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CROSSED SEMIMODULES AND CAT$^{\mathbf{1}}$-MONOIDS

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Abstract. The main idea of this paper is to introduce the notion of cat$^{\mathbf{1}}$-monoids and to prove that the category of crossed semimodules $\mathcal{C} = (A, B, \partial)$ where A is a group is equivalent to the category of cat$^{\mathbf{1}}$-monoids. This is a generalization of the well known equivalence between category of cat$^{\mathbf{1}}$-groups and that of crossed modules over groups.

1. Introduction

Crossed modules as defined by Whitehead [30, 31] have been widely used in homotopy theory [5], the theory of group representation (see [7] for a survey), in algebraic K-theory [13] and homological algebra [12, 15]. Crossed modules can be viewed as 2-dimensional groups [6]. In [9] Brown and Spencer proved that the category of internal groupoids within the groups, which are also called in [9] under the name of $\mathcal{G}$-groupoids and alternative names, quite generally used are group-groupoids [8] or 2-groups (see for example [2]) is equivalent to the category of crossed modules of groups. In [14] Loday defined an algebraic object called cat$^{\mathbf{1}}$-group as a group G with two endomorphisms s, t of G such that $st = t, ts = s$ and $[\operatorname{Ker} s, \operatorname{Ker} t] = 0$, where $[\operatorname{Ker} s, \operatorname{Ker} t]$ represents the commutator subgroup of G; and proved that the categories of cat$^{\mathbf{1}}$-groups and crossed modules are equivalent.

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In [25] Porter proved a similar result to one in [9] holds for certain algebraic categories, introduced by Orzech [23], which definition was adapted by him and called category of groups with operations. The equivalences of these categories given in [9] and [25] are very important and useful which enable to generalize some results on group-groupoids, equivalently $\mathrm{cat}^{\mathbf{1}}$-groups, to the more general internal groupoids for a certain algebraic category (see for example [1, 18, 19] and [21]). There are also other generalizations of the Brown-Spencer Theorem [9]. First one is the generalization to the category of monoids, that is the natural equivalence of the category of crossed semimodules satisfying certain conditions and that of Schreier internal categories within the category of monoids which is given by Patchkoria [24]. The other one is the generalization to the category of monoid with operations given in [22] by Martins-Ferreira et. al. In [24], Patchkoria also defined Schreier internal groupoids in the category of monoids which are naturally equivalent to the category of crossed semimodules $\mathcal{C} = (A, B, \partial)$ where A is a group. This is the special case of the equivalence given in [26] by Porter. Topological aspect of the results of Patchkoria is given in [28]. See [29] for 2-categorical approach to Schreier internal categories within monoids using Schreier 2-categories with one object. Equivalences given in [3, 9, 24, 26, 28] are very important and enable one to interpret special objects such as normal subobject, quotient object, etc. and special morphisms such as covering, lifting, etc. which are known in one of the categories in equivalences. See for example [1, 17, 19, 20, 27] for such interpretations.

The natural equivalence between crossed modules and $\mathrm{cat}^{\mathbf{1}}$-groups is useful for generalisation of crossed modules to higher dimensions (see [14]). In this note we introduce the notion of $\mathrm{cat}^{\mathbf{1}}$-monoids and prove the natural equivalence between the category of $\mathrm{cat}^{\mathbf{1}}$-monoids and the category of crossed semimodules $\mathcal{C} = (A, B, \partial)$ where A is a group. This is a new way of thinking crossed semimodules as simple algebraic structures.

2. Preliminaries

Let C be a finitely complete category. An *internal category* $\mathrm{D} = (D_0, D_1, d_0, d_1, \varepsilon, \mu)$ in C consists of a set of objects D_0 and a set of morphisms D_1 together with morphisms $d_0, d_1 \colon D_1 \to D_0$, $\varepsilon \colon D_0 \to D_1$ in C called the source, the target and the identity object maps, respectively,

$$D_1 \underset{t}{\overset{\overset{\varepsilon}{\longleftarrow}}{\underset{s}{\rightrightarrows}}} D_0$$

such that $d_0\varepsilon = d_1\varepsilon = 1_{D_0}$ and a morphism $\mu\colon D_1 \times_{D_0} D_1 \to D_1$ of $\mathcal{C}$ called the composition map (usually expressed as $\mu(f,g) = g \circ f$) where $D_1 \times_{D_0} D_1$ is the pullback of d_0, d_1 such that $\varepsilon d_0(f) \circ f = f = f \circ \varepsilon d_0(f)$ [16]. An *internal groupoid* in C is an internal category in which all morphisms are invertible up to groupoid composition.

An internal category (hence a groupoid) in the category Gp of groups is called group-groupoid. A group-groupoid can also be obtained as a group object in Cat.

A crossed module over groups is a pair of groups A, B with an action $\bullet\colon B \times A \to A$ of B on A denoted by $b \bullet a$ for $a \in A$ and $b \in B$ and a morphism $\partial\colon A \to B$ of groups satisfies the following conditions

[CM 1] $\partial(b \bullet a) = b\partial(a)b^{-1}$

[CM 2] $\partial(a) \bullet a_1 = aa_1a^{-1}$,

for $a, a_1 \in A$ and $b \in B$ [30, 31]. For the basic examples of crossed modules see [10].

The following theorem is proved in [9] by Brown and Spencer.

Theorem 2.1. *The category* GpGd *of group-groupoids is equivalent to the category* Cm *of crossed modules over groups.*

Let Mon denote the category of monoids. Recall from [24] that a Schreier internal category $\mathrm{M} = (M_0, M_1, d_0, d_1, \varepsilon, \mu)$ in Mon is an internal category which satisfies the Schreier condition: for any $f \in M_1$ there exists a unique $g \in \operatorname{Ker} d_0$ such that

$$f = g\varepsilon d_0(f).$$

A Schreier internal groupoid in Mon is a Schreier internal category in which all morphisms are invertible up to composition.

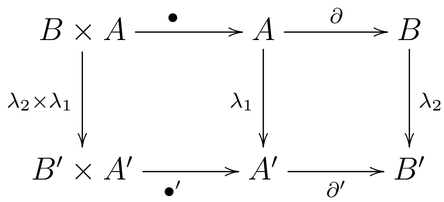
We recall the definition of crossed semimodules from [24]. A crossed semimodule $\mathcal{C} = (A, B, \partial)$ consists of a pair of monoids A, B and a homomorphism $\partial\colon A \to B$ of monoids with an action $\bullet\colon B \times A \to A$ of B on A satisfying

[CSM 1] $\partial(b \bullet a)b = b\partial(a)$

[CSM 2] $(\partial(a) \bullet a_1)a = aa_1$,

for $a, a_1 \in A$ and $b \in B$.

Let $\mathcal{C} = (A, B, \partial)$ and $\mathcal{C}' = (A', B', \partial')$ be crossed semimodules. A crossed semimodule morphism is a mapping $\lambda = (\lambda_1, \lambda_2)\colon \mathcal{C} \to \mathcal{C}'$ where $\lambda_1\colon A \to A'$ and $\lambda_2\colon B \to B'$ are monoid homomorphisms such that $\lambda_1(b \bullet a) = \lambda_2(b) \bullet' \lambda_1(a)$ and $\lambda_2\partial = \partial'\lambda_1$.



Hence crossed semimodules and their morphisms form a category which we denoted by Csm.

The following theorem and corollary are proved in [24]:

Theorem 2.2. *The category* SIC *of Schreier internal categories in* Mon *is equivalent to the category* Csm *of crossed semimodules.*

Note that this theorem is obtained as a special case of the equivalence given in [29].

Let Csm* denote the category of crossed semimodules $\mathcal{C} = (A, B, \partial)$ such that A is a group. Then we obtain the following corollary as a restriction of above equivalence.

Corollary 2.3. *The category* SIG *of Schreier internal groupoids in* Mon *is equivalent to the category* Csm*.

Note that this corollary is obtained as a special case of the equivalence given in [26]. Restricting this corollary to the case of groups, Theorem 2.1 is obtained as given in [9] and [24].

Let G be a group. We recall that from [10] a $\mathbf{cat^1}$-group (or 1-cat-group [14]) is a triple $\mathcal{G} = (G, s, t)$ with two group homomorphisms $s, t\colon G \to G$ called structural maps satisfying following conditions

[C1G 1] $st = t$ and $ts = s$

[C1G 2] $[\operatorname{Ker} s, \operatorname{Ker} t] = 1$ where $[\operatorname{Ker} s, \operatorname{Ker} t]$ is the commutator subgroup of $\operatorname{Ker} s$ and $\operatorname{Ker} t$.

A cat$^{\mathbf{1}}$-group morphism is a group homomorphism $f\colon (G,s,t) \to (G',s',t')$ where $\mathcal{G} = (G,s,t)$ and $\mathcal{G}' = (G',s',t')$ are cat$^{\mathbf{1}}$-groups such that $s'f = fs$ and $t'f = ft$.

$$\begin{array}{ccc} G & \overset{s}{\underset{t}{\rightrightarrows}} & G \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ G' & \overset{s'}{\underset{t'}{\rightrightarrows}} & G' \end{array}$$

Hence cat$^{\mathbf{1}}$-groups form a category which we denoted by $\text{Cat}^1\text{-Gp}$. For the categorical equivalence between crossed modules over groups and cat$^{\mathbf{1}}$-groups, see [10].

3. Cat$^{\mathbf{1}}$-monoids

Definition 3.1. A cat$^{\mathbf{1}}$-monoid is a triple $\mathcal{M} = (M,s,t)$ where M is a monoid and $s,t\colon M \to M$ are monoid homomorphisms of M satisfying

[C1M 1] $st = t$ and $ts = s$,
[C1M 2] $xy = yx$ for $x \in \operatorname{Ker} s,\ y \in \operatorname{Ker} t$,
[C1M 3] $\operatorname{Ker} s$ is a group and
[C1M 4] there exists a unique $\widetilde{m} \in \operatorname{Ker} s$ for all $m \in M$ such that $m = \widetilde{m}s(m)$.

Example 3.2. Consider the monoid $(\mathbb{Z}_4,\cdot)$ and the group $(\mathbb{Z}_5,\cdot)$. Then we construct a cat$^{\mathbf{1}}$-monoid $(\mathbb{Z}_4 \times \mathbb{Z}_5, s, t)$ with direct product where $s(\overline{x}^4,\overline{y}^5) = (\overline{x}^4,\overline{1}^5)$ and $t(\overline{x}^4,\overline{y}^5) = (\overline{y\cdot x}^4,\overline{1}^5)$. Clearly $\operatorname{Ker} s$ is a group. Since $(\overline{x}^4,\overline{y}^5) = (\overline{1}^4,\overline{y}^5)\cdot(\overline{x}^4,\overline{1}^5)$, all pairs satisfy the condition [C1M4].

Example 3.3. Every cat$^{\mathbf{1}}$-group is a cat$^{\mathbf{1}}$-monoid. Note that $\widetilde{m} = ms(m^{-1})$.

Proposition 3.4. *Given any cat$^{\mathbf{1}}$-monoid (M,s,t), we have*

1. $\operatorname{Im} s = \operatorname{Im} t$,
2. *s and t are identities on* $\operatorname{Im} s$,
3. *$s^2 = s$ and $t^2 = t$.*

Proof. The proof is straightforward. So it is ommitted. □

Definition 3.5. A morphism $f\colon (M,s,t) \to (M',s',t')$ of $\mathbf{cat^1}$-monoids is a morphism of monoids such that $s'f = fs$ and $t'f = ft$.

$$\begin{array}{ccc} M & \overset{s}{\underset{t}{\rightrightarrows}} & M \\ f\downarrow & & \downarrow f \\ M' & \overset{s'}{\underset{t'}{\rightrightarrows}} & M' \end{array}$$

Hence we can construct the category of $\mathbf{cat^1}$-monoids which we denoted by Cat^1-Mon.

Proposition 3.6. *Let $\mathcal{C} = (A,B,\partial)$ be an object of* Csm*. *Then $\mathcal{M} = (M,s,t)$ is a $\mathbf{cat^1}$-monoid where $M = B \ltimes A$ is the semi-direct product of monoids, $s(b,a) = (b,1)$ and $t(b,a) = (\partial(a)b,1)$.*

Proof. We define a functor $\psi\colon \text{Csm}^* \to \text{Cat}^1\text{-Mon}$ such that $\psi(A,B,\partial) = (M,s,t)$. The product of M is defined by

$$(b,a)(b',a') = (bb', a(b \bullet a'))$$

for $a,a' \in A,\ b,b' \in B$. Clearly s is a monoid homomorphism. We prove that t is a homomorphism of monoids.

$$\begin{aligned} t\Big((b,a)(b',a')\Big) &= t\Big(bb', a(b\bullet a')\Big) \\ &= \Big(\partial(a)\partial(b\bullet a')bb', 1\Big) \\ &= (\partial(a)b\partial(a')b', 1) \\ &= (\partial(a)b,1)(\partial(a')b',1) \\ &= t(b,a)t(b',a') \end{aligned}$$

It is easy to show that s and t satisfy [C1M1]. With respect to [C1M2], for elements $(1,a)\in \operatorname{Ker} s$ and $(b',a')\in \operatorname{Ker} t$, we have

$$\begin{aligned}(b',a')(1,a) &= (b',a'(b'\bullet a))\\ &= \Big(b',\big(\partial(a')\bullet(b'\bullet a)\big)a'\Big)\\ &= \Big(b',\big(\partial(a')b'\bullet a\big)a'\Big)\\ &= \Big(b',\big(1\bullet a\big)a'\Big)\\ &= (b',aa')\\ &= (b',a(1\bullet a'))\\ &= (1,a)(b',a').\end{aligned}$$

Clearly $\operatorname{Ker} s$ is a group. Since $(b,a)=(1,a)(b,1)=(1,a)s(b,a)$, that is $\widetilde{(b,a)}=(1,a)$ and so each element of M satisfies [C1M4]. □

PROPOSITION 3.7. *Let $\mathcal{M}=(M,s,t)$ be a cat$^{\mathbf{1}}$-monoid. Then $\gamma(M,s,t)=(A,B,\partial)$ is an object of* CSM* *where $A=\operatorname{Ker} s,\ B=\operatorname{Im} s,\ \partial=t|_{\operatorname{Ker} s}$ and an action of B on A is defined by $(n\bullet x)n=nx$, for $x\in\operatorname{Ker} s$ and $n\in\operatorname{Im} s$.*

Proof. We define a functor

$$\gamma\colon\ \text{CAT}^1\text{-MON}\to \text{CSM}^*$$

as a weak inverse of ψ such that $\gamma(M,s,t)=(A,B,\partial)$. First we show that $\bullet$ is an action of $\operatorname{Im} s$ on $\operatorname{Ker} s$. Let $n,n'\in\operatorname{Im} s$ and $x,x'\in\operatorname{Ker} s$. Since

$$(nn'\bullet x)nn' = nn'x = n(n'\bullet x)n' = \Big(n\bullet(n'\bullet x)\Big)nn',$$

under the condition [C1M4] we write

$$(nn')\bullet x = n\bullet(n'\bullet x).$$

Since

$$(n\bullet(xx'))n = nxx' = (n\bullet x)nx' = (n\bullet x)(n\bullet x')n,$$

under the condition [C1M4]

$$n\bullet(xx') = (n\bullet x)(n\bullet x').$$

Clearly

$$1\bullet x = (1\bullet x)1 = 1x = x.$$

Now we will verify [CSM1] and [CSM2]. Since $n = t(n) = s(n)$, by Proposition 3.4. we get

$$t(n \bullet x)n = t(n \bullet x)t(n) = t((n \bullet x)n) = t(nx) = t(n)t(x) = nt(x).$$

On the other hand

$$(t(x) \bullet x')x = (t(x) \bullet x')t(x)t(x^{-1})x = t(x)x't(x^{-1})x.$$

Since $t(x^{-1})x \in \operatorname{Ker} t$ and $x' \in \operatorname{Ker} s$, by [C1M2] we have

$$(t(x) \bullet x')x = t(x)x't(x^{-1})x = t(x)t(x^{-1})xx' = xx'.$$

□

As a corollary of Propositions 3.6 and 3.7 we can give the following theorem.

Theorem 3.8. *The categories* Cat^1-Mon *and* Csm* *are equivalent.*

Proof. Let $f\colon (M, s, t) \to (M', s', t')$ be a morphism of cat$^{\mathbf{1}}$-monoids, $n \in \operatorname{Im} s, x \in \operatorname{Ker} s$. Since $f(n) \in N'$, the condition [C1M4] allows us to write

$$f(n \bullet x)f(n) = f((n \bullet x)n) = f(nx) = f(n)f(x) = \Big(f(n) \bullet f(x)\Big)f(n)$$

and so under the same condition we have

$$f(n \bullet x) = f(n) \bullet f(x).$$

This means that $\gamma(f) = (f, f)$ is a morphism of crossed semimodules.

On the other hand, given a morphism $\lambda = (\lambda_1, \lambda_2)\colon \mathcal{C} \to \mathcal{C}'$ where $\mathcal{C} = (A, B, \partial)$ and $\mathcal{C}' = (A', B', \partial')$ are crossed semimodules such that A, A' are groups, $\psi(\lambda_1, \lambda_2) = (\lambda_2, \lambda_1)$ is a morphism of cat$^{\mathbf{1}}$-monoids as shown in the following diagram.

$$\begin{array}{ccc} B \ltimes A & \overset{s}{\underset{t}{\rightrightarrows}} & B \ltimes A \\ {\scriptstyle (\lambda_2,\lambda_1)}\downarrow & & \downarrow{\scriptstyle (\lambda_2,\lambda_1)} \\ B' \ltimes A' & \overset{s'}{\underset{t'}{\rightrightarrows}} & B' \ltimes A' \end{array}$$

We will verify that this diagram is commutative.

$$\begin{aligned}(\lambda_2,\lambda_1)s(b,a) &= (\lambda_2,\lambda_1)(b,1)\\ &= (\lambda_2(b),1)\\ &= s'(\lambda_2(b),\lambda_1(a))\\ &= s'(\lambda_2,\lambda_1)(b,a)\end{aligned}$$

$$\begin{aligned}(\lambda_2,\lambda_1)t(b,a) &= (\lambda_2,\lambda_1)(\partial(a)b,1)\\ &= (\lambda_2\partial(a)\lambda_2(b),1)\\ &= (\partial\lambda_1(a)\lambda_2(b),1)\\ &= t'(\lambda_2(b),\lambda_1(a))\\ &= t'(\lambda_2,\lambda_1)(b,a)\end{aligned}$$

A natural equivalence $S\colon 1_{\text{CAT}^1\text{-MON}} \to \psi\gamma$ is given via a mapping

$$S(M,s,t) = (\operatorname{Im} s \ltimes \operatorname{Ker} s, s', t')$$

such that $S_M(m) = (s(m),\widetilde{m})$ and $s'(n,x) = (n,1),\ t'(n,x) = (t(x)n,1)$ is an isomorphism. We will verify that S_M is a homomorphism.

$$\begin{aligned}S_M(mm_1) &= S_M\Big(\widetilde{m}s(m)\widetilde{m_1}s(m_1)\Big)\\ &= S_M\Big(\widetilde{m}\big(s(m)\bullet\widetilde{m_1}\big)s(m)s(m_1)\Big)\\ &= S_M\Big(\widetilde{m}\big(s(m)\bullet\widetilde{m_1}\big)s(mm_1)\Big)\\ &= \Big(s(mm_1),\widetilde{m}\big(s(m)\bullet\widetilde{m_1}\big)\Big)\\ &= \Big(s(m),\widetilde{m}\Big)\Big(s(m_1),\widetilde{m_1}\Big)\\ &= S_M(m)S_M(m_1)\end{aligned}$$

On the other hand, a natural equivalence $T\colon 1_{\text{CSM}^*} \to \gamma\psi$ is defined by

$$T(\mathcal{C}) = (\operatorname{Ker} s, s(B\ltimes A), t)$$

for $\mathcal{C} = (A,B,\partial)$ such that $T_{\mathcal{C}}(b) = (b,1),\ T_{\mathcal{C}}(a) = (1,a)$.

Other details are straightforward and so is omitted. □

Since every group is a monoid then the following equivalence given in [10, 14] is a consequence of Theorem 3.8.

Theorem 3.9. [10] *The category of crossed modules over groups is equivalent to the category of* $\mathrm{cat}^{\mathbf{1}}$*-groups.*

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