therefore, (2) yields from (4) and (5) for all positive integers $n \geq 1$.

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